LIPSCHITZ CONTINUITY OF LIPSCHITZ-KILLING CURVATURE DENSITIES AT INFINITY

SI TIEP DINH AND NHAN NGUYEN

Abstract. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^2 definable map in an o-minimal structure. We prove that the Lipschitz-Killing curvature density at infinity $\Lambda_k^{\lim}(f^{-1}(t), \infty)$ of the fibers is locally Lipschitz outside the set of asymptotic critical values of f for $k \geq 1$. For $k = 0$, it is locally Lipschitz outside the set of generalized critical values of f. This reinforces the recent result of Dutertre and Grandjean, where only continuity was achieved.

1. Introduction

Consider a C^2 definable map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ in a given o-minimal structure with $m < n$. Our primary focus lies in understanding the behavior of the fibers of f at infinity, which can be captured by the concept of asymptotic critical values. Recall that the *set of asymptotic critical values* of f is given by

$$K_\infty(f) = \left\{ \begin{array}{lcl} t \in \mathbb{R}^m & : & \text{there exists a sequence } x_l \to \infty \text{ such that} \\ & & f(x_l) \to y \;\text{ and }\; \|x_l\|\nu(d_x f(x_l)) \to 0 \end{array} \right\}$$

where ν is the Rabier function [27] defined by

$$\nu : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}, \quad \nu(A) = \inf_{\|\varphi\|=1} \|A^*(\varphi)\|,$$

with $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ being the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $A^* \in \mathcal{L}\left((\mathbb{R}^m)^*, (\mathbb{R}^n)^*\right)$ being the adjoint operator of A and $\varphi \in (\mathbb{R}^m)^*$. Remark that $\nu(A) = \operatorname{dist}(A, \Sigma)$ where Σ is the set of singular maps (maps with non-maximal rank) of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. In particular, when $m = 1$, $\nu(A) = \|A\|$ (see [20]).

It is worth noting that $K_\infty(f)$ is a closed definable set of dimension smaller than k containing $B_\infty(f)$, the set of bifurcation values at infinity of f (see [6],[19]) which is the set of all values t at which f is not topologically trivial at infinity. In particular, computing $K_\infty(f)$ is significantly easier than computing $B_\infty(f)$.

For the case of functions (i.e., $k = 1$), Grandjean [17, 18] explored the continuity of functions $t \mapsto K(t)$ and $t \mapsto |K|(t)$, which respectively denote the total curvature and the total absolute curvature of the fiber $f^{-1}(t)$. He established that these functions admit only a finite number of discontinuities. However, the precise nature of these discontinuities had remained uncharacterized. In a later work, Dutertre and Grandjean [14] pointed out that the discontinuities only appear on the *set of generalized critical values* of f defined as

$$K(f) = K_\infty(f) \cup K_0(f)$$

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where

$$K_0(f) = \{t \in \mathbb{R}^m : \text{ there exists } x \in f^{-1}(t) \text{ such that } \operatorname{rank}(d_x f) < m\}$$

is the *set of critical values* of f. Note that f is globally trivial on $\mathbb{R}^m \setminus K(f)$. Recently, in [8], Pham and the first author investigated the behavior of the tangent cone at infinity of the fibers of a polynomial f outside the set $K_\infty(f)$ and proved that the density of these tangent cones is locally Lipschitz.

In a different context, Dutertre and Grandjean [15] studied the continuity of Lipschitz-Killing curvature densities $\Lambda_k^{\lim}(\cdot,\infty)$, a generalized notion of density, at infinity for definable maps. They proved that $\Lambda_k^{\lim}(f^{-1}(t),\infty)$ is continuous on $\mathbb{R}^m \setminus K(f)$.

The study of Lipschitz-Killing measures on subanalytic sets was initiated by Fu [16] through the application of geometric measure theory, and later by Bröker and Kuppe [2] in the broader context of definable sets using stratified Morse theory. The localization of these concepts was independently introduced by Bernig–Bröcker [1] and Comte–Merle [4]. Although the notions introduced in these two papers are not identical, one can be expressed as a linear combination of the other. Comte and Merle [4] proved that local Lipschitz-Killing curvatures are continuous along the strata of a Verdier stratification. This result was later extended to Whitney stratifications by Valette and the second author [26], who further proved that if the stratification satisfies Verdier's condition, then these local curvatures are indeed locally Lipschitz. Additional notable contributions on this subject can be found in the works of Durtertre, see for example [9, 10, 11, 12, 13].

In this paper, we present a simple proof showing that $\Lambda_k^{\lim}(f^{-1}(t),\infty)$ is locally Lipschitz on $\mathbb{R}^m \setminus K_\infty(f)$ for $k = 1,\dots,n$. In addition, $\Lambda_0^{\lim}(f^{-1}(t),\infty)$ is locally Lipschitz on $\mathbb{R}^m \setminus K(f)$ (see Theorem 2.11). This generalizes the result of Dutertre and Grandjean in [15].

The idea of our proof is as follows: Let $c \in \mathbb{R}^m \setminus K_\infty(f)$. If $f^{-1}(c)$ is bounded, the result follows directly from definition (see Remark 2.4). We then may assume that $f^{-1}(c)$ is unbounded. It follows from Dutertre's works that, for a closed unbounded definable set $X \subset \mathbb{R}^n$,

$$\Lambda_k^{\lim}(X,\infty) = \Lambda_k^{\lim}(\overline{\varphi(X)},0) \text{ for } k = 1,\dots,n \tag{1.1}$$

and

$$\Lambda_0^{\lim}(X,\infty) = \chi(X) - 1 + \Lambda_0^{\lim}(\overline{\varphi(X)},0). \tag{1.2}$$

Moreover, there is a linear kinematic formula relating $\Lambda_k^{\lim}(\overline{\varphi(X)},0)$ and the local Lipschitz-Killing curvatures $\Lambda_k^{\mathrm{loc}}(\overline{\varphi(X)},0)$ defined by Comte and Merle [4]. Here, $\varphi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ is given by $\varphi(x) = \frac{x}{\|x\|^2}$. This allows us to shift the problem to the study of the continuity of $\Lambda_k^{\mathrm{loc}}(.,0)$.

Consider the map $\Phi = (f(x),\varphi(x))$. It is straightforward that the germs at 0 of $\overline{\varphi(f^{-1}(t))}$ and $\pi^{-1}(t) \cap \overline{\operatorname{Im}(\Phi)}$ coincide where $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the orthogonal projection. We show that $(\operatorname{Im}(\Phi), \mathbb{R}^m \times \{0\}^n)$ satisfies the (w)-regularity condition on $\mathbb{R}^m \setminus K_\infty(f)$. This is a surprising fact. Using [26, Proposition 4.4], we obtain that $\Lambda_k^{\mathrm{loc}}(\overline{\varphi(f^{-1}(t))},0)$ is locally Lipschitz on $\mathbb{R}^m \setminus K_\infty(f)$ for $k = 0,\dots,n$. As $k \geq 1$, it is obvious from Equality (1.1) that $\Lambda_k^{\lim}(X,\infty)$ is locally Lipschitz. When $k = 0$, since $\chi(f^{-1}(t))$ appears in Formula (1.2), restricting to $\mathbb{R}^m \setminus K_\infty(f)$ would not be enough to ensure the continuity of $\chi(f^{-1}(t))$. However, on $\mathbb{R}^m \setminus K(f)$, all fibers are of the same topological type, so $\chi(f^{-1}(t))$ is constant. The result finally follows.

Throughout the paper, we assume the reader's familiarity with the notion of o-minimal structures on $\mathbb{R}$. For more comprehensive details, we refer the reader to [5], [28], [24].

In the paper, we use the following notations:

- $\mathbb{B}^n_r$, $\overline{\mathbb{B}}^n_r$ and $\mathbb{S}^{n-1}_r$ respectively denote the n-dimensional open ball, the n-dimensional closed ball and the $(n-1)$-dimensional sphere in $\mathbb{R}^n$ with radius r centered at the origin. When $r = 1$, we use the notations $\mathbb{B}^n$, $\overline{\mathbb{B}}^n$ and $\mathbb{S}^{n-1}$, respectively.
- b_k and s_k denote the volumes of the unit k-dimensional ball and the unit k-dimensional sphere, respectively.
- $\mathbb{G}^k_n$ is the set of k-dimensional vector subspaces of $\mathbb{R}^n$ equipped with the $O(n)$-invariant density and g^k_n is its volume; $\mathbb{A}^k_n$ is the set of k-dimensional affine plane in $\mathbb{R}^n$.
- Let $X \subset \mathbb{R}^n$ be a definable set. We denote by $\overline{X}$ the closure of X in $\mathbb{R}^n$ and by $\chi(X)$ the Euler characteristic of X. The *link at infinity* of X, denoted by $\mathrm{Lk}^\infty(X)$, is the set $\mathrm{Lk}^\infty(X) = X \cap \mathbb{S}^{n-1}_R$ for R sufficiently large. If $0 \in \overline{X}$, then the *link at* 0 of X, denoted by $\mathrm{Lk}(X)$, is given by $\mathrm{Lk}(X) = X \cap \mathbb{S}^{n-1}_r$ for r sufficiently small.
- Given non-negative functions $f, g : X \to \mathbb{R}$, we write $f \lesssim g$ if there exists a positive constant C such that $f(x) \le Cg(x)$ for all $x \in X$. This constant C is referred to as a constant for the relation $\lesssim$.

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2. Lipschitz-Killing curvatures at infinity

2.1. **Lipschitz-Killing curvature densities.** Let us recall the definition of Lipschitz-Killing curvature densities based on the approach by Bröcker and Kuppe [2]. Additionally, we will review some related results from Dutertre [9, 10, 11, 12, 13].

Let X be a compact definable subset of $\mathbb{R}^n$ and let $\mathcal{S} = \{S_i\}_{i \in I}$ be a C^2 Whitney stratification of X. Note that the existence of such a stratification is guaranteed by [22, 25]. Fix a stratum S. For $k \in \{0, \dots, d_S\}$ where $d_S = \dim S$, we define $\lambda^S_k \colon S \to \mathbb{R}$ as

$$\lambda^S_k(x) = \frac{1}{s_{n-k-1}} \int_{\mathbb{S}^{n-1} \cap N_x S} \mathrm{ind}_{\mathrm{nor}}(v^*, X, x)\, \sigma_{d_S - k}(II_{x,v})\, dv,$$

where v^* is the linear form on $\mathbb{R}^n$ defined by $v^*(x) = \langle v, x \rangle$, $N_x S$ is the normal space to S at x in $\mathbb{R}^n$, $II_{x,v}$ is the second fundamental form on S in the direction of v and $\sigma_{d_S-k}(II_{x,v})$ is the $(d_S - k)$-th elementary symmetric function of its eigenvalues. The index $\mathrm{ind}_{\mathrm{nor}}(v^*, X, x)$ is defined as follows:

$$\mathrm{ind}_{\mathrm{nor}}(v^*, X, x) = 1 - \chi\left(X \cap N_x S \cap \mathbb{B}^n_r(x) \cap \{v^* = v^*(x) - \delta\}\right),$$

where $0 < \delta \ll r \ll 1$. For $k = d_S + 1, \dots, n$, set $\lambda^S_k(x) = 0$.

Given a Borel set $U \subset X$ and $k \in \{0, \dots, n\}$, we set

$$\Lambda_k(X, U) = \sum_{i \in I} \int_{S_i \cap U} \lambda^{S_i}_k(x) dx.$$

These measures $\Lambda_k(X, -)$ are called *Lipschitz-Killing measures* of X. Note that

$$\Lambda_{d+1}(X, U) = \cdots = \Lambda_n(X, U) = 0,$$

and $\Lambda_d(X, U) = \mathcal{H}_d(U)$ where $d = \dim X$ and $\mathcal{H}_d$ is the d-dimensional Hausdorff measure in $\mathbb{R}^n$. In particular, when $U = X$,

$$\Lambda_0(X, X) = \chi(X).$$

2.1. **Definition.** Let $(X,0) \subset (\mathbb{R}^n,0)$ be a closed definable germ. For $k=0,\dots,n$, *the k-th Lipschitz-Killing curvature density at* 0 of X is defined by:

$$\Lambda_k^{\lim}(X,0)=\lim_{r\to 0}\frac{\Lambda_k(X,X\cap\overline{\mathbb{B}}_r^n)}{b_k r^k}.$$

2.2. **Remark.** $\Lambda_0^{\lim}(X,0)=1$, $\Lambda_d^{\lim}(X,0)=\theta(X,0)$, $\Lambda_k^{\lim}(X,0)=0$ for all $k>d$ where $d=\dim(X,0)$ and $\theta(X,0)$ is the density at 0 of X.

2.3. **Definition.** Let X be a closed definable subset of $\mathbb{R}^n$. For $k=0,\dots,n$, the *k-th Lipschitz-Killing curvature density at infinity of X* is defined as:

$$\Lambda_k^{\lim}(X,\infty)=\lim_{R\to+\infty}\frac{\Lambda_k(X,X\cap\overline{\mathbb{B}}_R^n)}{b_k R^k}.$$

2.4. **Remark.** (i) $\Lambda_d^{\lim}(X,\infty)=\theta(X,\infty)$ if $d\geq 1$ and $\Lambda_k^{\lim}(X,\infty)=0$ for all $k>d$, where $d=\dim(X,\infty)$ and $\theta(X,\infty)$ is the density at infinity of X.

(ii) If X is bounded then $\Lambda_k^{\lim}(X,\infty)=0$ for all $k\geq 1$ and $\Lambda_0^{\lim}(X,\infty)=\chi(X)$.

The following two theorems are due to Durtertre [9, 10, 11], originally stated for closed semi-algebraic/subanalytic sets but also applicable to closed definable sets.

2.5. **Theorem** ([9, Corollary 5.7] and [10, Corollary 3.14])**.** *Let $X\subset\mathbb{R}^n$ be a closed definable set. We have*

$$\Lambda_0^{\lim}(X,\infty)=\chi(X)-\frac{1}{2}\chi(\mathrm{Lk}^\infty(X))-\frac{1}{2g_n^{n-1}}\int_{\mathbb{G}_n^{n-1}}\chi(\mathrm{Lk}^\infty(X\cap H))dH.$$

Furthermore for $k\in\{1,\dots,n-2\}$, we have

$$\begin{aligned}\Lambda_k^{\lim}(X,\infty)=&-\frac{1}{2g_n^{n-k-1}}\int_{\mathbb{G}_n^{n-k-1}}\chi\left(\mathrm{Lk}^\infty(X\cap H)\right)dH\\&+\frac{1}{2g_n^{n-k+1}}\int_{\mathbb{G}_n^{n-k+1}}\chi\left(\mathrm{Lk}^\infty(X\cap H)\right)dH.\end{aligned}$$

and

$$\begin{aligned}\Lambda_{n-1}^{\lim}(X,\infty)&=\frac{1}{2g_n^2}\int_{\mathbb{G}_n^2}\chi\left(\mathrm{Lk}^\infty(X\cap H)\right)dH,\\ \Lambda_n^{\lim}(X,\infty)&=\frac{1}{2g_n^1}\int_{\mathbb{G}_n^1}\chi\left(\mathrm{Lk}^\infty(X\cap H)\right)dH.\end{aligned}$$

Similarly, we also have the following.

2.6. **Theorem** ([11, Theorem 5.1])**.** *Let $(X,0)\subset(\mathbb{R}^n,0)$ be a closed definable germ. We have*

$$\Lambda_0^{\lim}(X,0)=1-\frac{1}{2}\chi(\mathrm{Lk}(X))-\frac{1}{2g_n^{n-1}}\int_{\mathbb{G}_n^{n-1}}\chi(\mathrm{Lk}(X\cap H))dH.$$

In addition, for $k\in\{1,\dots,n-2\}$,

$$\begin{aligned}\Lambda_k^{\lim}(X,0)=&-\frac{1}{2g_n^{n-k-1}}\int_{\mathbb{G}_n^{n-k-1}}\chi(\mathrm{Lk}(X\cap H))dH\\&+\frac{1}{2g_n^{n-k+1}}\int_{\mathbb{G}_n^{n-k+1}}\chi(\mathrm{Lk}(X\cap H))dH,\end{aligned}$$

and

$$\Lambda^{\lim}_{n-1}(X,0) = \frac{1}{2g_n^2}\int_{\mathbb{G}_n^2}\chi(\mathrm{Lk}(X\cap H))dH,$$
$$\Lambda^{\lim}_{n}(X,0) = \frac{1}{2g_n^1}\int_{\mathbb{G}_n^1}\chi(\mathrm{Lk}(X\cap H))dH.$$

2.2. **Comte-Merle's local Lipschitz-Killing curvatures and kinematic formulas.** Let X be a closed definable germ at 0. Comte and Merle [4] introduced, for each $k = 0,\dots,n$, a value referred to as *the k-th local Lipschitz-Killing curvature*, which is defined as follows:

$$\Lambda_k^{\mathrm{loc}}(X,0) = \lim_{r\to 0}\frac{1}{\beta(n,k)}\int_{P\in\mathbb{G}_n^k}\int_{x\in P}\chi\left(X\cap\pi_P^{-1}(x)\cap\overline{\mathbb{B}}_r^n\right)\,\mathrm{d}\mathcal{H}_k(x)\,dP$$

where $\beta(n,k) = \dfrac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma(\frac{1}{2})\Gamma\left(\frac{n+1}{2}\right)}$.

It follows from definition that $\Lambda_0^{\mathrm{loc}}(X,0) = 1$, $\Lambda_d^{\mathrm{loc}}(X,0) = \theta(X,0)$ and $\Lambda_k^{\mathrm{loc}}(X,0) = 0$ for every $k > d$, where $d = \dim X$.

It has been known from [2, Corollary 8.5] that if $X\subset\mathbb{R}^n$ is a compact definable set and $U\subset X$ is a Borel set, then for $k\in\{0,\dots,n\}$, we have

$$\Lambda_{n-k}(X,U) = \frac{1}{\beta(n,n-k)}\int_{\mathbb{A}_n^k}\Lambda_0(X\cap E, X\cap E\cap U)dE.$$

And this yields that

$$\Lambda_k^{\mathrm{loc}}(X,0) = \lim_{r\to 0}\frac{\Lambda_k\left(X\cap\overline{\mathbb{B}}_r^n, X\cap\overline{\mathbb{B}}_r^n\right)}{b_k r^k}.$$

Note that in general, $\Lambda_k\left(X, X\cap\overline{\mathbb{B}}_r^n\right)$ is different from $\Lambda_k\left(X\cap\overline{\mathbb{B}}_r^n, X\cap\overline{\mathbb{B}}_r^n\right)$.

In [4], the authors introduced the notion of the polar invariant $\sigma_k(X,0)$. To define this, consider $k\in 0,\dots,n$. For $P\in\mathbb{G}_n^k$, let $\pi_P : X\to P$ denote the orthogonal projection onto P. It was shown that for generic P in $\mathbb{G}_n^k$, there exists an open dense definable set $(K^P,0)\subset(P,0)$, along with its decomposition $K^P = \bigcup_{i=1}^{N_P} K_i^P$ such that the function

$$K_i^P\mapsto\chi_i^P = \lim_{r\to 0}\lim_{y\in K_i^P, y\to 0}\chi\left(\pi_P^{-1}(y)\cap X\cap\overline{\mathbb{B}}_r^n\right)$$

is well-defined for every i. The polar invariant $\sigma_k(X,0)$ is then given by

$$\sigma_k(X,0) = \frac{1}{s_k}\int_{\mathbb{G}_n^k}\sum_{i=1}^{N_P}\chi_i^P\cdot\theta\left(K_i^p,0\right)dP.$$

Note that $\sigma_0(X,0) = 1$ by definition. In addition, one has the following linear kinematic formula.

2.7. **Theorem** ([4, Theorem 3.1]). *Let $(X,0)\subset(\mathbb{R}^n,0)$ be a closed definable germ. Then, we have*

$$\begin{pmatrix}\Lambda_1^{loc}(X,0)\\ \vdots\\ \Lambda_n^{loc}(X,0)\end{pmatrix} = \begin{pmatrix}1 & m_1^2 & \dots & m_1^n\\ 0 & 1 & \dots & m_2^n\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{pmatrix}\cdot\begin{pmatrix}\sigma_1(X,0)\\ \vdots\\ \sigma_n(X,0)\end{pmatrix},$$

where $m_i^i = 1,\ m_i^j = \frac{b_j}{b_{j-i}b_i}\begin{pmatrix} j \\ i \end{pmatrix} - \frac{b_{j-1}}{b_{j-1-i}b_i}\begin{pmatrix} j-1 \\ i \end{pmatrix}$, *for* $i+1 \leq j \leq n$.

In another context, Dutertre [11, Theorem 5.6] proved the following.

2.8. **Theorem.** *For* $k \in \{0, \ldots, n-1\}$,

$$\Lambda_k^{\lim}(X,0) = \sigma_k(X,0) - \sigma_{k+1}(X,0).$$

In particular,

$$\Lambda_n^{\lim}(X,0) = \sigma_n(X,0).$$

This leads to the conclusion that the Lipschitz-Killing curvature densities $\Lambda_k^{\lim}(X,0)$ can be expressed as linear combinations of local Lipschitz-Killing curvatures $\Lambda_j^{\mathrm{loc}}(X,0)$, and the corresponding matrix is invertible. Utilizing this alongside [26, Proposition 4.4], we obtain the following.

2.9. **Proposition.** *Let X be a closed definable set in $\mathbb{R}^n$. Consider a C^2-Verdier stratification $\mathcal{S}$ of X. Assume that $S \subset \{0\}^{n-m} \times \mathbb{R}^m$ is a stratum of $\mathcal{S}$ and let π denote the orthogonal projection from $\mathbb{R}^n$ onto $\{0\}^{n-m} \times \mathbb{R}^m$. Then, for $k = 0, \ldots, n$, the function $t \mapsto \Lambda_k^{\lim}(X_t, 0)$ is locally Lipschitz along S where $X_t = \pi^{-1}(t) \cap X$.*

2.10. **Remark.** The existence of Verdier stratifications in o-minimal structures was proven by [23].

2.3. **Lipschitz continuity of Lipschitz-Killing curvature densities at infinity.** The main result of the paper is as follows.

2.11. **Theorem.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ ($n \geq 2$) be a C^2 definable map. Then, the function $t \mapsto \Lambda_k^{\lim}(f^{-1}(t), \infty)$ is locally Lipschitz on $\mathbb{R}^m \setminus K_\infty(f)$ if $k > 0$. And, the function $t \mapsto \Lambda_0^{\lim}(f^{-1}(t), \infty)$ is locally Lipschitz on $\mathbb{R}^m \setminus K(f)$.*

Before proving Theorem 2.11, we need some preparation. Consider the map

$$\varphi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad x \mapsto \varphi(x) = \frac{x}{\|x\|^2}.$$

2.12. **Lemma.** *Let $X \subset \mathbb{R}^n$ be an unbounded closed definable set. Then,*

$$\Lambda_0^{\lim}(X,\infty) = \chi(X) - 1 + \Lambda_0^{\lim}(\overline{\varphi(X)}, 0)$$

and

$$\Lambda_k^{\lim}(X,\infty) = \Lambda_k^{\lim}(\overline{\varphi(X)}, 0)$$

for $k = 1, \ldots, n$.

Proof. Observe that for each element H of $\mathbb{G}_n^k$ and $R > 0$, we have

$$\varphi(X \cap \mathbb{S}_R^{n-1} \cap H) = \varphi(X) \cap \mathbb{S}_{\frac{1}{R}}^{n-1} \cap H.$$

Therefore

$$\varphi(\mathrm{Lk}^\infty(X \cap H)) = \mathrm{Lk}(\varphi(X) \cap H).$$

Consequently,

$$\chi(\mathrm{Lk}^\infty(X \cap H)) = \chi(\mathrm{Lk}(\varphi(X) \cap H)) = \chi(\mathrm{Lk}(\overline{\varphi(X)} \cap H)).$$

By Theorem 2.5 and Theorem 2.6, we derive the desired equalities. □

Now let us recall the notion of (w)-regularity introduced by Verdier [29]. For a pair (Y, Z) of C^2 submanifolds in $\mathbb{R}^n$ satisfying $Y \subset \overline{Z}$, we say that (Y, Z) is (w)*-regular* if for any point $x \in Y$, there exist a neighborhood U_x of x in $\mathbb{R}^n$ and a constant $C > 0$ such that

$$\delta\left(T_y Y, T_z Z\right) \leqslant C\|y - z\| \quad \text{for all } y \in U_x \cap Y \text{ and } z \in U_x \cap Z.$$

Here, for linear subspaces A, B of $\mathbb{R}^n$,

$$\delta(A, B) = \sup_{v \in A, \|v\|=1} \|v - \pi_B(v)\|$$

where π_B is the orthogonal projection from $\mathbb{R}^n$ onto B.

Let $c \in \mathbb{R}^m \setminus K_\infty(f)$. Suppose that $f^{-1}(c)$ is unbounded. Then there is $r > 0$ sufficiently small such that $f^{-1}(t)$ is unbounded for all $t \in \mathbb{B}_r^m(c)$. It is easy to derive from the definition of $K_\infty(f)$ that if r is taken small enough then there are constants $R > 0$ and $\varepsilon > 0$ such that

$$\|x\|\nu(d_x f) > \varepsilon \ \text{ for all } \ x \in f^{-1}(\mathbb{B}_r^m(c)) \setminus \overline{\mathbb{B}}_R^n. \tag{2.1}$$

Consider the map

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \times \mathbb{R}^n, \qquad \Phi(x) = (f(x), \varphi(x)) = \left(f(x), \frac{x}{\|x\|^2}\right).$$

Observe that Φ is a C^2 injection. Additionally, $\operatorname{rank}(d_x\Phi) = n$ for all $x \neq 0$. This implies that $\Phi|_{\mathbb{R}^n \setminus \{0\}}$ is a C^2 embedding. Consequently, if we let

$$X = f^{-1}(\mathbb{B}_r^m(c)) \setminus \overline{\mathbb{B}}_R^n$$

which is an open subset of $\mathbb{R}^n$, then

$$Z = \Phi(X) \tag{2.2}$$

is a C^2 submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Let

$$Y = \mathbb{B}_r^m(c) \times \{0\}. \tag{2.3}$$

It is obvious that $Y \subset \overline{Z}$. The following lemma is the key to prove Theorem 2.11.

2.13. **Lemma.** *The pair (Y, Z) is (w)-regular.*

Proof. To prove the lemma, it suffices to show that the unit vector fields $\frac{\partial}{\partial t_i}$, $i = 1, \dots, k$ defined on Y can be extended to a rugose stratified vector field $v(t, u)$ on $Z \cup Y$ (see [3, Proposition 2]).

Fix i, $1 \leq i \leq k$. For $x \in X$, let $N_x f$ denote the orthogonal complement of $\ker d_x f$ in $T_x X = \mathbb{R}^n$. Since $d_x f : \mathbb{R}^n \to \mathbb{R}^m$ is surjective, the restriction $d_x f|_{N_x f} : N_x f \to \mathbb{R}^m$ is an isomorphism. We define

$$\xi(x) = (d_x f|_{N_x f})^{-1}\left(\frac{\partial}{\partial t_i}\right).$$

Then, ξ is a tangent vector field on X and $d_x f(\xi(x)) = \frac{\partial}{\partial t_i}$ (in other words, ξ is a lift of $\frac{\partial}{\partial t_i}$ by df).

2.14. **Claim.**

$$\|\xi(x)\| \leq \frac{1}{\nu(d_x f)}$$

Proof. It follows from [20, Proposition 2.3] that

$$\nu(d_xf) = \sup\{r \geq 0 : \mathbb{B}^m_r \subset d_xf(\overline{\mathbb{B}}^n)\} = \sup\{r \geq 0 : \mathbb{B}^m_r \subseteq d_xf(\overline{\mathbb{B}}^n \cap N_xf)\}.$$

This implies that

$$\overline{\mathbb{B}}^m_{\nu(d_xf)} \subset d_xf(\overline{\mathbb{B}}^n \cap N_xf),$$

and hence

$$\overline{\mathbb{B}}^m \subset d_xf\left(\overline{\mathbb{B}}^n_{\frac{1}{\nu(d_xf)}} \cap N_xf\right).$$

Therefore

$$(d_xf)^{-1}(\overline{\mathbb{B}}^m) \cap N_xf \subset \overline{\mathbb{B}}^n_{\frac{1}{\nu(d_xf)}} \cap N_xf.$$

Note that $\left\|\frac{\partial}{\partial t_i}\right\| = 1$ and $\xi(x) = (d_xf|_{N_xf})^{-1}(\frac{\partial}{\partial t_i})$. This yields that $\xi(x) \in \overline{\mathbb{B}}^n_{\frac{1}{\nu(d_xf)}} \cap N_xf$, and the claim follows. □

It follows from (2.1) and Claim 2.14 that

$$\|\xi(x)\| \leq \frac{\|x\|}{\varepsilon}. \tag{2.4}$$

Now we define

$$v(t,u) = \begin{cases} \dfrac{\partial}{\partial t_i} & \text{for } (t,u) \in Y \\ d_x\Phi\left(\xi(x)\right) = \dfrac{\partial}{\partial t_i} + d_x\varphi\left(\xi(x)\right) & \text{for } (t,u) \in Z \end{cases}$$

where $x = \Phi^{-1}(t,u) = \frac{u}{\|u\|^2}$ and $\varphi(x) = \frac{x}{\|x\|^2}$.

It is clear that $v(t,u)$ is an extension of $\frac{\partial}{\partial t_i}$. It remains to check the rugosity of v, which means that for each $a = (t,0) \in Y$, there exists a neighborhood U_a of a in $\mathbb{R}^m \times \mathbb{R}^n$ such that

$$\|v(z) - v(y)\| \lesssim \|z - y\|$$

for all $z \in Z \cap U_a$ and $y \in Y \cap U_a$.

Given $a_0 = (t_0, 0) \in Y$, choose a small neighborhood U of a_0 in $\mathbb{R}^{m+n}$. Let $z = (t_z, u_z) \in Z \cap U$ and $y = (t_y, 0) \in Y \cap U$. Setting $x = \Phi^{-1}(z)$, we have

$$\|v(z)-v(y)\| \leq \|v(z)-v(t_z,0)\|+\|v(t_z,0)-v(t_y,0)\| = \|v(z)-v(t_z,0)\| = \|d_x\varphi(\xi(x))\|. \tag{2.5}$$

Note that

$$d_x\varphi = \begin{bmatrix} \dfrac{\|x\|^2 - 2x_1^2}{\|x\|^4} & \dfrac{-2x_1x_2}{\|x\|^4} & \cdots & \dfrac{-2x_1x_n}{\|x\|^4} \\ \vdots & \vdots & & \vdots \\ \dfrac{-2x_nx_1}{\|x\|^4} & \dfrac{-2x_nx_2}{\|x\|^4} & \cdots & \dfrac{\|x\|^2 - 2x_n^2}{\|x\|^4} \end{bmatrix}.$$

Observe that the absolute value of each entry of $d_x\varphi$ is $\lesssim \frac{1}{\|x\|^2}$ so $\|d_x\varphi\| \lesssim \frac{1}{\|x\|^2}$. By (2.4), it yields that

$$\|d_x\varphi\left(\xi(x)\right)\| \lesssim \|d_x\varphi\|\|\xi(x)\| \lesssim \frac{1}{\|x\|}. \tag{2.6}$$

On the other hand,

$$\|z - y\| \geq \operatorname{dist}(z, Y) = \|z - (t_z, 0)\| = \|u_z\| = \|\varphi(x)\| = \frac{1}{\|x\|}.$$

This, together with (2.5) and (2.6), implies that

$$\|v(z) - v(y)\| \lesssim \|z - y\|.$$

Note that the constant for the relation $\lesssim$ depends only on U. Thus, Lemma 2.13 is proved. □

Proof of Theorem 2.11. Let $c \in \mathbb{R}^m \setminus K_\infty(f)$. If $f^{-1}(c)$ is bounded, the result is trivial by Remark 2.4 (ii). Now assume that $f^{-1}(c)$ is unbounded. Let r, ε and R be positive constants such that (2.1) holds. Let Y and Z be given by (2.3) and (2.2). By definition, the germs $(\overline{Z}_t, 0)$ and $(\overline{\varphi(f^{-1}(t)}, 0)$ coincide. Since (Y, Z) is (w)-regular in view of Lemma 2.13, according to Proposition 2.9, for $0 \le k \le n$, $\Lambda_k^{\lim}(\overline{Z}_t, 0)$ is locally Lipschitz on $\mathbb{B}_r^m(c)$, implying the same for $\Lambda_k^{\lim}(\overline{\varphi(f^{-1}(t))}, 0)$. Furthermore, by Lemma 2.12 we have:

$$\Lambda_0^{\lim}(f^{-1}(t), \infty) = \chi(f^{-1}(t)) - 1 + \Lambda_k^{\lim}(\overline{\varphi(f^{-1}(t))}, 0)$$

and

$$\Lambda_k^{\lim}(f^{-1}(t), \infty) = \Lambda_k^{\lim}(\overline{\varphi(f^{-1}(t))}, 0)$$

for $k = 1, \ldots, n$. Therefore, $\Lambda_k^{\lim}(f^{-1}(t), \infty)$ is also locally Lipschitz on $\mathbb{B}_r^m(c)$ for $k \ge 1$; in particular $\Lambda_k^{\lim}(f^{-1}(t), \infty)$ is locally Lipschitz at c. In addition, if $c \notin K_0(f)$, f is topologically trivial in a neighborhood of c. Thus $\chi(f^{-1}(t))$ is constant in that neighborhood. Since $\Lambda_0^{\lim}(\overline{\varphi(f^{-1}(t))}, 0)$ is locally Lipschitz at c, so is $\Lambda_0^{\lim}(f^{-1}(t), \infty)$. The theorem is proved. □

2.15. **Remark.** Globally topological fibration triviality does not necessarily imply the local Lipschitz continuity (or even the continuity) of Lipschitz-Killing curvature densities of the fibers. To illustrate this, consider the polynomial function (see [7, Example 3.1], [8, Example 3.3])

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \quad (x, y, z) \mapsto f(x, y, z) = z(x^2 + (xy - 1)^2).$$

It has been shown in [7, Example 3.1] that f is a globally trivial fibration and in [8, Example 3.3] that $0 \in K_\infty(f)$. Note that

$$f^{-1}(0) = \{z = 0\}.$$

Thus

$$\begin{aligned} \Lambda_2^{\lim}(f^{-1}(0), \infty) &= \Lambda_2^{\lim}(\{z = 0\}, \infty) = \Lambda_2^{\lim}(\overline{\varphi\{z = 0\}}, 0) \\ &= \Lambda_2^{\lim}(\{z = 0\}, 0) = \theta(\{z = 0\}, 0) = 1 \end{aligned} \tag{2.7}$$

where $\theta(\cdot, 0)$ is the density at 0. On the other hand, for $t \neq 0$, we have

$$\Lambda_2^{\lim}(f^{-1}(t), \infty) = \Lambda_2^{\lim}(\overline{\varphi(f^{-1}(t))}, 0) = \theta(\overline{\varphi(f^{-1}(t))}, 0) \ge \theta(C_0(\overline{\varphi(f^{-1}(t))}), 0) \tag{2.8}$$

where $C_0(\cdot)$ is the tangent cone at 0 and the last inequality follows from [21, Théorème 3.8]. It is not hard to see that for an unbounded definable set $X \subset \mathbb{R}^n$, $C_\infty(X) = C_0(\overline{\varphi(X)})$ where $C_\infty(X)$ denotes the tangent cone at infinity of X. Moreover, we have

$$C_\infty(f^{-1}(t)) = \begin{cases} C_\infty(f^{-1}(0) \cup \{x = 0,\ z \geqslant 0\} & \text{if } t > 0, \\ C_\infty(f^{-1}(t) \cup \{x = 0,\ z \leqslant 0\} & \text{if } t < 0. \end{cases}$$

Therefore

$$\theta(C_0(\overline{\varphi(f^{-1}(t))}), 0) = \theta(C_\infty(f^{-1}(t)), 0) = \frac{3}{2}.$$

Combining this and (2.8), we get

$$\Lambda_2^{\lim}(f^{-1}(t), \infty) \ge \frac{3}{2} \quad \text{for } t \neq 0.$$

So this, together with (2.7), implies that the function $t \mapsto \Lambda_2^{\lim}(f^{-1}(t), \infty)$ is not continuous at 0. In particular, it is not locally Lipschitz continuous at 0.

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Institute of Mathematics, VAST, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam
Email address: dstiep@math.ac.vn

FPT University, Danang, Vietnam
Email address: nguyenxuanvietnhan@gmail.com