Extragradient subgradient-type algorithms with adaptive step sizes for solving quasi-equilibrium problems in Hilbert spaces

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Abstract

We propose two iterative algorithms for solving pseudomonotone quasi-equilibrium problem in Hilbert spaces. The first one combines the subgradient method and the projection method with self-adaptive step sizes to generate a sequence of iterates that converges weakly to a solution of the problem. The second algorithm combines the first one with the Mann iteration scheme to obtain the strong convergence of the generated iterates. The convergence of our proposed algorithms requires a condition milder than a similar one assumed in existing iterative solution methods for quasi-equilibrium problem. Numerical experiments show that our algorithms are efficient and competitive to other extragradient-type, projection-type, and proximal point algorithms in solving the problem.

Keywords: Quasi-equilibrium problem, fixed point problem, pseudomonotone, quasidomonotone, extragradient method, projection method, Mann iteration scheme

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a closed convex subset of $\mathcal{H}$. We are interested in the following quasi-equilibrium problem:

$$\text{(QEP)} \qquad \text{Find } p \in T(p) \text{ such that } f(p,y) \geq 0 \text{ for all } y \in T(p),$$

where T is a multivalued mapping from C into itself and $f : C \times C \to \mathbb{R}$ is a bifunction such that $f(x,x) = 0$ for all $x \in C$.

There are two following important particular cases of (QEP).

- When $f(x,y) = \langle F(x), y - x \rangle$ where F is a mapping from C to $\mathcal{H}$, (QEP) becomes a convex quasi-variational inequality problem:

$$\text{(QVIP)} \qquad \text{Find } p \in T(p) \ \text{ such that } \langle F(p), x - p \rangle \geq 0 \text{ for all } y \in T(p).$$

- When $T(x) = C$ for all $x \in C$, (QEP) becomes the following equilibrium problem:

$$\text{(EP)} \qquad \text{Find } p \in C \ \text{ such that } f(p,y) \geq 0 \text{ for all } y \in C.$$

The (QVIP) was introduced in [1] in relation with an impulse control problem, while it is well-known that (EP) is a general model for a wide range of other important problems such as optimization, Kakutani fixed point, complementarity problem, variational inequality problem, and Nash equilibria problem (see e.g. [2**?** –4]). It is shown in [5, 6] that (QVIP) cannot be considered as an (EP). The (QEP) as well as its subclasses (QVIP) and (EP) have a numerous applications economics, engineering, and operations research (see e.g. [7–11]), therefore they are active research topics in literature. Perhaps the most well-studied application of (QEP) is the generalized Nash equilibrium problem (see e.g. [12–14]), which can be modeled in form of (QVIP) when its data sets are differentiable (see [15]).

So far, many iterative methods for solving (EP) have been proposed (in e.g. [16**?** –25] and the references therein). Since (QEP) includes (EP) as a subclass, it is natural to extend existence results as well as solution methods for the latter to the former. In fact, results on solution existence for (QEP) have been given in [26–28], and many solution approaches to (QEP) (and (QVIP) in particular) have been proposed such as the extragradient-type methods [15, 29**?** , 30], projection-like algorithms [31, 32], Newton-type method [7, 33], augmented Lagrangian method [34], proximal point method [9, 35, 36], and gap functions technique [37]. However, solving (QEP) is still a challenging task since it requires to solve simultaneously an equilibrium problem and a fixed point problem. For instance, in [15], the authors introduced a general class of algorithms for solving (QEP) in $\mathbb{R}^n$ with f being pseudomonotone. Their method is an extragradient-type one whose second step consists in finding a descent direction (or step size) by using a line search. The convergence of their algorithms is established under suitable conditions, especially the following one:

$$S^* := \{x \in \cap_{z \in C} T(z) \mid f(x,y) \geq 0 \ \forall y \in \cup_{z \in C} T(z)\} \neq \emptyset. \tag{1}$$

The condition (1) was originally used in [32] to obtain the convergence of their projection-like algorithms for solving (QVIP) arised from the generalized Nash equilibrium problem. When $T(x) = C$ for all $x \in C$, we have $\cap_{z \in C} T(z) = \cup_{z \in C} T(z) = C$, and consequently S^* is the solution set of (EP). Thus, (1) can be viewed as a generalization to (QEP) of the hypothesis that the solution set of (EP) is nonempty. As claimed in [15], although this hypothesis is considered as a mild condition when solving (EP), its generalization (1) for (QEP) is not easy to verify since it concerns the image of T on every point in C.

The condition that S^* is nonempty becomes indispensable for the convergence of existing iterative solution methods for monotone (QEP) and (QVIP), including extragradient-type, projection-type, and proximal point algorithms (see [29, 31, 35**?**]). In [30], an extragradient-type method modified from the one in [15] to solve (QEP) without monotonicity of bifunction of f. However, the authors in [30] imposed the following condition

$$S_* := \{x \in T(z) \mid f(y, x) \le 0 \ \forall y \in C\} \ne \emptyset$$

for the convergence of their algorithm. It is known that $S_* \subset S^*$ and in general $S_* \ne S^*$ when the bifunction f is nonmonotone. Therefore, the algorithm proposed in [30] cannot be performed when $S_* = \emptyset$ even though (QEP) has a solution and $S^* \ne \emptyset$.

The aim of this paper is to develop extragradient subgradient-type algorithms with self-adaptive step sizes to solve pseudomonotone (QEP) such that the convergence of the proposed algorithms is still guaranteed even when $S^* = \emptyset$. Our first proposed algorithm is a combination of the subgradient method and the projection method in each the step size is simply updated through previous iteration points without using any search procedure. Under suitable conditions imposed on the bifunction f and the multivalued mapping T, the sequence of iterates $\{x^k\}$ generated by the algorithm is proved to converge weakly to a solution of (QEP), provided that

$$T^* := \{x \in \cap_{k=0,1,\ldots} T(x^k) \mid f(x, y) \ge 0 \ \forall y \in \cup_{k=0,1,\ldots} T(x^k)\} \ne \emptyset. \quad (2)$$

Obviously $S^* \subset T^*$, so the condition (2) is milder than (1). We will reclaim the mildness of (2) in Example 11 by showing an example of (QEP), which is a generalized Nash equilibrium problem, where $T^* \ne \emptyset$ but $S^* = \emptyset$. For this example, the aforementioned algorithms that require (1) are no longer guaranteed to be convergent, but our approaches work well. Our second proposed algorithm combines the first one with the Mann iteration scheme for finding a fixed point of a multivalued mappings in [38], for that we obtain the strong convergence of the generated iterates to a solution of (QEP). We also present some numerical experiments with the aim to evaluate the performance of our proposed algorithms and compare with the most recent extragradient subgradient-type, projection-like, and proximal point methods for (QEP).

The organization of the paper is as follows. In Section 2, we collect some basic concepts and preliminary results. Section 3 is devoted to describing our proposed algorithms and proving their convergence results. The last section provides a number of numerical experiments to illustrate our algorithms and compare with others.

2 Preliminaries

In this section, we recall some preliminaries that are used to prove the main results of this paper. For any nonempty closed subsets A and B of $\mathcal{H}$, the *Hausdorff distance* between these subsets is defined by

$$d^H(A,B) := \max\{d(A,B), d(B,A)\}$$

where $d(A,B) := \sup_{a\in A}\inf_{b\in B}\|a-b\|$. Note that for unique-element sets $A=\{a\}$ and $B=\{b\}$ we have $d^H(A,B)=d(A,B)=\|a-b\|$. If, furthermore, A is convex, then the projection $P_A(x)$ of a point $x\in\mathcal{H}$ on A is defined by the unique solution of $\min_{y\in A}\|x-y\|$.

Definition 1 *(See e.g. [39]). Let $T:C\to 2^C$ be a multi-valued mapping in which $T(x)$ is a nonempty closed bounded subset of C for each $x\in C$.*

(i) T is said to be nonexpansive if for all $x,y\in C$ we have

$$d^H(T(x),T(y)) \le \|x-y\|.$$

(ii) T is said to be quasi-nonexpansive if $Fix(T) := \{p\in C \mid p\in T(p)\}\neq\emptyset$ and for each $x\in C, p\in Fix(T)$ we have $d^H(T(x),T(p))\le\|x-p\|$.

(iii) T is said to be $*$-nonexpansive if for all $x,y\in C$ and $u\in\Pi_T(x)$ there exists $v\in\Pi_T(y)$ such that $\|u-v\|\le\|x-y\|$, in which Π_T is defined by

$$\Pi_T(x) := \{z\in T(x) \mid \|x-z\| = d(x,T(x))\}.$$

It is obvious that any nonexpansive multi-valued mapping T with $Fix(T)\neq\emptyset$ is also quasi-nonexpansive. It is shown in the first example of [38] that a quasi-nonexpansive multi-valued mapping might not be nonexpansive. By definition, if T is $*$-nonexpansive, then Π_T is nonexpansive. In case T is convex valued, we have $\Pi_T(x)\equiv P_{T(x)}(x)$, and T is $*$-nonexpansive if

$$\|P_{T(x)}(x) - P_{T(y)}(y)\| \le \|x-y\|.$$

Lemma 2 (See [40], Lemma 1). *Let $T:C\to 2^C$ be a multi-valued mapping with nonempty closed bounded value such that $\Pi_T(x)\neq\emptyset$ for every $x\in C$. Then the following claims are equivalent.*

(i) $x\in Fix(T)$.

(ii) $\Pi_T(x)=\{x\}$.

(iii) $x \in Fix(\Pi_T)$.

Definition 3 *Let $T : C \to 2^C$ be a multi-valued mapping with nonempty closed bounded value.*

(i) *(See e.g. [41]) T is called demiclosed if for any sequence $\{x^k\} \subset C$, $x^k \rightharpoonup p \in C$ and $y^k \in T(x^k)$, $y^k \rightharpoonup p$ we have $p \in Fix(T)$.*

(ii) *(See e.g. [9], Definition 2.3) T is called M-continuous if it satisfies the following two conditions:*
(C1) for any sequences $\{x^k\}, \{y^k\} \subset C$ with $x^k \rightharpoonup x$, $y^k \in T(x^k)$ and $y^k \rightharpoonup y$ we have $y \in T(x)$,
(C2) for any sequence $\{x^k\} \subset C$ with $x^k \rightharpoonup x$ and for each $y \in T(x)$ there exists a sequence $\{y^k\} \subset C$ with $y^k \in T(x^k)$ such that $y^k \to y$.

(iii) *(See e.g. [38]) T is called hemicompact if for any sequence $\{x^k\} \subset C$ satisfying $d(x^k, T(x^k)) \to 0$ there exists a subsequence $\{x^{k_i}\}$ such that $x^{k_i} \to p \in C$ as $i \to +\infty$*

It is mentioned in [38] that the hemicompactness of T is satisfied when C is compact and T has bounded value. In [9] the authors remarked that the concept of M-continuity was originally introduced by Mosco in [42] for studying stable approximation of variational inequalities, and then used in [43] for studying quasi-variational inequality as well as in [44] for regularization of set-valued variational inequalities.

Definition 4 *(See [45]). A bifunction $f : C \times C \to \mathbb{R}$ is said to be*

(i) *strongly monotone with modulus $\eta > 0$ (shortly η-strongly monotone) on C if $f(x, y) + f(y, x) \leq -\eta\|x - y\|^2$ for all $x, y \in C$;*

(ii) *monotone on C if $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;*

(iii) *strictly monotone on C if $f(x, y) + f(y, x) < 0$ for all $x, y \in C$ with $x \neq y$;*

(iv) *pseudomonotne on C if $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in C$;*

(v) *quasimonotone on C, if $f(x, y) > 0 \Rightarrow f(y, x) \leq 0$ $\forall x, y \in C$.*

Let us now recall from e.g. [46] some basic concepts and results in convex analysis. The subdifferential of a convex function $g : C \to \mathbb{R} \cup \{+\infty\}$ at $p \in C$ is defined by

$$\partial g(p) = \{x^* \in \mathcal{H} \mid g(x) - g(p) \geq \langle x^*, x - p\rangle \ \forall x \in C\}.$$

If $\partial g(x) \neq \emptyset$, then g is called subdifferentiable at x. The function g is said to be subdifferentiable on C if it is subdifferentiable at every x in C. The outer normal cone N_C of C at $p \in C$ is defined by

$$N_C(p) = \{x^* \in \mathcal{H} \mid \langle x^*, x - p\rangle \leq 0 \ \forall x \in C\}.$$

Lemma 5 (See e.g. [46], Proposition 2.31) *Let $C \subset \mathcal{H}$ be convex and $g : C \to \mathbb{R}$ convex and subdifferentiable. Then p is an optimal solution of the following convex minimization problem*

$$\min\{g(x) \mid x \in C\}$$

if and only if $0 \in \partial g(p) + N_C(p)$.

Lemma 6 (See e.g. [47], Chapter 6) *Let C be a nonempty closed convex subset of $\mathcal{H}$ and P_C the projection mapping onto C. Then the following assertions hold.*

(i) For any $x \in \mathcal{H}$ we have $z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \le 0$ for all $y \in C$.
(ii) The mapping P_C is nonexpansive, i.e., we have $\|P_C(x) - P_C(y)\| \le \|x - y\|$ for all $x, y \in \mathcal{H}$.
(iii) For all $x \in \mathcal{H}$ and $z \in C$ we have

$$\|x - P_C(x)\|^2 + \|P_C(x) - z\|^2 \le \|x - z\|^2.$$

Lemma 7 (See [48], Lemma 2.47) *Let $C \subset \mathcal{H}$ be nonempty and $\{x^k\} \subset \mathcal{H}$ satisfy the following conditions:*

(i) for all $x \in C$ we have $\lim_{k\to+\infty} \|x^k - x\|$ exists;
(ii) every sequentially weak cluster point of $\{x^k\}$ is in C.

Then, the sequence $\{x^k\}$ converges weakly to a point in C.

It is easy and straightforward to obtain the following lemma.

Lemma 8 *For every $x, y \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ the following assertions hold.*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$.
(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$.

3 Projection algorithms for quasi-equilibrium problems

In this section, we construct two iterative algorithms for solving the quasi-equilibrium problem (QEP). In order to prove the convergence of the proposed algorithms, we need to use the following assumptions imposed on the bifunction f.

(A_1) $f(x, x) = 0$ for all $x \in C$, $f(x, y)$ is pseudomonotone on $C \times C$, and $f(\cdot, \cdot)$ is sequentially weakly upper semicontinuous on $C \times C$.
(A_1') $f(x, x) = 0$ for all $x \in C$, $f(x, y)$ is quasiomonotone on $C \times C$, $f(\cdot, y)$ is weakly upper semicontinuous on C for every $y \in C$, and $f(\cdot, \cdot)$ is sequentially weakly lower semicontinuous on $C \times C$.
(A_2) $f(x, \cdot)$ is convex and subdifferentiable on C.

(A_3) There exists $L > 0$ such that

$$d^H\left(\partial_2 f(x,x), \partial_2 f(y,y)\right) \leq L\|x-y\| \ \forall x,y \in C,$$

where $\partial_2 f(x,x) = \partial_2 f(x,\cdot)(x)$ is the subdifferential of $f(x,\cdot)$ at x, i.e.,

$$\partial_2 f(x,x) = \{\xi \in \mathcal{H} \mid \langle \xi, z-x\rangle \leq f(x,z) \ \forall z \in C\}.$$

Algorithm 1 An algorithm for solving (QEP)

Take an arbitrary starting point $x^0 \in C$. Take $\lambda_0 > 0$, $\nu \in (0,1)$, $L^* > L$, $\gamma \in (0,2)$, and control parameter sequences $\{\rho_i\}$, $\{\kappa_i\}$ satisfying conditions

$$\rho_i > 0, \quad \sum_{i=0}^{+\infty} \rho_i < +\infty, \quad \kappa_i > 0, \quad \sum_{i=0}^{+\infty} \kappa_i < +\infty. \tag{3}$$

for $k = 0, 1, 2, \dots$ **do**
    Compute $u^k \in \partial_2 f(x^k, x^k)$.
    if $u^k = 0$ **then**
        return x^k
    else
        Compute

$$y^k = P_{T(x^k)}(x^k - \lambda_k u^k). \tag{4}$$

        if $y^k = x^k$ **then**
            return x^k
        else
            Compute $v^k \in B\left(u^k, L^*\|x^k - y^k\|\right) \cap \partial_2 f(y^k, y^k)$.
            if $v^k = 0$ **then**
                return x^k
            else
                Compute

$$d^k = x^k - y^k - \lambda_k(u^k - v^k), \tag{5}$$

$$\tau_k = \begin{cases} (\gamma + \kappa_k)\frac{|\langle x^k - y^k, d^k\rangle|}{\|d^k\|^2} & \text{if } \|d^k\| \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

$$x^{k+1} = P_{T(x^k)}(x^k - \tau_k \lambda_k v^k), \tag{7}$$

$$\lambda_{k+1} = \begin{cases} \min\left\{\frac{\nu\|x^k - y^k\|}{\|u^k - v^k\|}, \lambda_k + \rho_k\right\} & \text{if } u^k - v^k \neq 0, \\ \lambda_k + \rho_k & \text{otherwise}. \end{cases} \tag{8}$$

            end if
        end if
    end if
end for

Our first algorithm proposed in this paper is presented in Algorithm 1. Hereafter, we use the notation $B(x, r)$ to denote the set of points $y \in \mathcal{H}$ such that $\|y - x\| \leq r$. We have some remarks concerning this algorithm.

Remark 9 *(i) Since $u^k \in \partial_2 f(x^k, x^k)$, if $u^k = 0$, then we have*

$$f(x^k, x) = f(x^k, x) - f(x^k, x^k) \geq \langle u^k, x - x^k\rangle = 0 \quad \forall x \in T(x^k) \subset C.$$

Hence in this case the algorithm terminates at iteration k and x^k is a solution of Problem (QEP).

(ii) Assume that $x^k = y^k$. Then, we get from (4) that $x^k = P_{T(x^k)}(x^k - \lambda_k u^k)$, which together with Lemma 6 (i) and $u^k \in \partial_2 f(x^k, x^k)$ implies that

$$0 \leq \langle u^k, x - x^k \rangle \leq f(x^k, x) - f(x^k, x^k) = f(x^k, x), \ \forall x \in T(x^k).$$

So, x^k is a solution of the problem (QEP).

(iii) We have from the assumption (A_2) that $\partial_2 f(y^k, y^k)$ is a nonempty, closed and convex set in $\mathcal{H}$, which together the assumption (A_3) implies that

$$\|u^k - P_{\partial_2 f(y^k,y^k)}(u^k)\| \leq d^H \left(\partial_2 f(x^k, x^k), \partial_2 f(y^k, y^k)\right) \leq L\|x^k - y^k\|.$$

Therefore, we can always choose v^k in line 11 of the algorithm statement. If $f(x, \cdot)$ is differentiable on C, then v^k is uniquely determined by $v^k = \nabla f(y^k, \cdot)(y^k)$, and so the proposed algorithm can be implemented without knowing the constant L which is mentioned in the assumption $(A3)$.

*(iv) In [**?**], The step sizes τ_k, and λ_k respectively can be updated via equations (5) and (8) without using the parameters κ_k and ρ_k. In Algorithm 1, we use the parameters κ_k and ρ_k to obtain larger step sizes.* □

As a key result in this section, we will show that Algorithm 1 is convergent when the following sets are nonempty:

$$T^* := \{x \in \cap_{k\in\mathbb{N}} T(x^k) \mid f(x, y) \geq 0 \ \forall y \in \cup_{k\in\mathbb{N}} T(x^k)\}$$
$$T_* := \{x \in \cap_{k\in\mathbb{N}} T(x^k) \mid f(y, x) \leq 0 \ \forall y \in \cup_{k\in\mathbb{N}} T(x^k)\}.$$

Example 10 *Consider the equilibrium problem (EP) and its associated Minty equilibrium problem:*

$$\text{(MEP)} \qquad \textit{Find } p \in C \ \textit{ such that } f(y, p) \leq 0 \textit{ for all } y \in C.$$

Suppose q and p are the solutions to the problems (MEP) and (EP), respectively. Let T be a multi-valued map such that $T(x)$ is a closed convex set in C and contains p, q and x for all $x \in C$. It is easy to see that $p, \ q \in \cap_{k\in\mathbb{N}} T(x^k)$, $f(p, y) \geq 0 \ \forall y \in C$ and $f(y, q) \leq 0 \ \forall y \in C$. Therefore, T^* and T_* are nonempty.

It is easy to see that the set S^* defined in (1) is a subset of both the set T^* defined above and the solution set $Sol(EP)$ of the equilibrium problem (EP). The following example shows an instance of (QEP) with twofold aims. First, for this problem instance we have $S^* = \emptyset$ while T^* and T_* are nonempty. Second, this is an instance of (QEP) whose solution set $Sol(QEP)$ is different from $Sol(EP)$.

Example 11 *Let $\mathcal{H} = \mathbb{R}^n$ and $C = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i \geq 1\}$ which is a closed polyhedron in the positive orthant of $\mathcal{H}$. The multi-valued mapping T is given by*

$$T(x) = \{z \in C \mid z \leq 2x\}.$$

Then $T(x)$ is a convex polytope for each $x \in C$, and T is M-continuous by definition. Now, let P and Q be $n \times n$ matrices with non-negative elements such that both Q and $P-Q$ are positive semidefinite. Let g be a convex, differentiable, and increasing function on $\mathbb{R}^n_+$ such that its gradient is Lipschitz continuous in the following sense

$$\|\nabla g(y) - \nabla g(x)\| \le L_1 \|y - x\| \quad \forall x, y \in \mathbb{R}^n_+,$$

where L_1 is a positive constant. For an increasing function g on $\mathbb{R}^n_+$ we mean that $g(x) \le g(y)$ for all $x, y \in \mathbb{R}^n_+$ with $x \le y$. In our numerical experiment presented in Section 4.1 we give an example of such a function g satisfying the mentioned conditions. We consider (QEP) in which the set C and the mapping T are given above, while the bifunction f is given by

$$f(x,y) = \langle Px + Qy + c, y - x\rangle + g(y) - g(x), \tag{9}$$

where $c \in \mathbb{R}^n_+$ satisfying

$$c_n > p_{1n} + q_{11} + c_1 - p_{nn} - q_{n1} + g(b) - g(a). \tag{10}$$

Here, $a = (0, \dots, 0, 1)$ is the last unit vector and $b = (1, 0, \dots, 0)$ is the first unit vector of $\mathbb{R}^n$. The bifunction f defined in (9) is a generalized form of the one in the well-known Cournot-Nash equilibrium model. It is obvious that $f(x,x) = 0$ for all $x \in \mathbb{R}^n$ and f is continuous on $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore, since $Q - P$ is negative semidefinite, for all $x, y \in \mathbb{R}^n$ we have

$$f(x,y) + f(y,x) = (y-x)^t (Q-P)(y-x) \le 0,$$

i.e., f is monotone on $\mathbb{R}^n$, hence it is pseudomonotone on C. So this bifunction satisfies Assumption (A1). Since Q is positive semidefinite, $\langle Px + Qy + c, y - x\rangle$ is convex quadratic with respect to y. This, together with the convexity and differentiability of g, implies that the bifunction f satisfies Assumption (A2). Note that

$$\partial_2 f(x,y) = \{Px + c - Q^t x + (Q + Q^t) y + \nabla g(y)\},$$

it follows that

$$\begin{aligned} d^H(\partial_2 f(x,x), \partial_2 f(y,y)) &= \|(P+Q)(x-y) + \nabla g(x) - \nabla g(y)\| \\ &\le \|(P+Q)(x-y)\| + \|\nabla g(x) - \nabla g(y)\| \\ &\le (\|P+Q\| + L_1)\|x - y\|. \end{aligned}$$

Hence f satisfies Assumption (A_3) with $L = \|P + Q\| + L_1$.

For the instance of (QEP) we are considering, we first observe that $S^* = \emptyset$. Indeed, it follows from definition of T that $T(a) = [a, 2a]$ which is the line segment connecting the points a and $2a$. Similarly, we have $T(b) = [b, 2b]$. By our choice, we have $a, b \in C$ and $[a, 2a] \cap [b, 2b] = \emptyset$, i.e., $T(a) \cap T(b) = \emptyset$. Since $S^* \subset T(a) \cap T(b)$, it must be empty.

Our second observation is that T^* and T_* are nonempty if we choose the starting point $x^0 = t_0 a$ for some $t_0 > 1$. Indeed, we have from the definition of T and (8) that the iterate x^1 is in the line segment $T(x^0) = [a, 2t_0 a]$, hence $x^1 = t_1 a$ for some t_1 satisfying $1 \le t_1 \le 2t_0$. By a simple induction, for each $k = 1, 2, \dots$ we have $T(x^k) = [a, 2t_k a]$ for some $t_k \in [1, 2t_{k-1}]$. Hence $a \in \cap_{k\in\mathbb{N}} T(x^k)$. Moreover, by the specified structure of the sets $T(x^k)$ with $k \in \mathbb{N}$, for any $y \in \cup_{k\in\mathbb{N}} T(x^k)$ we have $y \ge a$. This, together with the facts that g is increasing on $\mathbb{R}^n$ while all elements of P, Q, a, and c are nonnegative, gives us

$$f(a,y) = \langle Pa + Qy + c, y - a\rangle + g(y) - g(a) \ge 0 \quad \forall y \in \cup_{k\in\mathbb{N}} T(x^k)$$

$$f(y,a) = \langle Py + Qa + c, a - y\rangle + g(a) - g(y) \leq 0 \quad \forall y \in \cup_{k\in\mathbb{N}} T(x^k).$$

So we can conclude that a belongs to both T^* and T_*, which means that T^* and T_* are nonempty.

By similar arguments, for all $y \in T(a) = [a, 2a]$ we have

$$f(a,y) = \langle Pa + Qy + c, y - a\rangle + g(y) - g(a) \geq 0,$$

which shows that $a \in Sol(QEP)$. It follows from (10) that

$$f(a,b) = p_{1n} + q_{11} + c_1 - p_{nn} - q_{n1} - c_n + g(b) - g(a) < 0,$$

so $a \notin Sol(EP)$. Since $a \in T^*$, it follows that T^* is not a subset of $Sol(EP)$. Since $a \in Sol(QEP)$, it also follows that $Sol(QEP) \neq Sol(EP)$. □

In the rest of this section, we always assume that (A_1)-(A_3) hold and $T^* \neq \emptyset$. We first obtain the following important lemmas.

Lemma 12 *The sequence $\{\lambda_k\}$ generated in Algorithm 1 is convergent.*

Proof. Let $M := \sum_{k=0}^{+\infty} \rho_k$. We first show by induction that

$$\lambda_k \in \left[\min\left\{\frac{\nu}{L^*}, \lambda_0\right\}, \lambda_0 + M\right] \tag{11}$$

for all $k \geq 0$. This obviously holds for $k = 0$. Assume that (11) holds for some $k \in \mathbb{N}$, we will prove that λ_{k+1} also belongs to the interval. Indeed, if $u^k - v^k = 0$, then we have from (8) and positivity of ρ_k that $\lambda_{k+1} = \lambda_k + \rho_k \geq \min\left\{\frac{\nu}{L^*}, \lambda_0\right\}$. Otherwise, we have $u^k - v^k \neq 0$, and it follows from the choice of v^k that $\|u^k - v^k\| \leq L^*\|x^k - y^k\|$. Consequently, we obtain

$$\frac{\nu\|x^k - y^k\|}{\|u^k - v^k\|} \geq \frac{\nu\|x^k - y^k\|}{L^*\|x^k - y^k\|} = \frac{\nu}{L^*}, \tag{12}$$

which, together with (8), implies that

$$\lambda_{k+1} \geq \min\left\{\frac{\nu\|x^k - y^k\|}{\|u^k - v^k\|}, \lambda_k + \rho_k\right\} \geq \min\left\{\frac{\nu}{L^*}, \lambda_k + \rho_k\right\} \geq \min\left\{\frac{\nu}{L^*}, \lambda_0\right\}.$$

On the other hand, by using (8) and condition (3), we get

$$\lambda_{k+1} \leq \lambda_k + \rho_k \leq \ldots \leq \lambda_0 + \sum_{i=0}^{k} \rho_i \leq \lambda_0 + \sum_{i=0}^{+\infty} \rho_i = \lambda_0 + M.$$

Hence $\lambda_{k+1} \in \left[\min\left\{\frac{\nu}{L^*}, \lambda_0\right\}, \lambda_0 + M\right]$, which completes the induction proof for (11).

We next show that $\{\lambda_k\}$ converges to $\lambda \in \left[\min\left\{\frac{\nu}{L^*}, \lambda_0\right\}, \lambda_0 + M\right]$. Indeed, by setting $(\lambda_{k+1} - \lambda_k)^+ = \max\{0, \lambda_{k+1} - \lambda_k\}$ and $(\lambda_{k+1} - \lambda_k)^- = \max\{0, -(\lambda_{k+1} - \lambda_k)\}$ we have $\lambda_{k+1} - \lambda_k = (\lambda_{k+1} - \lambda_k)^+ - (\lambda_{k+1} - \lambda_k)^-$, and it follows that

$$\lambda_{k+1} - \lambda_0 = \sum_{i=0}^{k}(\lambda_{i+1} - \lambda_i) = \sum_{i=0}^{k}(\lambda_{i+1} - \lambda_i)^+ - \sum_{i=0}^{k}(\lambda_{i+1} - \lambda_i)^-. \tag{13}$$

It is easy to see from (3) and (8) that

$$\sum_{k=0}^{+\infty}(\lambda_{k+1} - \lambda_k)^+ \le \sum_{k=0}^{+\infty}\rho_k < +\infty. \tag{14}$$

If $\sum_{k=0}^{+\infty}(\lambda_{k+1} - \lambda_k)^- = +\infty$, then by taking the limit as $k \to \infty$ on both sides of (13) and using (14) we have $\lambda_k \to -\infty$, which contradicts (11). Hence we must have $\sum_{k=0}^{+\infty}(\lambda_{k+1}-\lambda_k)^- < +\infty$, which, together with (13) and (14), guarantees that $\lim_{k\to\infty}(\lambda_{k+1}-\lambda_0)$ is finite. So $\{\lambda_k\}$ is convergent. Since (11) holds for all $k \ge 0$, the limit $\lim_{k\to+\infty} \lambda_k = \lambda$ is also in $\left[\min\left\{\frac{\nu}{L^*}, \lambda_0\right\}, \lambda_0 + M\right]$. □

Lemma 13 *Let $\{x^k\}$, $\{y^k\}$, $\{\lambda_k\}$ be sequences generated by Algorithm 1. For any $p \in T^*$ there exists a nonnegative integer K such that*

$$\|x^{k+1} - p\|^2 \le \|x^k - p\|^2 - (\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{(\lambda_{k+1} - \nu\lambda_k)^2}{(\lambda_{k+1} + \nu\lambda_k)^2}\|x^k - y^k\|^2 \tag{15}$$

for all $k \ge K$.

Proof. For all $k \ge 0$ we have $f(p, y^k) \ge 0$ since $p \in T^*$. By pseudomonotonicity of f, we then get $f(y^k, p) \le 0$. Additionally, by $f(y^k, y^k) = 0$ and $v^k \in \partial_2 f(y^k, y^k)$, while noting that τ_k and λ_k are nonnegative, we obtain

$$\tau_k\lambda_k\langle v^k, y^k - p\rangle \ge \tau_k\lambda_k[f(y^k, y^k) - f(y^k, p)] \ge 0 \quad \forall k \ge 0. \tag{16}$$

Furthermore, since $T(x^k)$ is closed convex, by Lemma 6(i) and the definition of y^k in (4) we have

$$\langle x^k - y^k - \lambda_k u^k, y^k - x\rangle \ge 0 \quad \forall x \in T(x^k), k \ge 0, \tag{17}$$

which implies

$$\tau_k\langle x^k - y^k - \lambda_k u^k, y^k - x^{k+1}\rangle \ge 0 \quad \forall k \ge 0 \tag{18}$$

because $\tau_k \ge 0$ and $x^{k+1} \in T(x^k)$. Now, for all $k \ge 0$ we have

$$\|x^{k+1} - p\|^2$$

$$
\begin{aligned}
&= \|P_{T(x^k)}(x^k - \tau_k\lambda_k v^k) - p\|^2 && \text{(by (7))}\\
&\le \|x^k - \tau_k\lambda_k v^k - p\|^2 - \|x^k - \tau_k\lambda_k v^k - x^{k+1}\|^2 && \text{(by Lemma 6(iii))}\\
&= \|x^k - p\|^2 - 2\tau_k\lambda_k\langle x^k - p, v^k\rangle + 2\tau_k\lambda_k\langle x^k - x^{k+1}, v^k\rangle - \|x^{k+1} - x^k\|^2\\
&= \|x^k - p\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau_k\lambda_k\langle x^k - x^{k+1}, v^k\rangle + 2\tau_k\lambda_k\langle y^k - x^k, v^k\rangle\\
&\quad - 2\tau_k\lambda_k\langle x^k - p, v^k\rangle - 2\tau_k\lambda_k\langle y^k - x^k, v^k\rangle\\
&= \|x^k - p\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau_k\lambda_k\langle x^k - x^{k+1} + y^k - x^k, v^k\rangle\\
&\quad - 2\tau_k\lambda_k\langle x^k - p + y^k - x^k, v^k\rangle\\
&= \|x^k - p\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau_k\lambda_k\langle y^k - x^{k+1}, v^k\rangle - 2\tau_k\lambda_k\langle y^k - p, v^k\rangle\\
&\le \|x^k - p\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau_k\lambda_k\langle y^k - x^{k+1}, v^k\rangle && \text{(by (16))}\\
&\le \|x^k - p\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau_k\lambda_k\langle y^k - x^{k+1}, v^k\rangle\\
&\quad + 2\tau_k\langle y^k - x^{k+1}, x^k - y^k - \lambda_k u^k\rangle && \text{(by (18))}\\
&= \|x^k - p\|^2 - \|x^{k+1} - x^k\|^2 + 2\tau_k\langle y^k - x^{k+1}, x^k - y^k - \lambda_k(u^k - v^k)\rangle\\
&= \|x^k - p\|^2 - \|x^k - x^{k+1}\|^2 + 2\tau_k\langle y^k - x^{k+1}, d^k\rangle\\
&= \|x^k - p\|^2 - \|x^k - x^{k+1} - \tau_k d^k + \tau_k d^k\|^2 + 2\tau_k\langle y^k - x^{k+1}, d^k\rangle\\
&= \|x^k - p\|^2 - \|x^k - x^{k+1} - \tau_k d^k\|^2 - (\tau_k\|d^k\|)^2 - 2\tau_k\langle x^k - x^{k+1} - \tau_k d^k, d^k\rangle\\
&\quad + 2\tau_k\langle y^k - x^{k+1}, d^k\rangle\\
&= \|x^k - p\|^2 - \|x^k - x^{k+1} - \tau_k d^k\|^2 + \tau_k^2\|d^k\|^2 - 2\tau_k\langle x^k - x^{k+1}, d^k\rangle\\
&\quad + 2\tau_k\langle y^k - x^{k+1}, d^k\rangle\\
&= \|x^k - p\|^2 - \|x^k - x^{k+1} - \tau_k d^k\|^2 + \tau_k^2\|d^k\|^2 - 2\tau_k\langle x^k - y^k, d^k\rangle\\
&\le \|x^k - p\|^2 + \tau_k^2\|d^k\|^2 - 2\tau_k\langle x^k - y^k, d^k\rangle. && (19)
\end{aligned}
$$

We distinguish the two following cases.

Case 1: $d^k = 0$. In this case, we have from (19) that

$$\|x^{k+1} - p\| \le \|x^k - p\|. \tag{20}$$

Since $d^k = x^k - y^k - \lambda_k(u^k - v^k) = 0$, by (8) we see that

$$\|x^k - y^k\| = \lambda_k\|u^k - v^k\| \le \nu\frac{\lambda_k}{\lambda_{k+1}}\|x^k - y^k\|.$$

Consequently, we get

$$\left(1 - \frac{\nu\lambda_k}{\lambda_{k+1}}\right)\|x^k - y^k\| \le 0. \tag{21}$$

As shown in Lemma 12, the sequence $\{\lambda_k\}$ is convergent, so $\lim_{k\to\infty}(1 - \frac{\nu\lambda_k}{\lambda_{k+1}}) = 1 - \nu > 0$. Thus, there exists a nonnegative integer K_0 such that

$1 - \frac{\nu\lambda_k}{\lambda_{k+1}} > 0$ for all $k \geq K_0$. By (21), this implies $\|x^k - y^k\| = 0$ for all $k \in \{\ell \in \mathbb{N} \mid \ell \geq K_0, d^\ell = 0\}$. Together with (20), this means that (15) holds for all $k \in \{\ell \in \mathbb{N} \mid \ell \geq K_0, d^\ell = 0\}$.

Case 2: $d^k \neq 0$. In this case, for all $k \geq 0$ we see from (11) that $\lambda_k > 0$. Hence, by (8) we have

$$\lambda_{k+1} \leq \frac{\nu\|x^k - y^k\|}{\|u^k - v^k\|} \quad \Leftrightarrow \quad \|u^k - v^k\| \leq \frac{\nu}{\lambda_{k+1}}\|x^k - y^k\|.$$

By this inequality, on one hand we get

$$\begin{aligned}\langle x^k - y^k, d^k\rangle &= \langle x^k - y^k, x^k - y^k - \lambda_k(u^k - v^k)\rangle \\ &= \|x^k - y^k\|^2 - \lambda_k\langle x^k - y^k, u^k - v^k\rangle \\ &\geq \|x^k - y^k\|^2 - \lambda_k\|x^k - y^k\|\|u^k - v^k\| \\ &\geq \left(1 - \frac{\nu\lambda_k}{\lambda_{k+1}}\right)\|x^k - y^k\|^2 \\ &\geq 0\end{aligned} \tag{22}$$

for all $k \geq K_0$ specified above. On the other hand, for all $k \geq 0$ we get

$$\|d^k\| = \|x^k - y^k - \lambda_k(u^k - v^k)\| \leq \|x^k - y^k\| + \lambda_k\|u^k - v^k\| \leq \left(1 + \frac{\nu\lambda_k}{\lambda_{k+1}}\right)\|x^k - y^k\|,$$

or equivalently,

$$\|x^k - y^k\| \geq \frac{\lambda_{k+1}}{\lambda_{k+1} + \nu\lambda_k}\|d^k\|.$$

Combining this inequality with (22), we obtain

$$\langle x^k - y^k, d^k\rangle \geq \frac{\lambda_{k+1}(\lambda_{k+1} - \nu\lambda_k)}{(\lambda_{k+1} + \nu\lambda_k)^2}\|d^k\|^2 \geq 0 \tag{23}$$

for all $k \geq K_0$. Since $\lim_{k\to+\infty}(2 - \gamma - \kappa_k) = 2 - \gamma > 0$, there exists $K_1 > 0$ such that $2 - \gamma - \kappa_k > 0$ for all $k \geq K_1$. So for $k \geq K := \max\{K_0, K_1\}$ we have

$$\begin{aligned}&\|x^{k+1} - p\|^2 \\ &\leq \|x^k - p\|^2 + \tau_k^2\|d^k\|^2 - 2\tau_k\langle x^k - y^k, d^k\rangle && \text{(by (19))} \\ &\leq \|x^k - p\|^2 + (\gamma + \kappa_k)^2\frac{\langle x^k - y^k, d^k\rangle^2}{\|d^k\|^4}\|d^k\|^2 - 2(\gamma + \kappa_k)\frac{\langle x^k - y^k, d^k\rangle^2}{\|d^k\|^2} \\ & && \text{(by (6))} \\ &= \|x^k - p\|^2 - (\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{\langle x^k - y^k, d^k\rangle^2}{\|d^k\|^2}\end{aligned}$$

$$\leq \|x^k - p\|^2 - (\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{\lambda_{k+1}(\lambda_{k+1} - \nu\lambda_k)}{(\lambda_{k+1} + \nu\lambda_k)^2}\langle x^k - y^k, d^k\rangle \quad \text{(by (23))}$$

$$\leq \|x^k - p\|^2 - (\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{(\lambda_{k+1} - \nu\lambda_k)^2}{(\lambda_{k+1} + \nu\lambda_k)^2}\|x^k - y^k\|^2. \quad \text{(by (22))}$$

In conclusion, (15) is satisfied for all $k \geq K = \max\{K_0, K_1\}$. □

Remark 14 *In the proof of Lemma 13, the pseudomonotonicity of f and $p \in T^*$ are used to obtain the inequality (16). It is easy to see that if $p \in T_* \neq \emptyset$, then the inequality (16) still holds. Therefore, the assertion in Lemma 13 is still true when $p \in T_*$ without the need for pseudomonotonicity of f.*

We are now ready to prove the main convergence result of Algorithm 1.

Theorem 15 *Assume that (A_1)-(A_3) hold, T is M-continuous and closed convex valued, and $T^* \neq \emptyset$. Then every weak cluster point of the sequence $\{x^k\}$ generated in Algorithm 1 belongs to $Sol(QEP)$. In addition, if the bifunction f is strictly monotone on C, then the whole sequence $\{x^k\}$ converges weakly to some $\hat{p} \in Sol(QEP)$.*

Proof. Let K be specified as in the proof of Lemma 13 and p an arbitrary point in T^*. Then, for all $k \geq K$ we have $2 - \gamma - \kappa_k > 0$ and it follows from (15) that

$$\|x^{k+1} - p\| \leq \|x^k - p\|.$$

So the sequence $\{\|x^k - p\|\}$ converges, hence $\{x^k\}$ is bounded. We also have from (15) that

$$\left[(\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{(\lambda_{k+1} - \nu\lambda_k)^2}{(\lambda_{k+1} + \nu\lambda_k)^2}\right]\|y^k - x^k\|^2 \leq \|x^k - p\|^2 - \|x^{k+1} - p\|^2$$

for all $k \geq K$. Passing $k \to +\infty$ while noting the convergences of $\{\lambda_k\}$ and $\{\|x^k - p\|\}$, we have

$$\gamma(2 - \gamma)\frac{(1 - \nu)^2}{(1 + \nu)^2}\lim_{k\to+\infty}\|x^k - y^k\| \leq 0.$$

Using condition (3), this inequality implies

$$\lim_{k\to+\infty}\|x^k - y^k\| = 0. \tag{24}$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i}\}$ weakly converging to some $\hat{p}$. Thanks to (24), the sequence $\{y^{k_i}\}$ also converges weakly to $\hat{p}$. Since $y^k \in T(x^k)$, by the M-continuity of T we obtain that $\hat{p} \in T(\hat{p})$.

To show that $\hat{p} \in Sol(QEP)$, it is left to verify that $f(\hat{p}, z) \geq 0$ for all $z \in T(\hat{p})$. Indeed, since v^k is chosen in $B\left(u^k, L^*\|x^k - y^k\|\right)$, we have $\|u^k - v^k\| \leq L^*\|x^k - y^k\|$. It then follows from (24) that $\|u^k - v^k\| \to 0$ as $k \to +\infty$. As already shown in (17), we have

$$\langle x^{k_i} - y^{k_i} - \lambda_{k_i} u^{k_i}, y^{k_i} - x \rangle \geq 0 \quad \forall x \in T(x^{k_i}).$$

Together with $v^{k_i} \in \partial_2 f(y^{k_i}, y^{k_i})$, this implies that for all $x \in T(x^{k_i})$ we have

$$\begin{aligned}\langle x^{k_i} - y^{k_i}, x - y^{k_i} \rangle &\leq \lambda_{k_i} \langle u^{k_i}, x - y^{k_i} \rangle \\ &\leq \lambda_{k_i} (\langle v^{k_i}, x - y^{k_i} \rangle + \langle u^{k_i} - v^{k_i}, x - y^{k_i} \rangle) \\ &\leq \lambda_{k_i} f(y^{k_i}, x) + \lambda_{k_i} \langle u^{k_i} - v^{k_i}, x - y^{k_i} \rangle,\end{aligned}$$

and by positivity of λ_{k_i} we get

$$\frac{1}{\lambda_{k_i}} \langle x^{k_i} - y^{k_i}, x - y^{k_i} \rangle \leq f(y^{k_i}, x) + \langle u^{k_i} - v^{k_i}, x - y^{k_i} \rangle.$$

For each $z \in T(\hat{p})$, we deduce from the M-continuity of T that there exists a sequence $\{z^{k_i}\}$ such that $z^{k_i} \in T(x^{k_i})$ and $z^{k_i} \rightharpoonup z$. Taking $x = z^{k_i}$ in the last inequality, one has

$$\frac{1}{\lambda_{k_i}} \langle x^{k_i} - y^{k_i}, z^{k_i} - y^{k_i} \rangle \leq f(y^{k_i}, z^{k_i}) + \langle u^{k_i} - v^{k_i}, z^{k_i} - y^{k_i} \rangle.$$

Taking the limit of both sides of this inequality as $i \to +\infty$, while noting $\lim_{i \to +\infty} \|x^{k_i} - y^{k_i}\| = 0$, $\lim_{i \to +\infty} \|u^{k_i} - v^{k_i}\| = 0$, the weakly upper semicontinuity of the function $f(\cdot, \cdot)$, and the boundedness of $\{\lambda_k\}$, we come up with $f(\hat{p}, z) \geq 0$. Since z is chosen arbitrarily in $T(\hat{p})$, this completes the proof for $\hat{p} \in Sol(QEP)$.

Now, we assume that the bifunction f is strictly monotone on C. Before showing that the whole sequence $\{x^k\}$ converges weakly to $\hat{p}$, we claim that $T^* = \{\hat{p}\}$. We will prove this claim by taking an arbitrary $p \in T^*$ and then show that $p = \hat{p}$. Indeed, since $p \in T^*$, we have

$$f(p, y) \geq 0 \quad \forall y \in \cup_{k \in \mathbb{N}} T(x^k). \tag{25}$$

Keeping in mind that $x^{k_i} \rightharpoonup \hat{p}$ and $\hat{p} \in T(\hat{p})$, by M-continuity of T there exists a sequence $\{y^{k_i}\}$ such that $y^{k_i} \in T(x^{k_i})$ and $y^{k_i} \to \hat{p}$. Since $y^{k_i} \in T(x^{k_i})$, by (25) we have $f(p, y^{k_i}) \geq 0$ for all $i \in \mathbb{N}$. Taking $i \to +\infty$ and using the weakly upper semicontinuity of the function $f(\cdot, \cdot)$ we get $f(p, \hat{p}) \geq 0$. Due to strict monotonicity of f, we obtain

$$f(\hat{p}, p) \leq 0. \tag{26}$$

On the other hand, since $p \in T^*$, we have $p \in T(x^{k_i})$ for all $i \in \mathbb{N}$. This, together with the facts that $x^{k_i} \rightharpoonup \hat{p}$ and T is M-continuous, implies that $p \in T(\hat{p})$. As shown above, it follows that $f(\hat{p}, p) \geq 0$. Combining this inequality with (26) we come up with $f(\hat{p}, p) = 0$. Therefore, if $\hat{p} \neq p$, then

$$f(p, \hat{p}) = f(p, \hat{p}) + f(\hat{p}, p) < 0$$

where the last inequality is due to the strict monotonicity of f. However, this inequality contradicts the fact $f(p, \hat{p}) \geq 0$ we have shown above. Thus we must have $p = \hat{p}$, and consequently, $T^* = \{\hat{p}\}$.

It has been shown in the beginning of this proof that $\{\|x^k - p\|\}$ converges for any $p \in T^*$. Since $T^* = \{\hat{p}\}$, it means that the sequence $\{\|x^k - \hat{p}\|\}$ is convergent. Then, by applying Lemma 7 for $C := T^*$ and $x := \hat{p}$, we deduce that the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to $\hat{p}$. □

Theorem 16 *Assume that (A_1), (A_2') and (A_3) hold, T is M-continuous and closed convex valued, and $T_* \neq \emptyset$. If the mapping $\partial_2 f(x, x)$ satisfies Condition $(C1)$, then every weak cluster point of the sequence $\{x^k\}$ generated in Algorithm 1 belongs to $Sol(QEP)$.*

Proof. Let p be an arbitrary point in T^*. By the same arguments as in the proof of Theorem 16, we can prove that the sequence $\{\|x^k - p\|\}$ converges, the sequence $\{x^k\}$ is bounded, and

$$\lim_{k \to +\infty} \|x^k - y^k\| = 0. \tag{27}$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i}\}$ weakly converging to some $\hat{p}$. Thanks to (24), the sequence $\{y^{k_i}\}$ also converges weakly to $\hat{p}$. Since $y^k \in T(x^k)$, by the M-continuity of T we obtain that $\hat{p} \in T(\hat{p})$.

To show that $\hat{p} \in Sol(QEP)$, it is left to verify that $f(\hat{p}, z) \geq 0$ for all $z \in T(\hat{p})$. Indeed, since v^k is chosen in $B\left(u^k, L^*\|x^k - y^k\|\right)$, we have $\|u^k - v^k\| \leq L^*\|x^k - y^k\|$. It then follows from (27) that $\|u^k - v^k\| \to 0$ as $k \to +\infty$. As already shown in (17), we have

$$\langle x^{k_i} - y^{k_i} - \lambda_{k_i} u^{k_i}, y^{k_i} - x \rangle \geq 0 \quad \forall x \in T(x^{k_i}).$$

Together with $v^{k_i} \in \partial_2 f(y^{k_i}, y^{k_i})$, this implies that for all $x \in T(x^{k_i})$ we have

$$\begin{aligned}\langle x^{k_i} - y^{k_i}, x - y^{k_i} \rangle &\leq \lambda_{k_i} \langle u^{k_i}, x - y^{k_i} \rangle \\ &\leq \lambda_{k_i} (\langle v^{k_i}, x - y^{k_i} \rangle + \langle u^{k_i} - v^{k_i}, x - y^{k_i} \rangle),\end{aligned}$$

and by positivity of λ_{k_i} we get

$$\frac{1}{\lambda_{k_i}} \langle x^{k_i} - y^{k_i}, x - y^{k_i} \rangle \leq \langle v^{k_i}, x - y^{k_i} \rangle + \langle u^{k_i} - v^{k_i}, x - y^{k_i} \rangle.$$

For each $z \in T(\hat{p})$, we deduce from the M-continuity of T that there exists a sequence $\{z^{k_i}\}$ such that $z^{k_i} \in T(x^{k_i})$ and $z^{k_i} \rightharpoonup z$. Taking $x = z^{k_i}$ in the last inequality, one has

$$\frac{1}{\lambda_{k_i}}\langle x^{k_i} - y^{k_i}, z^{k_i} - y^{k_i}\rangle \le \langle v^{k_i}, z^{k_i} - y^{k_i}\rangle + \langle u^{k_i} - v^{k_i}, z^{k_i} - y^{k_i}\rangle.$$

Passing to the limit in the above inequality and taking into account that $\lim_{i\to\infty}\|x^{k_i} - y^{k_i}\| = 0$, $\lim_{i\to\infty}\|u^{k_i} - v^{k_i}\| = 0$, $\lim_{i\to\infty}\lambda_{k_i} = \lambda > 0$ and the sequence $\{y^k\}$ is bounded, we obtain

$$\liminf_{i\to\infty}\langle v^{k_i}, z^{k_i} - y^{k_i}\rangle \ge 0.$$

Let $\{\gamma_i\}$ be a positive sequence decreasing and $\gamma_i \to 0$ as $i \to \infty$. Then, for each $i \in \mathbb{N}$, there exists a smallest positive integer h_i such that

$$\left\langle v^{k_i}, z^{k_i} - y^{k_i}\right\rangle + \gamma_i > 0.$$

Observe that $\{k_i\}$ is increasing. If $v^{k_i} = 0$ then according to Remark 9 (i) and $v^{k_i} \in \partial_2 f(y^{k_i}, y^{k_i})$ we have y^{k_i} as a solution, so without loss of generality we assume $v^{k_i} \ne 0$ for all i. Setting $\varrho^{k_i} := \frac{1}{\|v^{k_i}\|^2}v^{k_i}$, we have $\langle v^{k_i}, \varrho^{k_i}\rangle = 1$ for every $i \in \mathbb{N}$ and

$$\left\langle v^{k_i}, z^{k_i} + \gamma_i\varrho^{k_i} - y^{k_i}\right\rangle > 0.$$

It follows from the above inequality, the quasimonotonicity of f and $v^{k_i} \in \partial_2 f(y^{k_i}, y^{k_i})$ that

$$f(y^{k_i}, z^{k_i}+\gamma_i\varrho^{k_i}) = f(y^{k_i}, z^{k_i}+\gamma_i\varrho^{k_i}) - f(y^{k_i}, y^{k_i}) \ge \left\langle v^{k_i}, z^{k_i} + \gamma_i\varrho^{k_i} - y^{k_i}\right\rangle > 0,$$

which together with the quasimonotonicity of f implies that

$$f(z^{k_i} + \gamma_i\varrho^{k_i}, y^{k_i}) \le 0, \ \forall i \in \mathbb{N}. \tag{28}$$

It is easy to see from the boundedness of $\{y^{k_i}\}$, the assumption $(A3)$ and $v^{k_i} \in \partial_2 f(y^{k_i}, y^{k_i})$ for all i that the sequence $\{v^{k_i}\}$ is bounded. Therefore, there exists a subsequence of $\{v^{k_i}\}$, still denote $\{v^{k_i}\}$, such that v^{k_i} converges weakly to v. Since $\partial_2 f(x, x)$ satisfies condition (C1) and the sequence $\{y^{k_i}\}$ converges weakly to $\hat{p}$, we have $v \in \partial_2 f(\hat{p}, \hat{p})$. Because the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 \le \|v\| \le \liminf_{i\to\infty}\|v^{k_i}\|.$$

If $v = 0$ then we have from Remark 9 (i) and $v \in \partial_2 f(\hat{p}, \hat{p})$ that $\hat{p}$ is a solution. So, we only consider $v \ne 0$. Then, we have from last inequality that

$0 \liminf_{i\to\infty} \|v^{k_i}\| > 0$. Hence, we have that

$$0 \le \limsup_{i\to\infty} \gamma_i \|\varrho^{k_i}\| = \limsup_{i\to\infty} \frac{\gamma_i}{\|v^{k_i}\|} \le \frac{\limsup_{i\to\infty} \gamma_i}{\liminf_{i\to\infty} \|v^{k_i}\|} = 0,$$

which implies that

$$\lim_{i\to\infty} \limsup_{i\to\infty} \gamma_i \|\varrho^{k_i}\| = 0. \tag{29}$$

It follows that $z^{k_i} + \gamma_i \varrho^{k_i} \rightharpoonup z$ as $i \to \infty$. Passing the limit into the inequality (28), using the weakly lower semicontinuity of the function $f(\cdot,\cdot)$ and $\lim_{i\to\infty} \gamma_i = 0$, we have

$$f(z,\hat{p}) \le 0, \ \forall z \in T(\hat{p}).$$

Since $f(\cdot, y)$ is weakly upper semicontinuous on C for every $y \in C$, the function f has the upper sign property ([**?**], Definition 2.1), and since the multivalued mapping $T(\cdot)$ has convex values, we obtain that $\hat{p} \in Sol(QEP)$ ([**?**], Proposition 3.1). □

Remark 17 *Let $\mathcal{H}$ be a finite dimensional Hilbert space. We have the following sufficient conditions for the mapping $\partial_2 f(x,x)$ to satisfy condition $(C1)$:*

(i) If f satisfies the assumption $(A2)$ and the function $f(\cdot, y)$ is upper semicontinuous on C then the mapping $\partial_2 f(x,x)$ satisfies Condition $(C1)$. Indeed, let $x^k \to x$, $u^k \in \partial_2 f(x^k, x^k)$ and $u^k \to u$ then we have

$$f(x^k, y) = f(x^k, y) - f(x^k, x^k) \ge \langle u^k, y - x^k \rangle, \forall y \in \mathcal{H}.$$

Taking the limit as $k \to +\infty$ on the last inequality and using the upper semicontinuity of $f(\cdot, y)$, we get $f(x,y) - f(y,y) \ge \langle u, y - x\rangle, \forall y \in \mathcal{H}$, and so $u \in \partial_2 f(x,x)$.

*(ii) Assume that f satisfies the assumption $(A2)$ and $\partial_2 f(x,x)$ is bounded for all $x \in C$. Then, we have from the assumption $(A3)$ that the mapping $\partial_2 f(x,x)$ satisfies Condition $(C1)$ ([**?**], Propositions 3 and 5, Chapter E). It is known that $\partial_2 f(x,x)$ is bounded at every $x \in \mathrm{int}(\mathrm{dom} f(x,\cdot))$ ([46], Theorem 2.6).* □

If $T(x) = C$ for all $x \in C$ then we have that T is M-continuous and closed convex valued, T^* coincides with the solution set of Problem (EP), and T_* becomes the solution set of the Minty equilibrium problem (MEP). If $f(\cdot, y)$ is upper semicontinuous for each $y \in C$ and $f(x,\cdot)$ is lower semicontinuous and convex for each $x \in C$, then $T_* \subseteq T^*$, add the assumption that $f(x,y)$ is pseudomonotone on C, we get $T^* = T_*$ ([**?**]). In general, the inclusion $T^* \subseteq T_*$ is false even if $f(x,y)$ is quasimonotone ([**?**]).

Applying Algorithm 1 with $T(x) = C$ for all $x \in C$, we get the following result for Problem (EP).

Proposition 18 *Assume that (A_1), (A_2') and (A_3) hold, $T(x) = C$ for all $x \in C$, and $T_* \neq \emptyset$. If the mapping $\partial_2 f(x,x)$ satisfies Condition $(C1)$, then the sequence $\{x^k\}$ generated in Algorithm 1 converges weakly to a solution of Problem (EP).*

Proof. Let p be an arbitrary point in T^*. From the proof of Theorem 16, we have that the sequence $\{\|x^k - p\|\}$ converges and every weak cluster point of the sequence $\{x^k\}$ generated in Algorithm 1 belongs to T_*. Then, by applying Lemma 7 for $C := T_*$ and $x := p$, we deduce that the sequence $\{x^k\}$ generated by Algorithm 1 converges weakly to a point $\hat{p} \in T_*$. Since $T_* \subseteq T^*$, $\hat{p}$ is a solution of Problem (EP). □

Remark 19 *To our knowledge, there are currently no results on algorithms specifically designed for quasi-equilibrium and equilibrium problems involving quasi-monotonic functions. In the case of the quasi-monotone inequal variation problem, the number of algorithms for this problem is not many. Some algorithms for equilibrium and quasi-equilibrium problems with function f do not have any generalized monotonicity as we can see in [30**? ?**]. However, these algorithms come at a cost because they have to use search procedures at each iteration.*

By Theorems 15 and 16, we only obtain weak convergence results of Algorithm 1. With the motivation of constructing a strongly convergent algorithm to solve (QEP), we exploit the fact that we need to solve simultaneously an equilibrium problem and a fixed point problem. This suggests us combining Algorithm 1 with the Mann iteration scheme for finding a fixed point of a multi-valued mapping (see [38]) to obtain Algorithm 2 described below. The strong convergence of Algorithm 2 is stated in Theorem 20.

Theorem 20 *Assume that (A_1)-(A_3) hold and $T^* \cap Fix(T) \neq \emptyset$. Assume furthermore that T is a hemicompact multi-valued mapping satisfying condition (C2) and Π_T is quasi-nonexpansive with $\Pi_T(x) \neq \emptyset$ for every $x \in C$. Then every weak cluster point of the sequence $\{x^k\}$ generated by Algorithm 2 belongs to $Sol(QEP)$. In addition, the whole sequence $\{x^k\}$ converges strongly to a solution of (QEP) if the bifunction f is strictly monotone on C and either T satisfies condition (C1) or the set S^* is nonempty.*

Proof. By the same arguments as in the proof of Lemma 12, all the elements of the sequence $\{\lambda_k\}$ generated in Algorithm 2 are in the interval $\left[\min\left\{\frac{\nu}{L^*}, \lambda_0\right\}, \lambda_0 + M\right]$ where $M = \sum_{k=0}^{+\infty} \rho_k$. Furthermore, the sequence $\{\lambda_k\}$ converges to some λ belonging to the same interval.

Let p be an arbitrary point in $T^* \cap Fix(T)$, and K determined as in the proof of Lemma 13. By the same arguments as in that proof, we get

$$\|w^k - p\|^2 \leq \|x^k - p\|^2 - (\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{(\lambda_{k+1} - \nu\lambda_k)^2}{(\lambda_{k+1} + \nu\lambda_k)^2}\|x^k - y^k\|^2 \tag{30}$$

for all $k \geq K$. Note that $p \in Fix(T)$, so we have $\Pi_T(p) = \{p\}$ by Lemma 2, and in addition we get for all $k \geq 0$ that

$$\begin{aligned}\|x^{k+1} - p\| &= \|\alpha_k W^k + (1-\alpha_k)w^k - p\| \\ &\leq \alpha_k \|W^k - p\| + (1-\alpha_k)\|w^k - p\| \\ &\leq \alpha_k d(\Pi_T(w^k), p) + (1-\alpha_k)\|w^k - p\| \\ &= \alpha_k d^H(\Pi_T(w^k), \Pi_T(p)) + (1-\alpha_k)\|w^k - p\| \\ &\leq \alpha_k \|w^k - p\| + (1-\alpha_k)\|w^k - p\| \\ &= \|w^k - p\|. \end{aligned} \tag{31}$$

Algorithm 2 An algorithm with Mann iteration for solving (QEP)

Take an arbitrary starting point $x^0 \in C$. Take $\lambda_0 > 0$, $\nu \in (0,1)$, $L^* > L$, $\gamma \in (0,2)$, $0 < \alpha < \beta < 1$, and control parameter sequences $\{\rho_i\}$, $\{\kappa_i\}$, $\{\alpha_i\}$ satisfying conditions

$$\rho_i > 0, \quad \sum_{i=0}^{+\infty} \rho_i < +\infty, \quad \kappa_i > 0, \quad \sum_{i=0}^{+\infty} \kappa_i < +\infty, \quad \alpha_i \in [\alpha, \beta].$$

for $k = 0, 1, 2, \ldots$ **do**
  Compute $u^k \in \partial_2 f(x^k, x^k)$.
  if $u^k = 0$ **then**
    return x^k
  else
    Compute $y^k = P_{T(x^k)}(x^k - \lambda_k u^k)$.
    if $y^k = x^k$ **then**
      return x^k
    else
      Compute $v^k \in B\left(u^k, L^*\|x^k - y^k\|\right) \cap \partial_2 f(y^k, y^k)$.
      if $v^k = 0$ **then**
        return x^k
      else
        Compute

$$\begin{aligned} d^k &= x^k - y^k - \lambda_k(u^k - v^k), \\ \tau_k &= \begin{cases} (\gamma + \kappa_k)\frac{|\langle x^k - y^k, d^k\rangle|}{\|d^k\|^2} & \text{if } \|d^k\| \neq 0, \\ 0 & \text{otherwise,} \end{cases} \\ w^k &= P_{T(x^k)}(x^k - \tau_k \lambda_k v^k), \\ x^{k+1} &= \alpha_k W_k + (1 - \alpha_k) w^k \quad \text{with } W^k \in \Pi_T(w^k), \\ \lambda_{k+1} &= \begin{cases} \min\left\{\frac{\nu\|x^k - y^k\|}{\|u^k - v^k\|}, \lambda_k + \rho_k\right\} & \text{if } u^k - v^k \neq 0, \\ \lambda_k + \rho_k & \text{otherwise.} \end{cases} \end{aligned}$$

      end if
    end if
  end if
end for

The last inequality above is by the assumption that Π_T is quasi-nonexpansive. By (30) and (31), we obtain

$$\|x^{k+1} - p\|^2 \leq \|x^k - p\|^2 - (\gamma + \kappa_k)(2 - \gamma - \kappa_k)\frac{(\lambda_{k+1} - \nu\lambda_k)^2}{(\lambda_{k+1} + \nu\lambda_k)^2}\|x^k - y^k\|^2$$

for all $k \geq K$, i.e., the sequences $\{x_k\}$, $\{y_k\}$, $\{\lambda_k\}$ generated in Algorithm 2 also satisfy (15). Thus, following the same arguments as in the begin of the proof of Theorem 15, we deduce that $\|x^{k+1} - p\| \leq \|x^k - p\|$ for all $k \geq K$, which implies the convergence of $\{\|x^k - p\|\}$ and the boundedness of $\{x^k\}$. We also deduce that $\|x^k - y^k\| \to 0$. By Lemma 8(i) we have

$$\begin{aligned}
\|x^{k+1} - p\|^2 &= \|\alpha_k W^k + (1-\alpha_k)w^k - p\|^2 \\
&= \alpha_k\|W^k - p\|^2 + (1-\alpha_k)\|w^k - p\|^2 - \alpha_k(1-\alpha_k)\|W^k - w^k\|^2 \\
&\leq \alpha_k(d(\Pi_T(w^k), p))^2 + (1-\alpha_k)\|w^k - p\|^2 - \alpha_k(1-\alpha_k)\|W^k - w^k\|^2 \\
&\leq \alpha_k(d^H(\Pi_T(w^k), \Pi_T(p)))^2 + (1-\alpha_k)\|w^k - p\|^2 - \alpha_k(1-\alpha_k)\|W^k - w^k\|^2 \\
&\leq \alpha_k\|w^k - p\|^2 + (1-\alpha_k)\|w^k - p\|^2 - \alpha_k(1-\alpha_k)\|W^k - w^k\|^2 \\
&\leq \|w^k - p\|^2 - \alpha_k(1-\alpha_k)\|W^k - w^k\|^2. \qquad (32)
\end{aligned}$$

For all $k \geq K$ we have $\|w^k - p\| \leq \|x^k - p\|$ due to (30), so by (32) we get

$$\|x^{k+1} - p\|^2 \leq \|x^k - p\|^2 - \alpha_k(1-\alpha_k)\|W^k - w^k\|^2.$$

Note that $\alpha_k \in [\alpha, \beta] \subset [0, 1]$ for all $k \geq 0$, it follows that

$$\alpha(1-\beta)\|W^k - w^k\|^2 \leq \alpha_k(1-\alpha_k)\|W^k - w^k\|^2 \leq \|x^k - p\|^2 - \|x^{k+1} - p\|^2,$$

which, together with $W^k \in \Pi_T(w^k)$, implies that

$$\alpha(1-\beta)(d(w^k, \Pi_T(w^k)))^2 \leq \|x^k - p\|^2 - \|x^{k+1} - p\|^2.$$

Since $[\alpha, \beta] \subset (0, 1)$, by taking the limit as $k \to +\infty$ on both sides of this inequality and keeping in mind that the sequence $\{\|x^k - p\|\}$ is convergent, we get

$$\lim_{k\to+\infty} d(w^k, \Pi_T(w^k)) = 0, \qquad (33)$$

which, together with $\Pi_T(w^k) \subseteq T(w^k)$, implies that

$$\lim_{k\to+\infty} d(w^k, T(w^k)) = 0. \qquad (34)$$

On the other hand, it follows from the definition of x^{k+1} in Algorithm 2 that

$$\|x^{k+1} - w^k\| = \alpha_k\|W^k - w^k\| = \alpha_k d(w^k, \Pi_T(w^k)).$$

Hence, by (33) we have

$$\lim_{k\to+\infty} \|x^{k+1} - w^k\| = 0. \qquad (35)$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i+1}\}$ weakly converging to some $\hat{p} \in C$. Together with (35), it implies that the sequence $\{w^{k_i}\}$ also converges weakly to $\hat{p}$. By (34) and the hemicompactness of T, there exists a subsequence $\{w^{k_{i_h}}\}$ strongly converging to $\hat{p}$. Then, by Lemma 2 and the triangle inequality, we have

$$\begin{aligned} d(\hat{p}, \Pi_T(\hat{p})) &\leq d(w^{k_{i_h}}, \hat{p}) + d(w^{k_{i_h}}, \Pi_T(w^{k_{i_h}})) + d^H(\Pi_T(w^{k_{i_h}}), \Pi_T(\hat{p})) \\ &\leq 2\|w^{k_{i_h}} - \hat{p}\| + d(w^{k_{i_h}}, \Pi_T(w^{k_{i_h}}))). \end{aligned} \tag{36}$$

Let $i \to +\infty$ in (36) and keeping (33) in mind, we obtain $d(\hat{p}, \Pi_T(\hat{p})) = 0$, which means $\hat{p} \in \Pi_T(\hat{p})$. By Lemma 2, one has $\hat{p} \in T(\hat{p})$. Since $\{x^{k_i+1}\}$ weakly converges to $\hat{p}$ and $\|x^k - y^k\| \to 0$, the sequence $\{y^{k_i+1}\}$ also converges weakly to $\hat{p}$. By the condition (C2) and the same arguments as in the end of the proof of Theorem 15, we obtain $f(\hat{p}, z) \geq 0$ for all $z \in T(\hat{p})$. Therefore, $\hat{p} \in Sol(QEP)$.

Now, assume that the bifunction f is strictly monotone on C and either T satisfies condition (C1) or the set S^* is nonempty. By the same arguments as in the proof of Theorem 15, we have $\hat{p} \in T^*$. Hence, $\hat{p} \in T^* \cap Fix(T)$, which implies that $\{\|x^k - \hat{p}\|^2\}$ has finite limit. By (35) and the fact that $\{w^{k_{i_h}}\}$ converges strongly to $\hat{p}$, we have $\{x^{k_{i_h}+1}\}$ also converges strongly to $\hat{p}$. Therefore, the sequence $\{x^k\}$ converges strongly to $\hat{p}$. □

Remark 21 *It is easy to see that $p \in Fix(T)$ for all $p \in S^*$. Hence, we have that $T^* \cap Fix(T) \neq \emptyset$ if $S^* \neq \emptyset$ because S^* is a subset of the set T^*.*

4 Numerical experiments

We did a number of numerical experiments to evaluate the performance of our first proposed algorithm (Algorithm 1) and compare its numerical behavior with some state-of-the-art algorithms for solving (QEP) involving pseudo-monotone equilibrium bifunctions. Namely, we chose version 1b of the hybrid extragradient algorithm (HEA1b for short) proposed in [15] and the proximal point method (PPM for short) proposed in [35]. It is worth noting that in the former paper the authors presented six variants of their algorithm and pointed out by their numerical experiments that Algorithm HEA1b outperforms the others in most of their tests.

All programs in our experiments were coded by using MATLAB R2020a and conducted on a PC Intel(R) Core(TM) i7-6700HQ CPU @ 2.60GHz 2.60GHz, 16.0 GB RAM. In all of our tests, we applied the stopping rule $\|x^{k+1} - x^k\| \leq 10^{-4}$. For experimenting Algorithm 1 proposed in this paper, we set $\lambda_0 = 0.5$, $\nu = 0.5$, $\gamma = 1$, and chose $\alpha_i = 0.5$, $\rho_i = \kappa_i = \frac{1}{i+1}$ for all $i \in \mathbb{N}$. For experimenting Algorithm HEA1b (resp., Algorithm PPM), we chose the same setting as in the experiments of [15] (resp., [35]). In the following subsections, we describe our tested instances and report the numerical results of the tests.

4.1 Experiment 1

Consider the (QEP) described in Example 11 with $g(x) = \sum_{i=1}^{n} x_i^2$. It is trivial to see that the function g is convex, differentiable, and increasing on $\mathbb{R}^n_+$, while its gradient $\nabla g(x) = 2x$ is Lipschitz continuous on $\mathbb{R}^n_+$. Therefore, this choice satisfies the conditions applied on the function g as stated in Example 11. Note that $g(b) - g(a) = 0$ where a and b are specified in the example.

For creating numerical instances of this problem, given the dimension n of the space $\mathcal{H} = \mathbb{R}^n$, we generated the matrices P and Q as follows. We first generated a diagonal matrix $D_Q \in \mathbb{R}^{n\times n}$ whose diagonal entries are random real values in $[0, 0.3]$, and a diagonal matrix $D_P \in \mathbb{R}^{n\times n}$ whose diagonal entries are random values in $[0.3, 1]$. We then generated a matrix $Z \in \mathbb{R}^{n\times n}$ whose entries are random values in $[0, 2]$. After that, we computed $Q = Z^t D_Q Z$ and $P = Z^t D_P Z$. In this way, it is easy to see that Q is positive semidefinite. Furthermore, since $D_P - D_Q$ is a diagonal matrix with nonnegative entries, it follows that $P - Q = Z^t(D_P - D_Q)Z$ is also positive semidefinite. In addition, since all entries of D_P, D_Q, Z are nonnegative, it is clear that all elements of both P and Q are nonnegative. Concerning vector c, we randomly generated non-negative values for $c_1, \ldots, c_{n-1}$, and then let $c_n = p_{1n} + q_{11} + c_1 - p_{nn} - q_{n1} + 1$, so the obtained vector c satisfies

$$c_n > p_{1n} + q_{11} + c_1 - p_{nn} - q_{n1} + g(b) - g(a).$$

For the instances of (QEP) in this experiment, it has been already shown in Example 11 that $S^* = \emptyset$, so the extragradient algorithm (HEA1b) and the proximal point method (PPM) are not guaranteed to be convergent, hence we only apply Algorithm 1. The projection performed in the experiment is solved using the MATLAB subroutine 'quadprog' by observing that the projection of a vector u on $T(x)$ is the optimal solution to the quadratic programming

$$\min \|u - z\|^2 \quad \text{s.t.} \quad \sum_{i=1}^{n} z_i \geq 1 \quad \text{and} \quad 0 \leq z_i \leq 2x_i \ \forall i = 1, \ldots, n.$$

For illustration, in the first generated instance, we set $n = 5$ and obtained

$$P = \begin{bmatrix} 5.9413 & 2.5584 & 2.5172 & 4.5852 & 4.6082 \\ 2.5584 & 2.1685 & 0.6480 & 2.1818 & 2.0165 \\ 2.5172 & 0.6480 & 1.7650 & 2.5874 & 2.0910 \\ 4.5852 & 2.1818 & 2.5874 & 4.8431 & 4.2424 \\ 4.6082 & 2.0165 & 2.0910 & 4.2424 & 4.5611 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.0159 & 0.4685 & 0.3725 & 0.6405 & 0.5837 \\ 0.4685 & 0.3472 & 0.1165 & 0.2973 & 0.2212 \\ 0.3725 & 0.1165 & 0.2473 & 0.3663 & 0.2803 \\ 0.6405 & 0.2973 & 0.3663 & 0.6432 & 0.5031 \\ 0.5837 & 0.2212 & 0.2803 & 0.5031 & 0.4926 \end{bmatrix},$$
$$c = (0.0399, 0.5880, 0.1125, 0.1292, 1.5192)^t.$$

Starting at $x^0 = (0, 0, 0, 0, 5)^t$, Algorithm 1 converges to the solution $(0, 0, 0, 0, 1)^t$ after 6 iterations and within 0.024 second. To confirm the efficiency of this algorithm on the considering problem, we generated more instances with different values of n, and for each of such instance we run 50 times of the algorithm with random starting points in the corresponding set C. In Table 1 we report the minimum, maximum, and average of the number of iterations as well as the CPU time in our tests.

Table 1: Results for Experiment 4.1.

n	Number of iterations			CPU time (seconds)		
	min	max	average	min	max	average
5	6	56	24.12	0.012	0.127	0.050
10	12	94	37.88	0.024	0.195	0.080
20	22	102	51.52	0.048	0.241	0.111
30	31	110	59.78	0.066	0.231	0.129
40	39	114	69.48	0.085	0.269	0.152

4.2 Experiment 2

In this experiment we considered the instance of (QEP) in which $\mathcal{H} = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $f(x, y) = y_1 - x_1 + y_2^2 - x_2^2$, $T(x) = \{y \in \mathbb{R}^2_+ \mid y_1 + y_2 = 1 + \frac{x_1}{1+x_1}\}$. This is a modification of Example 4.1 in [35], which is the QEP version of Example 4.4. in [49]. It is known that the solution set of this (QEP) consists of the single point $x^* = (1, \frac{1}{2})$. It is simple to verify that the bifunction f in this (QEP) satisfies conditions (A1)-(A3), especially that f is monotone on $C \times C$ and its partial differential $\partial_2 f(x, y) = (1, 2y_2)^t$ satisfies (A3) with $L = 2$. Note that the projection on $T(x)$ performed in this experiment has the following explicit formulation:

$$P_{T(x)}(u) = \begin{cases} (0, \beta)^t & \text{if } u_1 - u_2 \le -\beta, \\ (\beta, 0)^t & \text{if } u_1 - u_2 \ge \beta, \\ \left(\frac{1}{2}(\beta + u_1 - u_2), \frac{1}{2}(\beta - u_1 + u_2)\right)^t & \text{if } -\beta < u_1 - u_2 < \beta, \end{cases}$$

in which $\beta = 1 + \frac{x_1}{1+x_1}$.

For the convergence of Algorithm HEA1b, the authors in [15] impose a condition that $x \in T(x)$ for all $x \in C$. However, this condition does not hold in general for the multivalued mapping T defined above. Hence, Algorithm

HEA1b is not applicable in this experiment. In fact, if we start from $x^0 = (0,0)^t$, then following the description of this algorithm we get $y^0 = (0.5, 0.5)^t$, which leads to

$$f(z, x^0) - f(z, y^0) = -0.75 < c\|x^0 - y^0\|^2 = 0.25$$

for all $z \in C$, and consequently, Step 2 in Algorithm HEA1b does not terminate.

To see the performance of Algorithm 1 and Algorithm PPM on the (QEP) instance in this experiment, we randomly generated a list of 50 initial points in the box $[0,5] \times [0,5]$. Then, for each starting point in the list, we tested these algorithms and recorded their performance (the number of iterations and running time). In Table 2 we report the minimum, maximum, and average of the number of iterations as well as the CPU time of each of these algorithms over 50 runs. One can see from the result in this table that our algorithm needs more number of iterations than Algorithm PPM but outperforms the latter in sense of computation time. In fact, a closer look into the detail result shows that, in every run corresponding to any starting point in the above mentioned list, our algorithm is faster than Algorithm PPM. This can be explained by the fact that in each iteration of Algorithm PPM one needs to solve an auxiliary equilibrium problem, which is more time consuming than performing projection operations in Algorithm 1.

Table 2: Results for Experiment 4.2.

Algorithm	Number of iterations			CPU time (seconds)		
	min	max	average	min	max	average
Algorithm 1	20	34	29.68	0.0000442	0.0001394	0.0000665
Algorithm PPM	7	19	16.24	0.0000699	0.0001902	0.0000993

4.3 Experiment 3

The (QEP) instance in this experiment is taken from Example 2 in [15]. In this instance we have $\mathcal{H} = \mathbb{R}^2$, $C = (-\infty, 15] \times (-\infty, 15]$, $f(x,y) = \langle F(x), y - x\rangle$ in which $F(x) = (2x_1 + \frac{8}{3}x_2 - 34, 2x_2 + \frac{5}{4}x_1 - 24.25)$, $T(x) = T_1(x_2) \times T_2(x_1)$ in which

$$T_1(x_2) = \{y_1 \in \mathbb{R} \mid 0 \le y_1 \le 10, y_1 \le 15 - x_2\},$$
$$T_2(x_1) = \{y_2 \in \mathbb{R} \mid 0 \le y_2 \le 10, y_2 \le 15 - x_1\}.$$

One can check that the bifunction f in this (QEP) satisfies conditions (A1)-(A3). Particularly, simple computations give us

$$f(x,y) + f(y,x) = \langle F(x) - F(y), y - x\rangle$$

$$= -2\left(\left((y_1 - x_1) + \frac{47}{48}(y_2 - x_2)\right)^2 + \frac{95}{48^2}(y_2 - x_2)^2\right),$$

hence f is strictly monotone on $C \times C$, and its partial differential $\partial_2 f(x,x) = F(x)$ satisfies (A3) with $L = \frac{\sqrt{643}}{4}$. It is worth noting that the projection on $T(x)$ performed in this experiment has the following explicit formulation:

$$P_{T(x)}(y) = (P_{T_1(x_2)}(y_1), P_{T_2}(x_1)(y_2)),$$

in which

$$P_{T_1(x_2)}(y_1) = \begin{cases} \min\{10, 15 - x_2\} & \text{if } y_1 \geq \min\{10, 15 - x_2\}, \\ 0 & \text{if } y_1 \leq 0, \\ y_1 & \text{if } y_1 \in [0, \min\{10, 15 - x_2\}], \end{cases}$$

$$P_{T_2(x_1)}(y_2) = \begin{cases} \min\{10, 15 - x_1\} & \text{if } y_2 \geq \min\{10, 15 - x_1\}, \\ 0 & \text{if } y_2 \leq 0, \\ y_2 & \text{if } y_2 \in [0, \min\{10, 15 - x_1\}]. \end{cases}$$

4.4 Experiment 4

Taken from Example 4 in [15].

This example is the same as the one in Experiment 3 above, except that

$$T_2(x_1) = \{y_2 \in \mathbb{R} \mid 0 \leq y_2 \leq 10\}.$$

Formula of of projection on $T(x)$:

$$P_{T(x)}(y) = (P_{T_1(x_2)}(y_1), P_{T_2}(x_1)(y_2)),$$

in which $P_{T_1(x_2)}(y_1)$ is as in Experiment 3 above, and

$$P_{T_2(x_1)}(y_2) = \begin{cases} 10 & \text{if } y_2 \geq 10, \\ 0 & \text{if } y_2 \leq 0, \\ y_2 & \text{if } y_2 \in [0, 10]. \end{cases}$$

4.5 Experiment 5

Taken from Example 1 in [15].

Consider (QEP) in which $H = \mathbb{R}^5$, $C = \{x \in [-5,5]^5 \mid \sum_{i=1}^5 x_i \geq -1\}$, $f(x,y) = \langle Px + Qy + q, y - x\rangle$ in which

$$P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix},$$

$T(x) = T_1(x) \times T_2(x) \times T_3(x) \times T_4(x) \times T_5(x)$ in which

$$T_i(x) = \{y_i \in \mathbb{R} \mid y_i + \sum_{1 \leq j \leq 5, j \neq i} x_j \geq -1\} \quad \text{for } i = 1, \ldots, 5.$$

Check conditions (A1)-(A3):

- $f(x,x) = 0$ for all $x \in C$; $f(x,y) + f(y,x) = (y-x)^t(Q-P)(y-x) \leq 0$ for all $x, y \in C$ since

$$Q - P = \begin{bmatrix} -1.5 & -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & -1 & -1.8 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

  is negative definite (by Sylvester's criterion on determinants of principal minors of $Q - P$), hence f is strictly monotone on $C \times C$, which implies that f is pseudomonotone on $C \times C$; f is continuous on $C \times C$.
- $f(x,\cdot)$ is quadratic, and since Q is positive definite, we have $f(x,\cdot)$ is convex and continuous on C; its partial differential is $\partial_2 f(x,y) = Px + q - Q^t x + (Q + Q^t)y$, so $\partial_2 f(x,x) = (P+Q)x + q$.
- For all $x, y \in C$ we have

$$\rho(\partial_2 f(x,x), \partial_2 f(y,y)) = \|(P+Q)x - (P+Q)y\| \leq \|P+Q\|\|x-y\|.$$

  Hence (A3) is satisfied with $L = \|P+Q\|$ which is the largest eigenvalue of $P+Q$. Since

$$P + Q = \begin{bmatrix} 4.7 & 3 & 0 & 0 & 0 \\ 3 & 5.2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 0 \\ 0 & 0 & 3 & 4.8 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

  has 5 eigenvalues 1.8983, 1.9396, 5, 7.9017, 7.9604, we have $L = 7.9604$ and we can take $L^* = 8$.

Formula of projection on $T(x)$:

$$P_{T(x)}(y) = (P_{T_1(x)}(y_1), P_{T_2(x)}(y_2), P_{T_3(x)}(y_3), P_{T_4(x)}(y_4), P_{T_5(x)}(y_5)),$$

in which each $T_i(x)$ is the ray $[-1 - \sum_{j\in\{1,\ldots,5\}\setminus\{i\}} x_j, +\infty) \subset \mathbb{R}$, hence

$$P_{T_i(x)}(y_i) = \begin{cases} y_i & \text{if } y_i \geq -1 - \sum_{j\in\{1,\ldots,5\}\setminus\{i\}} x_j, \\ -1 - \sum_{j\in\{1,\ldots,5\}\setminus\{i\}} x_j & \text{if } y_i < -1 - \sum_{j\in\{1,\ldots,5\}\setminus\{i\}} x_j. \end{cases}$$

4.6 Experiment 6

Taken from Example 2 in [15].

Consider (QEP) in which $H = \mathbb{R}^5$, $C = [1, 150]^5$, $f(x, y) = \langle F(x), y - x\rangle$ in which $F(x) = (F_1(x), F_2(x), F_3(x), F_4(x), F_5(x))$ where

$$\begin{aligned}
F_1(x) &= 10 + \left(\frac{x_1}{5}\right)^{\frac{1}{1.2}} + \left(\frac{5000}{Q}\right)^{\frac{1}{1.1}} \left(\frac{x_1}{1.1Q} - 1\right), \\
F_2(x) &= 8 + \left(\frac{x_2}{5}\right)^{\frac{1}{1.1}} + \left(\frac{5000}{Q}\right)^{\frac{1}{1.1}} \left(\frac{x_2}{1.1Q} - 1\right), \\
F_3(x) &= 6 + \frac{x_3}{5} + \left(\frac{5000}{Q}\right)^{\frac{1}{1.1}} \left(\frac{x_3}{1.1Q} - 1\right), \\
F_4(x) &= 4 + \left(\frac{x_4}{5}\right)^{\frac{1}{0.9}} + \left(\frac{5000}{Q}\right)^{\frac{1}{1.1}} \left(\frac{x_4}{1.1Q} - 1\right), \\
F_5(x) &= 2 + \left(\frac{x_5}{5}\right)^{\frac{1}{0.8}} + \left(\frac{5000}{Q}\right)^{\frac{1}{1.1}} \left(\frac{x_5}{1.1Q} - 1\right),
\end{aligned}$$

with $Q = x_1 + x_2 + x_3 + x_4 + x_5$, the mapping $T : C \to C$ is determined by $T(x) = (T_1(x), T_2(x), T_3(x), T_4(x), T_5(x))$ in which for each $i = 1, \ldots, 5$ we have

$$T_i(x) = \{y_i \in [1, 150] \mid y_i + \sum_{j\in\{1,\ldots,5\}, j\neq i} x_j \leq 700\}.$$

Check conditions (A1)-(A3):

- $f(x, x) = 0$ for all $x \in C$; $f(x, y) + f(y, x) = \langle F(x) - F(y), y - x\rangle \leq 0$ for all $x, y \in C$ due to the monotonicity of F (since F is the gradient of a convex function), hence f is monotone on $C \times C$, which implies that f is pseudomonotone on $C \times C$; f is continuous on $C \times C$.
- $f(x, \cdot)$ is affine, hence it is convex and continuous on C; its partial differential is $\partial_2 f(x, y) = F(x)$, so $\partial_2 f(x, x) = F(x)$.

- Since F is continuous on the compact set C, it is Lipschitz on C. Therefore, for all $x, y \in C$ we have

$$\begin{aligned} &\rho(\partial_2 f(x,x), \partial_2 f(y,y)) \\ =&\|F(x) - F(y)\| \\ \leq&L\|x - y\|, \end{aligned}$$

Hence (A3) is satisfied with L is the Lipschitz constant of F.

Formula of projection on $T(x)$:

$$P_{T(x)}(y) = (P_{T_1(x)}(y_1), P_{T_2}(x)(y_2), P_{T_3(x)}(y_3), P_{T_4(x)}(y_4), P_{T_5(x)}(y_5)),$$

in which for each $i = 1, \dots, 5$ we have

$$P_{T_i(x)}(y_i) = \begin{cases} 1 & \text{if } y_i \leq 1, \\ u_i & \text{if } y_i \geq u_i, \\ y_i & \text{if } y_i \in [0, u_i], \end{cases}$$

where $u_i = \min\{150, 700 - \sum_{j \in \{1,\dots,5\}, j \neq i} x_j\}$.

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