THE VANISHING OF THE FIRST TIGHT HILBERT COEFFICIENT FOR BUCHSBAUM RINGS

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Abstract. We prove that if the first tight Hilbert coefficient vanishes then ring is F-rational provided it is a Buchsbaum local ring satisfying the (S_2) condition.

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1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d such that $\widehat{R}$ is reduced and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then for $n \gg 0$, $\ell(R/\overline{I^{n+1}})$ agrees with a polynomial in n of degree d, and we have integers $\overline{e}_0(I), \ldots, \overline{e}_d(I)$ such that

$$\ell(R/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I).$$

These integers $\overline{e}_i(I)$ are called the normal Hilbert coefficients of I.

It is well-known that $\overline{e}_0(I)$ is the Hilbert-Samuel multiplicity of I, which is always a positive integer. The first coefficient $\overline{e}_1(I)$ is sometimes called the normal Chern coefficient of I. It was proved by Goto-Hong-Mandal [5] that when $\widehat{R}$ is unmixed, $\overline{e}_1(I) \geq 0$ for all $\mathfrak{m}$-primary ideals $I \subseteq R$ (which answers a question posed by Vasconcelos [15]). If $(R, \mathfrak{m})$ is a Noetherian local ring such that $\widehat{R}$ is reduced and (S_2), then $\overline{e}_1(Q) = 0$ for some parameter ideal $Q \subseteq R$ if and only if R is regular and $\nu(\mathfrak{m}/Q) \leq 1$, see [9].

In [4], it was shown that when R is reduced ring of characteristic $p > 0$, for $n \gg 0$, the function $\ell(R/(I^{n+1})^*)$ also agrees with a polynomial of degree d, and one can define the tight Hilbert coefficients $e_0^*(I), \ldots, e_d^*(I)$ in a similar way (see Section 2 for more details). It is easy to see that $\overline{e}_1(I) \geq e_1^*(I)$. Recently, Ma-Quy proved that $e_1^*(Q) \geq 0$ for any parameter ideal $Q \subseteq R$ under mild assumptions, see [9, Theorem 1.2]. The following conjecture was asked by Huneke in [2, 9].

Conjecture 1.1. *If R is excellent and has characteristic $p > 0$ (such that $\widehat{R}$ is reduced and (S_2)), and $e_1^*(Q) = 0$ for some (and hence for all) parameter ideal $Q \subseteq R$, then R is F-rational.*

This conjecture is true for Cohen-Macaulay local rings in [4, Theorem 4.4], and tight Buchsbaum local rings in [2, Corollary 4.6]. We will prove this conjecture for Buchsbaum local rings in Theorem 3.4. We also prove that the condition (S_2) of the ring cannot be omitted in Proposition 3.7.

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2. Preliminaries

2.1. **Hilbert coefficients.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then for all $n \gg 0$ we have

$$\ell(R/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I),$$

where $e_0(I), \cdots, e_d(I)$ are all integers, and are called the Hilbert coefficients of I.

Now we suppose $\widehat{R}$ is reduced. This condition guarantees that $R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \cdots$ is module-finite over $R[It]$. As a consequence of this fact, one can show that for all $n \gg 0$, $\ell(R/\overline{I^{n+1}})$ agrees with a polynomial in n and one can write

$$\ell(R/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I),$$

where the integers $\overline{e}_0(Q), \cdots, \overline{e}_d(Q)$ are called the normal Hilbert coefficients. It is well-known that $e_0(I) = \overline{e}_0(I)$ agrees with the Hilbert-Samuel multiplicity $e(I, R)$ of I.

Now in characteristic $p > 0$, the tight closure of an ideal $I \subseteq R$, introduced by Hochster-Huneke, is defined as follows:

$$I^* := \{x \in R \mid \text{there exists } c \in R \setminus \cup_{\mathfrak{p} \in \min(R)} \mathfrak{p} \text{ such that } cx^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0\}.$$

A local ring $(R, \mathfrak{m})$ of prime characteristic $p > 0$ is called F-rational if every ideal generated by a system of parameters is tightly closed, i.e. $Q^* = Q$ for every parameter ideal Q. In general, tight closure is always contained in the integral closure, i.e. $I^* \subseteq \bar{I}$. We know that $R \oplus I^*t \oplus (I^2)^*t^2 \oplus \cdots$ is an $R[It]$-algebra that is also module-finite over $R[It]$: the fact that it is an $R[It]$-algebra follows from the fact that $(I^a)^*(I^b)^* \subseteq (I^{a+b})^*$ for all a, b, and that it is module-finite over $R[It]$, it is an $R[It]$-subalgebra of $R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \cdots$, and the latter is module-finite over $R[It]$ (note that $R[It]$ is Noetherian). Based on the discussion above, one can show that for all $n \gg 0$, $\ell(R/(I^{n+1})^*)$ also agrees with a polynomial in n, and we write

$$\ell(R/(I^{n+1})^*) = e_0^*(I)\binom{n+d}{d} - e_1^*(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d^*(I),$$

for all $n \gg 0$ (see [7] for more general results). We call the integers $e_0^*(I), \ldots, e_d^*(I)$ the tight Hilbert coefficients with respect to I. It is easy to see that $e_0^*(I) = e(I, R)$ is still the Hilbert-Samuel multiplicity of I, and that we always have $\overline{e}_1(I) \geq e_1^*(I) \geq e_1(I)$ by comparing the coefficients of n^{d-1} and note that $I^n \subseteq (I^n)^* \subseteq \overline{I^n}$ for all n. The main result of this paper is inspired by the following results.

Theorem 2.1 ([15, 11, 3])**.** *Let Q be a parameter ideal of R. Then $e_1(Q) \leq 0$. Moreover, R is Cohen-Macaulay if and only if $e_1(Q) = 0$ for some Q provided R is unmixed.*

Theorem 2.2 ([5, 9])**.** *Let $(R, \mathfrak{m})$ be a local ring such that $\widehat{R}$ is reduced and equidimensional. Then for all parameter ideals Q we have $\overline{e}_1(Q) \geq 0$. Moreover, if $\overline{e}_1(Q) = 0$ for some parameter ideal Q, then R is regular and $\nu(\mathfrak{m}/Q) \leq 1$ provided $\widehat{R}$ is reduced and (S_2).*

Theorem 2.3 ([9, 4])**.** *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p such that $\widehat{R}$ is reduced and equidimensional. Then for all parameter ideals Q we have $e_1^*(Q) \geq 0$. Moreover, if $e_1^*(Q) = 0$ for some parameter ideal Q, then R is F-rational provided R is Cohen-Macaulay.*

2.2. **Buchsbaum and tight Buchsbaum rings.**

We recall definition of Buchsbaum rings.

Definition 2.4. A local ring $(R, \mathfrak{m})$ is called *Buchsbaum* if $\ell(R/Q) - e_0(Q)$ is independent of the choice of parameter ideal Q.

Remark 2.5. Let $(R, \mathfrak{m})$ be a Buchsbaum local ring of dimension d and Q any parameter ideal. We have

$$\ell(R/Q) - e_0(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(R)).$$

Hilbert function of parameter ideals in Buchsbaum rings can be understood in terms of local cohomology, see [13, Corollary 3.2], [14, Corollary 4.2].

Theorem 2.6. *Let R be Buchsbaum and Q any parameter ideal of R. Then for $n \geq 0$,*

$$\ell(R/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{n+d-i}{d-i}, \text{ where}$$

$$e_i(Q) = (-1)^i \sum_{j=0}^{d-i} \binom{d-i-1}{j-1} \ell(H^j_{\mathfrak{m}}(R)) \text{ for all } i = 1, 2, \dots, d.$$

Recently, Ma and the third author in [8] considered the analogy of Buchsbaum rings in characteristic p as follows.

Definition 2.7. An unmixed local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is called *tight Buchsbaum* if $e_0(Q) - \ell(R/Q^*)$ is independent of the choice of parameter ideal Q.

The following result is proved in [8, Theorem 4.3] (see also [12] for partial result).

Remark 2.8. Let $(R, \mathfrak{m})$ be a tight Buchsbaum local ring of dimension d and Q any parameter ideal. We have

$$\ell(R/Q) - \ell(R/Q^*) = \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)}),$$

and

$$e_0(Q) - \ell(R/Q^*) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)}),$$

where $0^*_{H^d_{\mathfrak{m}}(R)}$ denotes the tight closure of the zero submodule of $H^d_{\mathfrak{m}}(R)$.

The tight Hilbert function of tight Buchsbaum ring is studied by Dubey, Quy and Verma in [2].

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a tight Buchsbaum local ring of dimension d, and Q a parameter ideal. Then for all $n \geq 0$*

$$\ell(R/(Q^{n+1})^*) = \sum_{i=0}^{d}(-1)^i e_i^*(Q)\binom{n+d-i}{d-i}, \text{ where}$$

(1) $e_1^*(Q) = e_0(Q) - \ell(R/Q^*) + e_1(Q)$ *and* $e_j^*(Q) = e_j(Q) + e_{j-1}(Q)$ *for all* $2 \leq j \leq d$,
(2) $e_1^*(Q) = \sum_{i=2}^{d-1}\binom{d-2}{i-2}\ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)})$,
(3) $e_i^*(Q) = (-1)^{i-1}\left[\sum_{j=0}^{d-i}\binom{d-i-1}{j-2}\ell(H^j_{\mathfrak{m}}(R)) + \ell(H^{d-i+1}_{\mathfrak{m}}(R))\right]$ *for* $i = 2, \ldots, d$.

Every tight Buchsbaum local ring is Buchsbaum. Thus the aim of this paper is to prove a generalization of the following result.

Corollary 2.10. *Let $(R, \mathfrak{m})$ be a tight Buchsbaum local ring that satisfies (S_2) condition. Then R is F-rational if and only if for some (and hence for any) parameter ideal Q have $e_1^*(Q) = 0$*

2.3. **Limit closure of a parameter ideal.** For the limit closure of parameter ideals we refer to [1, 10].

Definition 2.11. Let $Q = (x_1, x_2, \ldots, x_d)$ be a parameter ideal in a Noetherian local ring $(R, \mathfrak{m})$. The *limit closure* of Q is defined as

$$Q^{\lim_R} = \bigcup_{n \geq 0}(x_1^{n+1}, \ldots, x_d^{n+1}) :_R (x_1 \cdots x_d)^n.$$

We will write $Q^{\lim}$ if R is clear from the context.

The limit closure $Q^{\lim}$ is independent of the choice of $x_1, x_2, \ldots, x_d$ since $Q^{\lim}/Q$ is the kernel of the natural map $R/Q \to H^d_{\mathfrak{m}}(R)$.

Remark 2.12. (1) ([8, Lemma 3.3]) If R is Buchsbaum, then for every parameter ideal Q we have

$$\ell(Q^{\lim}/Q) = \sum_{i=0}^{d-1}\binom{d}{i}\ell(H^i_{\mathfrak{m}}(R)).$$

(2) Let $(R, \mathfrak{m})$ be an equidimensional local ring of characteristic p. We have $Q^{\lim} \subseteq Q^*$.

3. The main results

The following lemma follows from [6, Theorem 8.20] or [2, Lemma 3.1] that plays an important role for main result.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and Q a parameter ideal. Then Q^n/Q^nQ^* is a free R/Q^*-module of rank $\binom{n+d-1}{d-1}$.*

Lemma 3.2. *Let $(R,\mathfrak{m})$ be an equidimensional excellent local ring of prime characteristic p and Q a parameter ideal. Then*

$$e_1^*(Q) \geq e_0(Q) - \ell(R/Q^*) + e_1(Q).$$

Proof. For all large enough n we have:

$$\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+d}{d} - e_1^*(Q)\binom{n+d-1}{d-1} + \text{ lower degree terms }.$$

Since $Q^nQ^* \subseteq (Q^{n+1})^*$ and by Lemma 3.1 so

$$\begin{aligned}\ell(R/(Q^{n+1})^*) &\leq \ell(R/Q^nQ^*) = \ell(R/Q^n) + \ell(Q^n/Q^nQ^*)\\ &\leq \ell(R/Q^n) + \ell(R/Q^*)\binom{n+d-1}{d-1}.\end{aligned}$$

Moreover

$$\begin{aligned}&\ell(R/Q^n)\\ &= e_0(Q)\binom{n+d-1}{d} - e_1(Q)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(Q)\\ &= e_0(Q)\left(\binom{n+d}{d} - \binom{n+d-1}{d-1}\right) - e_1(Q)\left(\binom{n+d-1}{d-1} - \binom{n+d-2}{d-2}\right) + \cdots + (-1)^d e_d(Q)\\ &= e_0(Q)\binom{n+d}{d} - (e_0(Q)+e_1(Q))\binom{n+d-1}{d-1} + \text{ lower degree terms }.\end{aligned}$$

Hence

$$\ell(R/(Q^{n+1})^*) \leq e_0(Q)\binom{n+d}{d} - (e_0(Q) - \ell(R/Q^*) + e_1(Q))\binom{n+d-1}{d-1} + \text{ lower degree terms }.$$

Therefore $e_1^*(Q) \geq e_0(Q) - \ell(R/Q^*) + e_1(Q)$. □

Lemma 3.3. *Let $(R,\mathfrak{m})$ be a Buchsbaum local ring that satisfies (S_2) condition and Q a parameter ideal. Then*

$$e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) \geq 0.$$

Moreover, if the equality occurs for some Q then R is Cohen-Macaulay.

Proof. Since R is Buchsbaum, by Theorem 2.6 for all parameter ideal Q we have

$$e_1(Q) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}\ell(H^i_{\mathfrak{m}}(R)).$$

Moreover,

$$e_0(Q) - \ell(R/Q^{\lim}) = \sum_{i=1}^{d-1}\binom{d-1}{i-1}\ell(H^i_{\mathfrak{m}}(R)).$$

Therefore

$$e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = \sum_{i=1}^{d-1}\binom{d-2}{i-2}\ell(H^i_{\mathfrak{m}}(R)) \geq 0$$

for all Q. The equality occurs iff $H^i_{\mathfrak{m}}(R) = 0$ for all $i = 2, \ldots, d-1$. Moreover, the if ring is (S_2) then $H^0_{\mathfrak{m}}(R) = H^1_{\mathfrak{m}}(R) = 0$. Hence, R is Cohen-Macaulay. □

The main result of this paper is as follows.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be an excellent reduced Buchsbaum local ring of prime characteristic p that satisfies (S_2) condition and Q a parameter ideal. Then R is F-rational iff $e_1^*(Q) = 0$.*

Proof. We need only to prove that if $e_1^*(Q) = 0$ then R is F-rational. By Lemma 3.2,

$$e_1^*(Q) \geq e_0(Q) - \ell(R/Q^*) + e_1(Q).$$

Since $Q^{\lim} \subseteq Q^*$,

$$e_0(Q) - \ell(R/Q^*) + e_1(Q) \geq e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) \geq 0.$$

Thus

$$e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = 0.$$

By Lemma 3.3, R is Cohen-Macaulay. By Theorem 2.3, see also [7, Corollary 4.9], we have R is F-rational. □

The main result of this paper claims that the two below questions have affirmative answers for Buchsbaum rings.

Question 3.5. *Let $(R, \mathfrak{m})$ be an excellent reduced and equidimensional local ring of prime characteristic p that satisfies the (S_2) condition. Is it true that for any parameter ideal Q we have $e_0(Q) - \ell(R/Q^*) + e_1(Q) \geq 0$. Moreover, R is F-rational iff for some (and hence for all) parameter ideal Q we have $e_0(Q) - \ell(R/Q^*) + e_1(Q) = 0$.*

Question 3.6. *Let $(R, \mathfrak{m})$ be a local ring that satisfying the (S_2) condition. Is it true that $e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) \geq 0$ for any parameter ideal Q. Moreover, R is Cohen-Macaulay iff for some (and hence for all) parameter ideal Q we have $e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = 0$.*

We next show that the (S_2) condition can not be removed in Theorem 3.4. First, we recall some facts about S_2-ification of the ring by [6]. Suppose R is an unmixed local ring. We shall say that a ring S is an S_2-ification of R if it lies between R and its total quotient ring, is module-finite over R, is (S_2) as an R-module, and has the property that for every element $s \in S \setminus R$, the ideal $D(s)$, defined as $\{r \in R \mid rs \in R\}$, has height at least two. If $(R, \mathfrak{m})$ is an unmixed ring then R has S_2-ification.

Proposition 3.7. *Let $(R, \mathfrak{m})$ be an excellent, unmixed local ring of dimension 2, $Q = (x, y)$ a parameter ideal.*

(1) Then

$$e_0(Q) - \ell(R/Q^{\lim}) = \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{(x, y)H^1_{\mathfrak{m}}(R)}\right).$$

(2) If x is a superficial element then we have

$$e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{(x,y)H^1_{\mathfrak{m}}(R)}\right) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{(x)H^1_{\mathfrak{m}}(R)}\right).$$

Therefore $e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) \leq 0$ for any parameter ideal Q. Moreover, if $QH^1_{\mathfrak{m}}(R) = 0$ (in this case Q is standard), then $e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = 0$.

Proof. (1) The first assertion is due to [1, Theorem 1.4], however in order to be more convenient, we show again the proof of the assertion. Let S is S_2-ification of R. Set $T = S/R$, then $\dim(S) = \dim(R) = 2$. Since S is (S_2) and $\dim(T) \leq 0$ so S is Cohen-Macaulay and T is $\mathfrak{m}$-torsion. The short exact sequence

$$0 \to R \to S \to T \to 0$$

induces the following exact sequence

$$H^0_{\mathfrak{m}}(S) \to H^0_{\mathfrak{m}}(T) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(S).$$

Hence $T = H^1_{\mathfrak{m}}(R)$. The short exact sequence

$$0 \to R \to S \to T \to 0$$

also leads to the following short exact sequence

$$0 \to R/(QS \cap R) \to S/QS \to T/QT \to 0.$$

By [1, Theorem 6.2], $Q^{\lim} = Q^{\lim_S} \cap R = QS \cap R$ since S is Cohen-Macaulay. Putting all together we have

$$\begin{aligned}\ell(R/Q^{\lim}) &= \ell(S/QS) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{QH^1_{\mathfrak{m}}(R)}\right)\\ &= e_0(Q,S) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{QH^1_{\mathfrak{m}}(R)}\right)\\ &= e_0(Q,R) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{QH^1_{\mathfrak{m}}(R)}\right).\end{aligned}$$

(2) Set $R_1 = R/(x)$, $\overline{R}_1 = R_1/H^0_{\mathfrak{m}}(R_1)$ then $\dim(R_1) = \dim(\overline{R}_1) = 1$, and $\overline{R}_1$ is Cohen-Macaulay. Consider the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(R_1) \to R_1 \to \overline{R}_1 \to 0.$$

Since $H^0_{\mathfrak{m}}(R_1)$ is $\mathfrak{m}$-torsion then for large n

$$y^n R_1 \cap H^0_{\mathfrak{m}}(R_1) = y^n(H^0_{\mathfrak{m}}(R_1) : y^n) = y^n H^0_{\mathfrak{m}}(R_1) = 0.$$

So

$$0 \to H^0_{\mathfrak{m}}(R_1) \to R_1/y^n R_1 \to \overline{R}_1/y^n\overline{R}_1 \to 0.$$

for $n \gg 0$. We have

$$\ell(R_1/y^n R_1) - \ell(\overline{R}_1/y^n\overline{R}_1) = \ell\left(H^0_{\mathfrak{m}}(R_1)\right).$$

For large n,

$$e_0(QR_1, R_1)\binom{n}{1} - e_1(QR_1) - e_0(Q\overline{R}_1)\binom{n}{1} = \ell(H^0_{\mathfrak{m}}(R_1)).$$

In other words, $e_1(QR_1, R_1) = -\ell(H^0_{\mathfrak{m}}(R_1))$. Because of superficial property of x, $e_1(QR_1, R_1) = e_1(Q, R)$ so $e_1(Q, R) = -\ell(H^0_{\mathfrak{m}}(R_1))$. The short exact sequence

$$0 \to R \xrightarrow{.x} R \to R_1 \to 0,$$

induces the following

$$0 \to H^0_{\mathfrak{m}}(R_1) \to H^1_{\mathfrak{m}}(R) \xrightarrow{.x} H^1_{\mathfrak{m}}(R) \to \frac{H^1_{\mathfrak{m}}(R)}{xH^1_{\mathfrak{m}}(R)} \to 0.$$

Thus

$$\ell\left(H^0_{\mathfrak{m}}(R_1)\right) = \ell\left((0 : x)_{H^1_{\mathfrak{m}}(R)}\right) = \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{xH^1_{\mathfrak{m}}(R)}\right).$$

Therefore, $e_1(Q, R) = -\ell\left(\frac{H^1_{\mathfrak{m}}(R)}{xH^1_{\mathfrak{m}}(R)}\right)$. Hence

$$e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{(x, y)H^1_{\mathfrak{m}}(R)}\right) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{(x)H^1_{\mathfrak{m}}(R)}\right).$$

The last assertion is clear. The proof is complete. □

Remark 3.8. If $(R, \mathfrak{m})$ is a Buchsbaum unmixed local ring of dimension two, then $e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = 0$ for any parameter ideal Q.

We provide some concrete examples.

Example 3.9. Let $S = k[[X, Y, Z, T]]$ be the formal power series of four variables. Let $I = (X, Y)^2$, $J = (Z, T)^2$ and $R = S/I \cap J$. Let x, y, z, w be the image of X, Y, Z, W in R. Notice that S/I and S/J are Cohen-Macaulay of dimension $\dim(S/I) = \dim(S/J) = 2$. Since R is unmixed of dimension $\dim(R) = 2$, so R is generalized Cohen-Macaulay. Let $\mathfrak{m} = (x, y, z, t)R$ be the maximal ideal of R. The short exact sequence

$$0 \to S/(I \cap J) \to S/I \oplus S/J \to S/(I + J) \to 0$$

induces the following exact sequence

$$H^0_{\mathfrak{m}}(S/I) \oplus H^0_{\mathfrak{m}}(S/J) \to H^0_{\mathfrak{m}}(S/(I + J)) \to H^1_{\mathfrak{m}}(S/I \cap J) \to H^1_{\mathfrak{m}}(S/I) \oplus H^1_{\mathfrak{m}}(S/J).$$

Since S/I and S/J are Cohen-Macaulay of dimension 2 we have

$$H^1_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(S/(I + J)) \cong S/(I + J) = k[[X, Y, Z, T]]/((X, Y)^2 + (Z, T)^2).$$

Hence $\mathfrak{m}H^1_{\mathfrak{m}}(R) \neq 0$, the ring is not Buchsbaum. It is not difficult to check that $Q = (x+z, y+t)R$ is a parameter ideal of R. Let a be a superficial element of R with respect to Q. Notice that $\ell(S/(I + J + (a))) \geq 4$. Therefore, by Proposition 3.7 we have

$$e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) = \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{QH^1_{\mathfrak{m}}(R)}\right) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{aH^1_{\mathfrak{m}}(R)}\right) \leq 3 - 4 < 0.$$

Hence, the (S_2) condition is important for Lemma 3.3 and Theorem 3.4.

Example 3.10. Let $R = \mathbb{F}_p[[X^5, X^4Y, XY^4, Y^5]]$ with the maximal ideal $\mathfrak{m} = (X^5, X^4Y, XY^4, Y^5)$ whose S_2-ification is $S = \mathbb{F}_p[[X^5, X^4Y, X^3Y^2, X^2Y^3, XY^4, Y^5]]$, $\dim(R) = \dim(S) = 2$. It should be note that S is a Veronese subring of the local regular ring $\mathbb{F}_p[[X,Y]]$, S is F-rational so that $H^0_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(S) = 0$ and $0^*_{H^2_{\mathfrak{m}}(S)} = 0$. Set $N = S/R$ then

$$N \cong \mathbb{F}_pX^2Y^3 \oplus \mathbb{F}_pX^3Y^2 \oplus \mathbb{F}_pX^3Y^7 \oplus \mathbb{F}_pX^7Y^3.$$

The short exact sequence $0 \to R \to S \to N \to 0$ induces the following exact sequence

$$0 \to H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(S) \to H^0_{\mathfrak{m}}(N) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(S) \to$$

$$\to H^1_{\mathfrak{m}}(N) \to H^2_{\mathfrak{m}}(R) \to H^2_{\mathfrak{m}}(S) \to H^2_{\mathfrak{m}}(N).$$

Since N has finite length so that it is $\mathfrak{m}$-tosion and S is F-rational then

$$H^0_{\mathfrak{m}}(R) = 0, \quad H^1_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(N) \cong N, \quad H^2_{\mathfrak{m}}(R) \cong H^2_{\mathfrak{m}}(S).$$

Moreover, $\mathfrak{m}H^1_{\mathfrak{m}}(R) \neq 0$ since $X^5.X^2Y^3 = X^7Y^3 \neq 0$ in $H^1_{\mathfrak{m}}(R)$. Hence R is not Buchsbaum but it is generalized Cohen-Macaulay. We have $f = aX^5 + bY^5$ with $a, b \in \mathbb{F}_p$ and $a, b \neq 0$ is a superficial element of R and $Q = (X^5, Y^5) = (f, Y^5)$ is a parameter ideal of R. It is obvious that

$$(f)H^1_{\mathfrak{m}}(R) \cong \mathbb{F}_pX^7Y^3 \oplus \mathbb{F}_pX^3Y^7 \cong QH^1_{\mathfrak{m}}(R).$$

Since $Q^*/Q^{\lim} \subseteq 0^*_{H^2_{\mathfrak{m}}(R)} \cong 0^*_{H^2_{\mathfrak{m}}(S)} \cong 0$ and by Proposition 3.7,

$$\begin{aligned} e_0(Q) - \ell(R/Q^*) + e_1(Q) &= e_0(Q) - \ell(R/Q^{\lim}) + e_1(Q) \\ &= \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{QH^1_{\mathfrak{m}}(R)}\right) - \ell\left(\frac{H^1_{\mathfrak{m}}(R)}{(f)H^1_{\mathfrak{m}}(R)}\right) \\ &= 2 - 2 = 0. \end{aligned}$$

Moreover S is a module finite over R, we have $(Q^n)^* = (Q^nS)^* \cap R$. On the other hand, S is F-rational (in fact, F-regular), we have $(Q^n)^* = Q^nS \cap R$ for all n. Thus we get the following short exact sequence

$$0 \to R/(Q^{n+1})^* \to S/Q^{n+1}S \to N \to 0$$

for all $n \gg 0$. Hence $\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+2}{2} - \ell(N) = e_0(Q)\binom{n+2}{2} - 4$. Hence $e_1^*(Q) = 0$ but R is not F-rational.

Remark 3.11. The main result of this paper should be true for mixed characteristic when we replace $e_1^*(Q)$ by $e_1^B(Q)$ for any big Cohen-Macaulay ring B of R. We will consider this topic in the future.

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