A note on the number of complex roots of a polynomial of one variable

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Abstract

The purpose of this article is twofold. The first aim is to find the number of complex solutions of a polynomial of one variable counting multiplicity. Then, our second aim is to approximate these solutions.

1 Introduction

From the Fundamental Theorem of Algebra, we deduce that every non-constant, single variable, degree n polynomial with real coefficients has counted with multiplicity, exactly n complex roots (see [3]). Mathematicians found the formula for the solution of a polynomial of degree less than 5 (see [2]). However, for an equation of degree greater than or equal to 5, we can't find a general formula of roots by radicals (see [4]). In this article, we want to find the number of complex solutions of a polynomial of one variable counting multiplicity. Then, we can approximate these solutions numerically. First of all, we list here some results about the information on the roots of a polynomial see for example [3, 6].

Theorem 1.1 (Descartes' Rule [3]). The number of positive real roots of an equation with real coefficients is either equal to the number of its variations of sign or is less than that number by a positive even integer. A root of multiplicity m is here counted as m roots.

This rule can be derived as a corollary of the Budan's Theorem below.

Theorem 1.2 (Budan's Theorem [3]). Let a and b be real numbers, $a < b$, neither a root of $f(x) = 0$, an equation of degree n with real coefficients. Let V_a denote the number of variations of the sign of

$$f(x); f'(x); f''(x); \ldots ; f^{(n)}(x)$$

for $x = a$, after vanishing terms have been deleted. Then $V_a - V_b$ is either the number of real roots of $f(x) = 0$ between a and b or exceeds the number of those roots by a positive even integer. A root of multiplicity m is here counted as m roots.

Given a square-free polynomial $f(x)$, denote:

- $f_0(x) = f(x)$.
- $f_1(x) = f'(x)$.
- $f_2(x) = -\text{rem}(f_0(x), f_1(x)) = f_1(x)q_0(x) - f_0(x)$, where $\text{rem}(f_0(x), f_1(x))$ is the remainder when polynomial long division is used to divide $f_0(x)$ by $f_1(x)$, and $q_0(x)$ is the quotient of said long division.

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- $f_3(x) = -\text{rem}(f_1(x), f_2(x)) = f_2(x)q_1(x) - f_1(x)$.
- In general, we define f_k inductively as $-\text{rem}(f_{k-2}(x), f_{k-1}(x)) = f_{k-1}(x)q_{k-2}(x) - f_{k-2}(x)$.
- Repeat this process until we arrive at some m such that $\text{rem}(f_{m-1}, f_m) = 0$, where $f_m(x) \neq 0$.

The finite sequence $f_0, f_1, ..., f_m$ is called a Sturm sequence or a Sturm chain. Using this definition, we can locate the distinct real roots of a polynomial.

THEOREM 1.3 (STURM THEOREM [6]). Let $f(x) = 0$ be a polynomial equation and let

$$f_0(x), f_1(x), \ldots, f_m(x)$$

be a Sturm sequence of the polynomial $f(x)$. Then for any two real numbers $a < b$, neither of which is a root of $f(x) = 0$, the number of distinct roots of $f(x) = 0$ lying between a and b equals the difference $g_f(a) - g_f(b)$ between the variations of the signs of the Sturm functions at $x = a$ and $x = b$.

2 Main result

In this section, we use the Sturm theorem to deduce a result on the number of complex solutions of a polynomial of one variable. For simplicity, for a polynomial $f(x)$ of degree n, we denote

- Rf: the number of real distinct roots of $f(x)$.
- $R^k f$: the number of real roots of multiple greater than or equal to k of $f(x)$.
- Cf: the number of complex distinct roots of $f(x)$.
- $C^k f$: the number of complex roots of multiple greater than or equal to k of $f(x)$.

We have the following theorem.

THEOREM 2.1. Consider $f(x) \in P_n[x]$ be a polymomial of degree $n \geq 1$. Let $f_0, f_1, \ldots f_m, f_{m+1} = 0$ be its Sturm sequence. Then, if f_m is a constant, then f only has simple roots. If f_m is a non-contant polynomial then f has multiple roots. Denote $f_{m_0}, f_{m_1}, ..., f_{m_s}, f_{m_{s+1}}$ be the Sturm sequence of f_m. Then:

1. The number of real distinct roots of $f(x)$ is

$$R^1 f = Rf = g_f(-\infty) - g_f(+\infty) = g_{f_0/f_m}(-\infty) - g_{f_0/f_m}(+\infty).$$

2. The number of complex distinct roots of $f(x)$

$$C^1 f = Cf = \deg \frac{f_0}{f_m} - g(-\infty) + g(+\infty) = n - \deg f_m - Rf.$$

3. The number of real roots of $f(x)$ of multiplicity 1 is

$$g(-\infty) - g(+\infty) - g_{f_m}(-\infty) + g_{f_m}(+\infty) = Rf - Rf_m$$

where $g_{f_m}(x)$ the number of changes sign of Sturm sequence of $f_m(x)$.

4. The number of complex roots of multiple greater than or equal to 1 of $f(x)$ is

$$n - 2\deg f_m + \deg f_{m_s} - R^1 f + R^1 fm$$

where $f_{m_0}, f_{m_1}, \ldots, f_{m_s}, f_{m_{(s+1)}} = 0$ is Sturm sequence of f_m.

Proof. 1. This is derived directly from the Sturm theorem since the multiplicity does not affect the difference $g_f(-\infty) - g_f(+\infty)$.

2. Since the polynomial f_0/f_m only contains simple roots of f so the numbers of distinct complex roots of f is

$$Cf = \deg \frac{f_0}{f_m} - g(-\infty) + g(+\infty).$$

3. If f_m (the greatest common divisor of f and its derivative) is a non-constant polynomial then $f(x)$ has roots with multiple greater than 1. We construct the Sturm sequence of f_m. Then $g_{f_m}(-\infty) + g_{f_m}(+\infty)$ is the numbers of distinct real roots of f_m. This is the number of roots with multiple greater than 1 of the $f(x)$. So, the number of real roots with multiplicity 1 is

$$g(-\infty) - g(+\infty) - g_{f_m}(-\infty) + g_{f_m}(+\infty) = Rf - Rf_m.$$

4. The number of distinct complex roots (not counting multiplicity) of f_m is

$$Cf_m = \deg \frac{f_m}{f_{m_s}} - g_{f_m}(-\infty) + g_{f_m}(+\infty) = \deg f_m - \deg f_{m_s} - Rf_m.$$

This is the number of complex roots of multiplicity greater than or equal 2 of $f(x)$. Thus, the the number of complex roots of multiplicity 1 of $f(x)$ is:

$$Cf - Cf_m = n - 2\deg f_m + \deg f_{m_s} - R^1 f + R^1 fm.$$

□

REMARK 2.2. Continue the Sturm sequence of f_{m_s} and so on, we will get the numbers of complex roots with higher multiplicity.

3 Numerical examples

In this section, we will use Newton's method[2] and the modified Newton's methodto approximate the solution of $f(x)$. Newton's method is a root-finding algorithm that produces successively better approximations to the roots of a real-valued function. The most basic version starts with a real-valued function f, its derivative f', and an initial guess x_0 for a root of f. If f satisfies certain assumptions and the initial guess is close, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

[2]also known as the Newton–Raphson method

is a better approximation of the root than x_0. The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently precise value is reached. If we know the multiplicity of roots, we can use the modified Newton's Method.

THEOREM 3.1 ([5]). If f is $(m+1)$-times continuously differentiable on $[a,b]$, which contains a root r of multiplicity $m > 1$, then Modified Newton's Method

$$x_{i+1} = x_i - \frac{mf(x_i)}{f'(x_i)}$$

converges locally and quadratically to r.

REMARK 3.2. When $m = 1$, this theorem becomes the original Newton's Method.

Example 3.3. Consider the following polynomial of degree 8:

$$f(x) = x^8 + 12x^7 + 68x^6 + 236x^5 + 550x^4 + 884x^3 + 964x^2 + 660x + 225 = 0.$$

The Sturm sequence of f is:

$$\begin{aligned}
&f_0(x) = f(x) = x^8 + 12x^7 + 68x^6 + 236x^5 + 550x^4 + 884x^3 + 964x^2 + 660x + 225\\
&f_1(x) = f'(x) = 8x^7 + 84x^6 + 408x^5 + 1180x^4 + 2200x^3 + 2652x^2 + 1928x + 660\\
&f_2(x) = -5x^6 - 48x^5 - 215x^4 - 560x^3 - 903x^2 - 864x - 405\\
&f_3(x) = x^5 + 5x^4 + 10x^3 + 6x^2 - 7x - 15\\
&f_4(x) = x^4 + 6x^3 + 16x^2 + 22x + 15\\
&f_5(x) = 0.
\end{aligned}$$

We have the following table:

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$g(x)$
$-\infty$	$+$	$-$	$-$	$-$	$+$	2
∞	$+$	$+$	$-$	$+$	$+$	2

So, the number of distinct real roots of $f(x)$ is

$$Rf = g(-\infty) - g(+\infty) = 2 - 2 = 0.$$

This means $f(x)$ has no real roots. So, all 8 roots are complex. We see that $f_4(x)$ is the greatest common divisor of f and f'. Thus, the number of distinct complex roots of $f(x)$ is

$$Cf = n - \deg f_4 - Rf = 8 - 4 - 0 = 4.$$

Next, we consider the sturm sequence of $f_4(x)$:

$$f_{4_0}(x) = f_4(x) = x^4 + 6x^3 + 16x^2 + 22x + 15$$
$$f_{4_1}(x) = 4x^3 + 18x^2 + 32x + 22$$
$$f_{4_2}(x) = -\frac{5}{4}x^2 - \frac{9}{2}x - \frac{27}{4}$$
$$f_{4_3}(x) = \frac{64}{25}x - \frac{64}{25}$$
$$f_{4_3}(x) = \frac{25}{2}$$
$$f_{4_4}(x) = 0.$$

We have

x	$f_{4_0}(x)$	$f_{4_1}(x)$	$f_{4_2}(x)$	$f_{4_3}(x)$	$f_{4_4}(x)$	$g(x)$
$-\infty$	$+$	$-$	$-$	$-$	$+$	2
∞	$+$	$+$	$-$	$+$	$+$	2

From the Sturm sequence of $f_4(x)$, we see that f_4 does not have multiple roots and from the table above we see that f_4 has no distinct real roots. Thus, f_4 have 4 distinct complex roots of the form: $a \pm ib$ và $c \pm id$ where $(a, b) \neq (c, d)$.

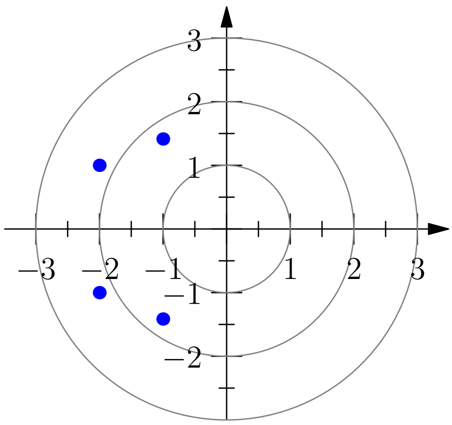
To approximate solutions of f, we use the modified Newton's methodwith $m = 2$. With intial guess $x_0 = 1 + i$, after 10 iterations, we get the result $x_{10} = -2 - i$. If we choose $x_0 = i$ after 7 iterations, we get the result : $x_7 = -1.0000 + 1.4142i$. If we use the Newton method, we need 17 iteration to get the same result as shown in the following tables:

k	Newton's method	Modified Newton's method
0	1+i	1+i
1	0.6210 + 0.9088i	0.2420 + 0.8175i
2	0.2766 + 0.8390i	-0.3888 + 0.7324i
3	-0.0403 + 0.7925i	-0.9568 + 0.7888i
4	-0.3364 + 0.7738i	-1.4765 + 1.1562i
5	-0.6163 + 0.7919i	-2.1792 - 0.7662i
6	-0.8792 + 0.8633i	-1.9572 - 0.9262i
7		-1.9989 - 1.0084i
8		-1.9999 - 1.0000i
9		-2.0000 - 1.0000i
10		-2.0000 - 1.0000i
17	-1.0000 + 1.4142i	
18	-1.0000 + 1.4142i	

and

k	Newton's method	Modified Newton's Method
0	i	i
1	-0.2800 + 0.9600i	-1.0334 + 1.0182i
2	-0.5371 + 0.9555i	-1.2977 + 1.4562i
3	-0.7684 + 0.9959i	-1.0302 + 1.2350i
4	-0.9588 + 1.0925i	-1.0413 + 1.4598i
5	-1.0662 + 1.2451i	-1.0048 + 1.4114i
6	-1.0419 + 1.3617i	-1.0000 + 1.4142i
7	-1.0190 + 1.3912i	-1.0000 + 1.4142i
8	-1.0091 + 1.4033i	
9	-1.0045 + 1.4089i	
10	-1.0022 + 1.4116i	
17	-1.0000 + 1.4142i	
18	-1.0000 + 1.4142i	

All 4 solutions are sketched in the following figure:

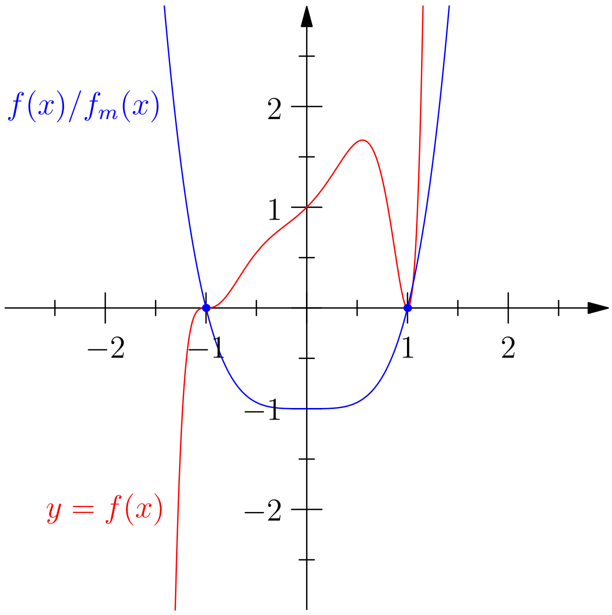


Example 3.4. Consider the polynomial equation of degree 11:

$$f(x) = x^{11} + x^{10} + x^9 + x^8 - 2x^7 - 2x^6 - 2x^5 - 2x^4 + x^3 + x^2 + x + 1 = 0,$$

The Sturm sequence of $f(x)$ is

$$
\begin{aligned}
f_0(x) &= f(x) = x^{11} + x^{10} + x^9 + x^8 - 2x^7 - 2x^6 - 2x^5 - 2x^4 + x^3 + x^2 + x + 1 \\
f_1(x) &= p'(x) = 11x^{10} + 10x^9 + 9x^8 + 8x^7 - 14x^6 - 12x^5 - 10x^4 - 8x^3 + 3x^2 + 2x + 1 \\
f_2(x) &= -x^9 - 2x^8 + 8x^7 + 8x^6 + 10x^5 + 12x^4 - 8x^3 - 8x^2 - 9x - 10 \\
f_3(x) &= -x^8 + 2x^4 - 1 \\
f_4(x) &= -x^7 - x^6 - x^5 - x^4 + x^3 + x^2 + x + 1 \\
f_5(x) &= 0.
\end{aligned}
$$



We have the following table:

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$g(x)$
$-\infty$	$-$	$+$	$+$	$-$	$+$	3
∞	$+$	$+$	$-$	$-$	$-$	1

Thus, f has $3-1=2$ distinct real solution (not counting multiplicity). The number of complex distinct roots of f is

$$11-7-2=2.$$

So f has two complex roots (not counting multiplicity) of the form $a \pm ib$. To get the information about multiplicity, we consider the Sturm sequence of f_4: $f_4(x)$:

$$\begin{aligned}
f_{4_0}(x) &= f_4(x) = -x^7 - x^6 - x^5 - x^4 + x^3 + x^2 + x + 1 \\
f_{4_1}(x) &= f_4'(x) = -7x^6 - 6x^5 - 5x^4 - 4x^3 + 3x^2 + 2x + 1 \\
f_{4_2}(x) &= x^5 + 2x^4 - 4x^3 - 4x^2 - 5x - 6 \\
f_{4_3}(x) &= x^4 - 1 \\
f_{4_4}(x) &= x^3 + x^2 + x + 1 \\
f_{4_5}(x) &= 0.
\end{aligned}$$

So, we have the following table:

x	$f_{4_0}(x)$	$f_{4_1}(x)$	$f_{4_2}(x)$	$f_{4_3}(x)$	f_{4_4}	$g(x)$
$-\infty$	$+$	$-$	$-$	$+$	0	2
∞	$+$	$+$	$+$	$+$	0	0

Thus, $f_4(x)$ has $2-0=2$ distinct real roots (not counting multiplicity). Then, we deduce that the roots f have multiplicity greater than or equal to 2. Then, f has no simple real root and f two complex roots with multiplicity greater than or equal to 2. We can continue to calculate the Sturm sequence of f_{4_3}. However, in this particular case, we have $f_{4_3}(x) = (x^2+1)(x+1)$.Thus, it has one simple root $x = -1$ and two complex roots $x = \pm i$. Therefore, $x = -1$ is a root with multiplicity 3, and $x = \pm i$ are also roots with multiplicity 3. The other real root is multiple 2. Then $f(x) = (x+i)^3(x-i)^3(x+1)^3(x-x_0)^2$. Divide $f(x)$ by $(x+i)^3(x-i)^3(x+1)^3 = (x+1)^3(x+1)^3$ we derive $x_0 = 1$. So, we have all the roots

- $x = 1$ (multiple 2)
- $x = -1$ (multiple 3)
- $x = \pm i$ (multiple 3)

In the general case, when it is not easy to factor $f_{4_3}(x)$, we continue this process to get the result.

4 Conclusion

The main result of the article is to state the theorem for calculating the number of complex solutions with corresponding multiples of any n degree algebraic equation. Then, we can accurately calculate the solution in special cases or approximate the solution with Newton's algorithm with known multiples.

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