NONLINEAR RAYLEIGH-TAYLOR INSTABILITY FOR INCOMPRESSIBLE VISCOUS FLUIDS REVISITED

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ABSTRACT. The goal of this paper is to prove the viscous Rayleigh-Taylor instability in a horizontal periodic domain with infinite height, extending the inviscid result of Guo and Hwang [4]. Using the spectral analysis obtained by Lafitte and the author [13], we show the existence of infinitely many unstable solutions to the linearized equations. Hence, we are able to construct a large class of initial data to approximate the nonlinear equations, refining Grenier's method [6] and then to prove the nonlinear instability. Our result improves the previous one of Jiang, Jiang and Ni [9] by using a strong notion of instability.

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1. INTRODUCTION

The governed equations are the gravity-driven incompressible Navier–Stokes equation:

$$\begin{cases} \partial_t \rho + \text{div}(\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + \text{div}(\rho \vec{u} \otimes \vec{u}) + \nabla P = \mu \Delta \vec{u} - \rho g \vec{e}_3, \\ \text{div} \vec{u} = 0, \end{cases} \tag{1.1}$$

where $t \geqslant 0, x = (x_1, x_2, x_3) \in (2\pi L\mathbb{T})^2 \times \mathbf{R}$ ($\mathbb{T}$ be the usual 1D torus and $L > 0$ be the length of periodicity). The unknowns $\rho := \rho(x,t)$, $\vec{u} := \vec{u}(x,t)$ and $P := P(x,t)$ denote respectively the density, the velocity and the pressure of the fluid, while $\mu > 0$ is the viscosity coefficient, $g > 0$ is the gravity constant and $\vec{e}_3 = (0,0,1)^T$. Let $\rho_0 > 0$ be a C^1-function depending only on x_3 and let P_0 be another function of x_3 given by $P_0' = -g\rho_0$ with $' = d/dx_3$. Hence, $(\rho_0(x_3), \vec{0}, P_0(x_3))$ is an equilibrium to Eq. (1.1). Of interest of this paper is to study the nonlinear instability of Eq. (1.1) around the above steady state, assuming that

$$0 < \rho_- < \rho_+ < +\infty \quad \text{with} \lim_{x_3 \to \pm\infty} \rho_0(x_3) = \rho_\pm, \tag{1.2}$$

since we are interested in Rayleigh-Taylor instability.

The Rayleigh–Taylor (RT) instability, studied first by Lord Rayleigh in [16] and then Taylor [18] is well known as a gravity-driven instability in two semi-infinite inviscid and incompressible fluids when the heavy one is on top of the light one. It has attracted much attention due to both its

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physical and mathematical importance. Two applications of worth mentioning are implosion of inertial confinement fusion capsules [14] and core-collapse of supernovae [15]. For a detailed physical comprehension of the RT instability, we refer to three survey papers [8, 19, 20].

Mathematically speaking, for the inviscid and incompressible regime in the neighborhood of such steady state, the linear instability has been studied by Lafitte et al. [2, 3, 11, 7] thanks to the variational structure of the linearized system. Guo and Hwang [4] used the celebrated framework of Grenier [6] to prove the nonlinear instability. After that, Lafitte developed these above results in [12] considering the density profile of quasi-isobaric type ($\rho_- = 0$). Tan and Xu in [17] extend the result of Guo and Hwang to a slab domain studying the effect of boundary conditions.

For the viscous problem (1.1), Jiang, Jiang and Ni [9] studied the linear instability of such steady state $(\rho_0(x_3), \vec{0}, P_0(x_3))$ satisfying (1.2) and that $\rho_0' \in C_0^\infty$ is locally positive. To study the linear instability, the authors followed a modified variational approach of Guo and Tice [5] for a RT problem with two compressible channel flows. Hence, the authors [9, Theorem 1.1 and Remark 1.3] introduced a weak notion of instability and proved the nonlinear RT instability in that sense.

Very recently, assuming that the density profile ρ_0 satisfies (1.2) and that ρ_0' is a nonnegative function of class $C_0^0(\mathbf{R})$, the author and Lafitte [13] initiate another approach to study the linear instability. The method is based on the spectral theory of compact and self-adjoint operators and the given result proves the existence of infinitely many normal mode solutions to the linearized equations (see (2.2)). For readers convenience, we refer to Theorem 2.1 below for the statement. This paper is a continuation of [13], passing the linear instability to the nonlinear instability by refining the framework of Grenier [6]. Precisely, having infinitely many normal mode solutions to the linearized equations at hands, we illustrate the nonlinear RT instability with *a wide class of initial data*. We refer the statement to Theorem 2.3. Let us point out that our main result uses the strong notion of instability, see [6, Definition 2.1] and thus sharpen the previous result of Jiang, Jiang and Ni [9].

This paper is organized as follows. In Section 2, we formulate the problem with the governing equations by following [13]. After that, we present the known result, Theorem 2.1, on the spectral analysis of the linearized equations (2.2). The second part is to present the nonlinear result and introduce the strategy of the proof.

2. THE GOVERNED EQUATIONS AND MAIN RESULTS

Let us formulate the main problem. In the vicinity $(\rho_0(x_3), \vec{0}, P_0(x_3))$, the quantities

$$\sigma = \rho - \rho_0, \quad \vec{u} = \vec{u} - \vec{0}, \quad p = P - P_0$$

satisfy the following nonlinear perturbation equations

$$\begin{cases} \partial_t \sigma + \vec{u} \cdot \nabla(\rho_0 + \sigma) = 0, \\ (\rho_0 + \sigma)\partial_t \vec{u} + (\rho_0 + \sigma)\vec{u} \cdot \nabla \vec{u} + \nabla p = \mu \Delta \vec{u} - g\sigma \vec{e}_3, \\ \mathrm{div} \vec{u} = 0. \end{cases} \tag{2.1}$$

Hence, instead of proving the nonlinear RT instability to (1.1), we move to prove the nonlinear instability of trivial solution to the nonlinear equations (2.1). For this purpose, in the first part, we represent the spectral analysis in [13] and deduce further property of the maximal growth rate to the linearized equations (2.2) below. In the second part, we introduce the strategy of the proof to nonlinear instability result, following Grenier's idea [6] with a refinement.

2.1. **The linear instability.** Omitting the nonlinear terms in (2.1), we obtain the following linearized equations,

$$\begin{cases} \partial_t\sigma + \rho_0' u_3 = 0, \\ \rho_0\partial_t\vec{u} + \nabla p = \mu\Delta\vec{u} - g\sigma\vec{e}_3, \\ \mathrm{div}\vec{u} = 0. \end{cases} \tag{2.2}$$

Since ρ_0 depends only on x_3, we continue the analysis into normal modes as in [1, Chapter X, Section 91]. The linear instability amounts to the investigation of normal mode solution to (2.2) of the form

$$\begin{aligned} \sigma(t,x) &= e^{\lambda t}\cos(k_1x_1 + k_2x_2)\zeta(x_3), \\ u_1(t,x) &= e^{\lambda t}\sin(k_1x_1 + k_2x_2)\psi(x_3), \\ u_2(t,x) &= e^{\lambda t}\sin(k_1x_1 + k_2x_2)\theta(x_3), \\ u_3(t,x) &= e^{\lambda t}\cos(k_1x_1 + k_2x_2)\phi(x_3), \\ p(t,x) &= e^{\lambda t}\cos(k_1x_2 + k_2x_2)q(x_3), \end{aligned} \tag{2.3}$$

where $\mathbf{k} = (k_1, k_2) \in (L^{-1}\mathbb{Z})^2$, $\lambda = \lambda(\mathbf{k}) \in \mathbb{C}\backslash\{0\}$ and $\mathrm{Re}\lambda \geqslant 0$. In this case, such a λ is a growth rate of the instability or is a characteristic value of the linearized problem (see [1, Chapter X, Sections 92-93]).

Let $k = |\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$, substituting (2.3) into (2.2), we arrive at the following system

$$\begin{cases} \lambda\zeta + \rho_0'\phi = 0, \\ \lambda\rho_0\psi - k_1q + \mu(k^2\psi - \psi'') = 0, \\ \lambda\rho_0\theta - k_2q + \mu(k^2\theta - \theta'') = 0, \\ \lambda\rho_0\phi + q' + \mu(k^2\phi - \phi'') + g\zeta = 0, \\ k_1\psi + k_2\theta + \phi' = 0. \end{cases} \tag{2.4}$$

We directly see $\zeta = -\frac{\rho_0'\phi}{\lambda}$. Hence, $(2.4)_4$ becomes

$$\lambda^2\rho_0\phi + \lambda q' + \lambda\mu(k^2\phi - \phi') = g\rho_0'\phi. \tag{2.5}$$

We multiply $(2.4)_2$ by k_1 and $(2.4)_3$ by k_2, then use $(2.4)_4$ to obtain the equality

$$\lambda^2\rho_0\phi' + k^2\lambda q + \lambda\mu(k^2\phi' - \phi''') = 0.$$

Deriving this equation, and replacing $\lambda q'$ thanks to (2.5), we get the fourth-order ordinary equation:

$$\lambda^2(\rho_0k^2\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi, \tag{2.6}$$

The investigation of normal mode solutions (2.3) for fixed k amounts to finding regular solutions $\phi \in H^4(\mathbf{R})$ of (2.6). These solutions physically decay to zero at $\pm\infty$, i.e. ϕ satisfies

$$\lim_{x_3\to\pm\infty}\phi(x_3) = 0. \tag{2.7}$$

For any increasing density profile ρ_0, we necessarily have the uniform boundedness of λ in k, following [13, Lemma 2.1].

Lemma 2.1. *Let ρ_0 be increasing, all characteristic values λ for Eq. (2.6)-(2.7) in $H^4(\mathbf{R})$ are always real and satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$, with $L_0 := (\|\frac{\rho_0'}{\rho_0}\|_{L^\infty(\mathbf{R})})^{-1}$ being the characteristic length of density profile.*

In [13], the author and Lafitte consider the density profile ρ_0 satisfying further that

$$\rho_0' \text{ is a nonnegative function of class } C_0^0(\mathbf{R}), \quad \mathrm{supp}(\rho_0') = [-a, a], \tag{2.8}$$

and that outside $(-a, a)$,

$$\rho_0(x_3) = \begin{cases} \rho_- & \text{as } x_3 \in (-\infty, -a], \\ \rho_+ & \text{as } x_3 \in [a, +\infty), \end{cases} \tag{2.9}$$

with $0 < \rho_- < \rho_+$. With the above profile, the following theorem is proven in [13, Theorem 2.1], showing the existence of infinitely many characteristic values λ.

Theorem 2.1. *Let* $\mathbf{k}$ *be fixed and let* ρ_0 *satisfy* (2.8) *and* (2.9). *There exists an infinite sequence* $(\lambda_n, \phi_n)_{n\geqslant 1}$ *with* $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ *and* $\phi_n \in H^4(\mathbf{R})$ *satisfying* (2.6)-(2.7). *In addition,* λ_n *decreases towards 0 as* n *goes to* ∞.

As an intermediate result of Theorem 2.1, we obtain an infinite sequence of normal mode solutions to the linearized equations.

Proposition 2.1. *Let* $\mathbf{k}$ *be fixed and let* $\rho_0 \in C^\infty(\mathbf{R})$ *satisfy* (2.8) *and* (2.9). *There exists an infinite sequence of normal mode solutions* $(\sigma_j, \vec{u}_j, p_j)(t, \mathbf{k}, x) \in H^\infty(\Omega)$ $(j \geqslant 1)$ *of the form* (2.3) *with* $\lambda = \lambda_j(\mathbf{k})$ *to the linearized equations* (2.2).

Following Lemma 2.1, we have

$$\Lambda := \sup_{\mathbf{k}\in(L^{-1}\mathbb{Z})^2\backslash\{0\}} \lambda_1(\mathbf{k}) \leqslant \sqrt{\frac{g}{L_0}}, \tag{2.10}$$

Our first theorem is to prove that Λ defined above is the sharp exponential growth rate for the linearized equations (2.2) in the following sense.

Theorem 2.2. *Let* $\rho_0 \in C^\infty(\mathbf{R})$ *satisfy* (2.8) *and* (2.9). *Let* $(\sigma, \vec{u})$ *be an arbitrary solution of the linearized equations* (2.2) *with an associated pressure* p. *For any* $s \in \mathbb{N}$ *and for* $0 < \zeta \ll 1$, *we denote*

$$\Lambda_s = \Lambda + s(\zeta + \sqrt{\zeta}).$$

The following inequality thus holds

$$\|\sigma(t)\|_{H^s}^2 + \|\vec{u}(t)\|_{H^s}^2 + \mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\vec{u}(\tau)\|_{H^s}^2 d\tau \leqslant C_{s,\zeta,\Lambda} e^{2\Lambda_s t}(\|\sigma(0)\|_{H^s}^2 + \|\vec{u}(0)\|_{H^{s+1}}^2). \tag{2.11}$$

2.2. **The nonlinear instability.** Once the linear instability is proven, we aim at showing the nonlinear instability by refining Grenier's method [6] as follows.

From (2.10), there exists a $\mathbf{k}_0 \in (L^{-1}\mathbb{Z})^2\backslash\{0\}$ such that $\lambda_1(\mathbf{k}_0) > \frac{\Lambda}{2}$. Fix $\mathbf{k} = \mathbf{k}_0$ and we obtain from Theorem 2.1 infinitely many normal mode solutions $(\sigma_j, \vec{u}_j, p_j)(t, x) \in H^\infty(\Omega)$ $(j \geqslant 1)$. Let integer $s \geqslant 3$ be given and we choose $\zeta > 0$ sufficiently small such that

$$\Lambda > \lambda_1 > \frac{\Lambda_s}{2}.$$

We thus split the sequence $(\lambda_n)_{n\geqslant 1}$ as follows

$$\Lambda > \lambda_1 > \cdots > \lambda_{\mathsf{M}} > \frac{\Lambda_s}{2} > \lambda_{\mathsf{M}+1} > \cdots > \tag{2.12}$$

Note that M is finite due to the limit $\lambda_j \to 0$ as $j \to \infty$. Choosing constants c_j being chosen such that

$$\text{at least one of } \mathsf{c}_j \ (1 \leqslant j \leqslant \mathsf{M}) \text{ is non-zero} \tag{2.13}$$

and

$$\frac{1}{2}|\mathsf{c}_{j_m}|\|\vec{u}_{j_m}\|_{L^2} > \sum_{j\geqslant j_m+1} |\mathsf{c}_j|\|\vec{u}_j\|_{L^2}, \text{ where } j_m := \min\{j : 1 \leqslant j \leqslant \mathsf{M}, \mathsf{c}_j \neq 0\}. \tag{2.14}$$

For any integer N, we formulate a linear combination of those above normal mode solutions

$$(\sigma_{(\mathsf{N})}, \vec{u}_{(\mathsf{N})}, p_{(\mathsf{N})})(t,x) := \sum_{j=1}^{\mathsf{N}} \mathsf{c}_j(\sigma_j, \vec{u}_j, p_j)t, x), \tag{2.15}$$

to set $\delta(\sigma_{(\mathsf{N})}, \vec{u}_{(\mathsf{N})}, p_{(\mathsf{N})})$ with $0 < \delta \ll 1$ as the first-order approximate solution to the nonlinear equations (2.1). After that (see Lemma 4.1), we construct higher-order approximations. Using such an approximate solution $(\sigma^a, \vec{u}^a, p^a)$ and constructing the classical energy estimates in Lemma 4.1, the nonlinear instability thus follows..

Theorem 2.3. *Let $\rho_0 \in C^\infty(\mathbf{R})$ satisfy (2.8) and (2.9). For any integer N, we can find a positive constant m_0 and two positive constants δ_0 and ε_0 sufficiently small so that for any $\delta \in (0, \delta_0)$, the nonlinear equations (2.1) with the initial data*

$$\delta \sum_{j=1}^{\mathsf{N}} \mathsf{c}_j(\sigma_j, \vec{u}_j, p_j)(x)$$

satisfying (2.13)-(2.14) has a unique local strong solution $(\sigma^\delta, \vec{u}^\delta)$ with an associated pressure q^δ such that

$$\|\vec{u}^\delta(T^\delta)\|_{L^2} \geqslant m_0\epsilon_0, \tag{2.16}$$

where $T^\delta \in (0, T_{\max})$ is given by

$$\delta \sum_{j=1}^{\mathsf{N}} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \delta \sum_{j=j_m}^{\mathsf{N}} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \epsilon_0.$$

Throughout this paper, we write $a \lesssim Cb$ for a generic constant C depending only physical parameters. We frequently use the interpolation inequality (for any integer s and real $\nu > 0$)

$$\|v\|_{H^s} \lesssim \|v\|_{L^2}^{\frac{1}{s+1}} \|v\|_{H^{s+1}}^{\frac{s}{s+1}} \leqslant \nu \|v\|_{H^{s+1}} + C_s \nu^{-s} \|v\|_{L^2}. \tag{2.17}$$

3. THE MAXIMAL GROWTH RATE

In this part, we show that Λ defined as in (2.10) is the maximal growth rate of the linearized equations (2.2), i.e. to prove Theorem 2.2. We begin with the variational formulation of the largest characteristic value λ_1.

Lemma 3.1. *Let $\mathcal{B}_{a,k,\lambda}$ is given by (3.3) and let (λ_1, ϕ_1) be found from Theorem 2.1. We have that*

$$\frac{1}{gk^2} = \max_{\phi \in H^2((-a,a))} \frac{\int_{-a}^{a} \rho_0' \phi^2}{\lambda_1 \mathcal{B}_{a,k,\lambda_1}(\phi,\phi)}, \tag{3.1}$$

and the extremal problem (3.1) is attained by ϕ_1 restricted on $(-a,a)$ up to a constant.

Furthermore, let us define the following bilinear form on $H^2(\mathbf{R})$,

$$\mathbb{B}_{k,\lambda}(\phi,\theta) := \lambda \int_{\mathbf{R}} \rho_0(k^2\phi\theta + \phi'\theta') + \mu \int_{\mathbf{R}} ((\phi'' + k^2\phi)(\theta'' + k^2\theta) + 4k^2\phi'\theta').$$

Hence, we have

$$\frac{1}{gk^2} = \max_{\phi \in H^2(\mathbf{R})} \frac{\int_{\mathbf{R}} \rho_0' \phi^2}{\lambda_1 \mathbb{B}_{k,\lambda_1}(\phi,\phi)}. \tag{3.2}$$

The extremal problem (3.2) is attained by ϕ_1 up to a constant.

To do that, we recall [13, Propositions 4.2, 4.3].

Proposition 3.1. *Let $\tau_\pm = \sqrt{k^2 + \lambda\rho_\pm/\mu}$.*

(1) *Let us denote by*

$$BV_{-a,k,\lambda}(\vartheta,\varrho) := \mu\begin{pmatrix} k\tau_-(k+\tau_-)\vartheta(-a)\varrho(-a) - k\tau_-\vartheta'(-a)\varrho(-a) \\ \qquad - k\tau_-\vartheta(-a)\varrho'(-a) + (k+\tau_-)\vartheta'(-a)\varrho'(-a)\end{pmatrix}$$

and by

$$BV_{a,k,\lambda}(\vartheta,\varrho) := \mu\begin{pmatrix} k\tau_+(k+\tau_+)\vartheta(a)\varrho(a) - k\tau_+\vartheta'(a)\varrho(a) \\ \qquad - k\tau_+\vartheta(a)\varrho'(a) + (k+\tau_+)\vartheta'(a)\varrho'(a)\end{pmatrix}.$$

Hence,

$$\begin{aligned}\mathcal{B}_{a,k,\lambda}(\vartheta,\varrho) := BV_{a,k,\lambda}(\vartheta,\varrho) + BV_{-a,k,\lambda}(\vartheta,\varrho) + \lambda\int_{-a}^{a}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho') \\ + \mu\int_{-a}^{a}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho).\end{aligned} \tag{3.3}$$

is a continuous and coercive bilinear form on $H^2((-a,a))$.

(2) *Let* $(H^2((-a,a)))'$ *be the dual space of* $H^2((-a,a))$ *associated with the norm* $\sqrt{\mathcal{B}_{a,k,\lambda}(\cdot,\cdot)}$, *there exists a unique operator*

$$Y_{a,k,\lambda} \in \mathcal{L}(H^2((-a,a)), (H^2((-a,a)))'),$$

that is also bijective, such that, for all $\vartheta, \varrho \in H^2((-a,a))$,

$$\mathcal{B}_{a,k,\lambda}(\vartheta,\varrho) = \langle Y_{a,k,\lambda}\vartheta, \varrho\rangle. \tag{3.4}$$

(3) *For all* $\vartheta \in H^2((-a,a))$, *we have*

$$Y_{a,k,\lambda}\vartheta = \lambda(\rho_0 k^2\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta) \quad \text{in } \mathcal{D}'((-a,a)).$$

(4) *Let* $f \in L^2((-a,a))$ *be given, there exists a unique solution* $\vartheta \in H^2((-a,a))$ *of*

$$Y_{a,k,\lambda}\vartheta = f \text{ in } (H^2((-a,a)))', \tag{3.5}$$

then $\vartheta \in H^4((-a,a))$ *and satisfies the boundary conditions at* $x_3 = -a$,

$$\begin{cases} k\tau_-\vartheta(-a) - (k+\tau_-)\vartheta'(-a) + \vartheta''(-a) = 0, \\ k\tau_-(k+\tau_-)\vartheta(-a) - (k^2 + k\tau_- + \tau_-^2)\vartheta'(-a) + \vartheta'''(-a) = 0,\end{cases} \tag{3.6}$$

and at $x_3 = a$,

$$\begin{cases} k\tau_+\vartheta(a) + (k+\tau_+)\vartheta'(a) + \phi''(a) = 0, \\ -k\tau_+(k+\tau_+)\phi(a) - (k^2 + k\tau_+ + \tau_+^2)\vartheta'(a) + \vartheta'''(a) = 0.\end{cases} \tag{3.7}$$

Proof of Lemma 3.1. We divide the proof into two parts, proving (3.1) and (3.2), respectively.

Part 1. We show that (3.1) holds. For all $\lambda > 0$, we solve the variational problem

$$\alpha_1(\lambda,k) = \max\Big(\int_{-a}^{a}\rho_0'\phi^2\Big|\phi \in H^2((-a,a)), \quad \lambda\mathcal{B}_{a,k,\lambda}(\phi,\phi) = 1\Big). \tag{3.8}$$

Let us define the Lagrangian functional

$$L_{\mathcal{B}}(\nu,\phi) = \int_{-a}^{a}\rho_0'\phi^2 - \nu(\lambda\mathcal{B}_{a,k,\lambda}(\phi,\phi) - 1). \tag{3.9}$$

It follows from the Lagrange multiplier theorem that the extrema of the quotient

$$\frac{\int_{-a}^{a}\rho_0'\phi^2 dx_3}{\lambda\mathcal{B}_{a,k,\lambda}(\phi,\phi)}$$

are necessarily obtained at the stationary points $(\nu_\star, \phi_\star)$ of $L_{\mathcal{B}}$, which satisfy

$$\lambda \mathcal{B}_{a,k,\lambda}(\phi_\star, \phi_\star) = 1, \qquad \int_{-a}^{a} \rho_0' \phi_\star \theta - \lambda \nu_\star \mathcal{B}_{a,k,\lambda}(\phi_\star, \theta) = 0, \tag{3.10}$$

for all $\theta \in H^2((-a,a))$. Restricting $\theta \in C_0^\infty((-a,a))$, one deduces from (3.10) that $\phi_\star$ has to satisfy

$$\lambda \nu_\star Y_{a,k,\lambda} \phi_\star = \rho_0' \phi_\star \tag{3.11}$$

in a weak sense. Using bootstrap argument, we further get that $\phi_\star \in H^4((-a,a))$ and satisfies (3.11) and the boundary conditions (3.6)-(3.7). Hence, all stationary points $(\nu_\star, \phi_\star)$ of $L_{\mathcal{B}}$ satisfy that $\lambda\nu_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{a,k,\lambda} = \mathcal{M} Y_{a,k,\lambda}^{-1} \mathcal{M}$ ($\mathcal{M} = \sqrt{\rho_0'}$) from $L^2((-a,a))$ to itself, with

$$\mathcal{M}^{-1} Y_{a,k,\lambda} \phi_\star = \frac{1}{\lambda \nu_\star} \mathcal{M} \phi_\star \in L^2((-a,a))$$

being an associated eigenfunction. That implies

$$\alpha_1(\lambda, k) \leqslant \lambda^{-1} \gamma_1(\lambda, k). \tag{3.12}$$

Meanwhile, since the operator $S_{a,k,\lambda}$ is self-adjoint and positive, we thus obtain that

$$\gamma_1(\lambda, k) = \sup_{\phi \in L^2((-a,a))} \frac{\langle S_{a,k,\lambda}\phi, \phi\rangle}{\|\phi\|^2_{L^2((-a,a))}}.$$

Hence, for all $\phi \in L^2((-a,a))$ and for $\psi = Y_{a,k,\lambda}^{-1} \mathcal{M}\phi \in H^4((-a,a))$, we have

$$\langle Y_{a,k,\lambda}\psi, \psi\rangle = \langle S_{a,k,\lambda}\phi, \phi\rangle,$$

which yields

$$\gamma_1(\lambda, k)\langle Y_{a,k,\lambda}\psi, \psi\rangle \leqslant \frac{\langle S_{a,k,\lambda}\phi, \phi\rangle^2}{\|\phi\|^2_{L^2((-a,a)}} \leqslant \|S_{a,k,\lambda}\phi\|^2_{L^2((-a,a))}.$$

This yields

$$\gamma_1(\lambda, k) \leqslant \sup\Big\{ \frac{\|\mathcal{M}\psi\|^2_{L^2((-a,a))}}{\langle Y_{a,k,\lambda}\psi, \psi\rangle} | \psi \in H^4((-a,a)) \text{ and } \mathcal{M}^{-1} Y_{a,k,\lambda}\psi \in L^2((-a,a)) \Big\}.$$

Owing to (3.4), we have that

$$\gamma_1(\lambda, k) \leqslant \sup\Big\{ \frac{\int_{-a}^{a} \rho_0' \psi^2}{\mathcal{B}_{a,k,\lambda}(\psi, \psi)} | \psi \in H^4((-a,a)) \text{ and } \mathcal{M}^{-1} Y_{a,k,\lambda}\psi \in L^2((-a,a)) \Big\}.$$

We thus obtain

$$\lambda^{-1}\gamma_1(\lambda, k) \leqslant \alpha_1(\lambda, k) \tag{3.13}$$

The two inequalities (3.12) and (3.13) tell us that $\alpha_1(\lambda_1, k) = \lambda^{-1}\gamma_1(\lambda_1, k)$ for all $\lambda > 0$, from which we deduce $\alpha_1(\lambda_1, k) = \frac{1}{gk^2}$ and the extremal problem (3.1) is attained by the function $\phi_1|_{(-a,a)}$ up to a constant.

Part 2. We prove that (3.2) holds. We set

$$\alpha_2(\lambda, k) = \max_{\phi \in H^2(\mathbf{R})} \Big(\int_{\mathbf{R}} \rho_0' \phi^2 \Big| \lambda \mathbb{B}_{k,\lambda}(\phi, \phi) = 1 \Big).$$

and consider the Lagrangian functional

$$L_{\mathbb{B}}(\omega, \phi) = \int_{\mathbf{R}} \rho_0' \phi^2 - \omega(\mathbb{B}_{k,\lambda}(\phi, \phi) - 1).$$

Thanks to Lagrange multiplier theorem again, the extrema of the quotient

$$\frac{\int_{\mathbf{R}} \rho_0' \phi^2}{\lambda \mathbb{B}_{k,\lambda}(\phi, \phi)}$$

are necessarily obtained at the stationary points $(\omega_\star, \Phi_\star) \in \mathbf{R}_+ \times H^2(\mathbf{R})$ of $L_{\mathbb{B}}$, which satisfy

$$\lambda \mathbb{B}_{k,\lambda}(\Phi_\star, \Phi_\star) = 1, \quad \int_{\mathbf{R}} \rho_0' \Phi_\star \theta - \lambda \omega_\star \mathbb{B}_{k,\lambda}(\Phi_\star, \theta) = 0 \tag{3.14}$$

for all $\theta \in H^2(\mathbf{R})$. Restricting $\theta \in C_0^\infty(\mathbf{R})$, one deduces from (3.14) that $\Phi_\star$ has to satisfy

$$\mu((\Phi_\star'')'' - 2k^2 \Phi_\star'' + k^4 \Phi_\star) + \lambda(k^2 \rho_0 \Phi_\star - (\rho_0 \Phi_\star')') = \frac{1}{\lambda \omega_\star} \rho_0' \Phi_\star \quad \text{in } \mathcal{D}'(\mathbf{R}). \tag{3.15}$$

Using bootstrap argument, we further get that $\Phi_\star \in H^4(\mathbf{R})$ and $\Phi_\star$ decays to 0 at infinity. Since $\text{supp}\rho_0' = [-a, a]$, we use [13, Proposition 3.1] to deduce that $\Phi_\star$ on $(-a, a)$ is a solution of

$$\lambda \omega_\star Y_{a,k,\lambda}(\Phi_\star|_{(-a,a)}) = \rho_0' \Phi_\star|_{(-a,a)} = \mathcal{M}^2 \Phi_\star|_{(-a,a)}$$

satisfying the boundary conditions (3.6)-(3.7). Set

$$\tilde{\Phi} = \mathcal{M}^{-1} Y_{a,k,\lambda}(\Phi_\star|_{(-a,a)}) = \frac{1}{\lambda \omega_\star} \mathcal{M} \Phi_\star|_{(-a,a)} \in L^2((-a,a)), \tag{3.16}$$

it yields

$$\lambda \omega_\star \tilde{\Phi} = \mathcal{M} Y_{a,k,\lambda}^{-1} \mathcal{M} \tilde{\Phi} = S_{a,k,\lambda} \tilde{\Phi}.$$

That means $\lambda \omega_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{a,k,\lambda}$ from $L^2((-a,a))$ to itself, with $\tilde{\Phi} \in L^2((-a,a))$ (defined as in (3.16)) being an associated eigenfunction. Hence, we get

$$\lambda \alpha_2(\lambda, k) \leqslant \gamma_1(\lambda, k). \tag{3.17}$$

Let us recall the function ϕ_1 from Theorem 2.1. One thus has

$$\alpha_2(\lambda, k) \geqslant \frac{\int_{\mathbf{R}} \rho_0' \phi_1^2}{\lambda \mathbb{B}_{k,\lambda}(\phi_1, \phi_1)}. \tag{3.18}$$

Note that from [13, Proposition 3.1],

$$\phi_1(x_3) = \begin{cases} A_1^- e^{k(x_3+a)} + A_2^- e^{\sqrt{k^2 + \frac{\lambda_1 \rho_-}{\mu}}(x_3+a)} & \text{as } -\infty < x_3 < -a, \\ A_1^+ e^{-k(x_3-a)} + A_2^+ e^{-\sqrt{k^2 + \frac{\lambda_1 \rho_+}{\mu}}(x_3-a)} & \text{as } a < x_3 < +\infty, \end{cases}$$

Hence, the direct computations show that

$$\mathbb{B}_{k,\lambda}(\phi_1, \phi_1) = \mathcal{B}_{a,k,\lambda}(\phi_1|_{(-a,a)}, \phi_1|_{(-a,a)}), \tag{3.19}$$

and we keep in mind the assumption $\text{supp}\rho_0' = [-a, a]$. Then, from (3.18) and (3.19), we have

$$\alpha_2(\lambda, k) \geqslant \frac{\int_{-a}^a \rho_0' \phi_1^2}{\lambda \mathcal{B}_{a,k,\lambda}(\phi_1|_{(-a,a)}, \phi_1|_{(-a,a)})}.$$

It then follows

$$\alpha_2(\lambda_1, k) \geqslant \frac{\int_{-a}^a \rho_0' \phi_1^2}{\lambda_1 \mathcal{B}_{a,k,\lambda_1}(\phi_1|_{(-a,a)}, \phi_1|_{(-a,a)})} = \frac{1}{gk^2}. \tag{3.20}$$

Combining (3.17) and (3.20) gives us that $\alpha_2(\lambda_1, k) = \frac{1}{gk^2}$ and the extremal problem (3.2) is attained by ϕ_1 up to a constant. We finish the proof of Proposition 3.1. □

Thanks to Lemma 3.1, we exploit the Fourier transform to obtain the following lemma, whose proof is similar to [10, page 1882]. Hence we omit the details here.

Lemma 3.2. *For any $\vec{u}$ such that $\text{div}\vec{u} = 0$, there holds*

$$\Lambda^2 \int_\Omega \rho_0 |\vec{u}|^2 + \Lambda \mu \int_\Omega |\nabla \vec{u}|^2 \geqslant g \int_\Omega \rho_0' |\vec{u}|^2. \tag{3.21}$$

Now let $(\sigma, \vec{u})$ be a solution to the linearized equations (2.2) with an associated pressure p, we prove Theorem 2.2. In advance, we show the two following lemmas.

Lemma 3.3. *There holds*

$$\|\partial_t\vec{u}(t)\|_{H^s} \lesssim \|\sigma(t)\|_{H^s}. \tag{3.22}$$

Proof. Since $\mathrm{div}\partial_t\vec{u} = 0$, we apply Young's inequality to get that

$$\int_\Omega \rho_0|\partial_t\vec{u}|^2 = \int_\Omega g\sigma\partial_t u_3 \leqslant \nu\|\partial_t u_3\|_{L^2}^2 + C_\nu\|\sigma\|_{L^2}^2.$$

Taking $\nu > 0$ sufficiently small, we have

$$\|\partial_t\vec{u}\|_{L^2} \lesssim \|\sigma\|_{L^2}. \tag{3.23}$$

Next, for any $\alpha \in \mathbb{N}^3$ with $|\alpha| \leqslant s$, we get

$$\partial^\alpha(\rho_0\partial_t\vec{u}) + \partial^\alpha\nabla p - \mu\Delta\partial^\alpha\vec{u} = \partial^\alpha(g\sigma\vec{e}_3).$$

Multiplying $\partial^\alpha\partial_t u$ on both sides of the resulting equation and then integrating by parts to obtain

$$\int_\Omega \rho_0|\partial^\alpha\partial_t\vec{u}|^2 + \mu\int_\Omega|\nabla\partial^\alpha\vec{u}|^2 = \int_\Omega g\partial^\alpha\sigma\partial^\alpha\partial_t u_3 + \int_\Omega(\rho_0\partial^\alpha\partial_t\vec{u} - \partial^\alpha(\rho_0\partial_t\vec{u}))\cdot\partial^\alpha\partial_t\vec{u}.$$

Note that

$$\partial^\alpha(\rho_0\partial_t\vec{u}) - \rho_0\partial^\alpha\partial_t\vec{u} = \sum_{0\neq\gamma\leqslant\alpha}\partial^\gamma\rho_0\partial^{\alpha-\gamma}\partial_t\vec{u}.$$

Using (2.17), we have

$$\int_\Omega(\rho_0\partial^\alpha\partial_t\vec{u} - \partial^\alpha(\rho_0\partial_t\vec{u}))\cdot\partial^\alpha\partial_t\vec{u} \lesssim \sum_{0\neq\gamma\leqslant\alpha}\|\partial^{\alpha-\gamma}\partial_t\vec{u}\|_{L^2}\|\partial^\alpha\partial_t\vec{u}\|_{L^2} \lesssim \|\partial_t\vec{u}\|_{L^2}^{\frac{1}{s+1}}\|\partial_t\vec{u}\|_{H^s}^{\frac{2s+1}{s+1}}.$$

Owing to Young's inequality and (3.23), one obtains for any $\nu > 0$,

$$\int_\Omega(\rho_0\partial^\alpha\partial_t\vec{u} - \partial^\alpha(\rho_0\partial_t\vec{u}))\cdot\partial^\alpha\vec{\partial}_t u \leqslant \nu\|\partial_t\vec{u}\|_{H^s}^2 + C_\nu\|\partial_t\vec{u}\|_{L^2}^2 \leqslant \nu\|\partial_t\vec{u}\|_{H^s}^2 + C_\nu\|\sigma\|_{L^2}^2. \tag{3.24}$$

Similarly,

$$\int_\Omega g\partial^\alpha\sigma\partial^\alpha\partial_t u_3 \leqslant \nu\|\partial_t u_3\|_{H^s}^2 + C_\nu\|\sigma\|_{H^s}^2. \tag{3.25}$$

As a result of (3.24) and (3.25), we sum up over α to observe

$$\rho_-\|\partial_t\vec{u}\|_{H^s}^2 \leqslant C_s\nu\|\partial_t\vec{u}\|_{H^s}^2 + C_\nu\|\sigma\|_{H^s}^2.$$

Let $\nu > 0$ be sufficiently small, the inequality (3.22) follows. Proof of Lemma 3.3 is complete. □

Lemma 3.4. *There holds*

$$\|\partial_t^2\vec{u}\|_{H^s} \lesssim \|\vec{u}\|_{H^s}. \tag{3.26}$$

Proof. We take the derivative in time of the second line of (2.2) to get that

$$\rho_0\partial_t^2\vec{u} + \nabla\partial_t p - \mu\Delta\partial_t\vec{u} = g\rho_0' u_3\vec{e}_3. \tag{3.27}$$

We multiply (3.27) by $\partial_t^2\vec{u}$ and use Young's inequality to get that

$$\int_\Omega \rho_0|\partial_t^2\vec{u}|^2 + \mu\int_\Omega|\nabla\partial_t\vec{u}|^2 = \int_\Omega g\rho_0' u_3\partial_t^2 u_3 \leqslant \nu\int_\Omega|\partial_t^2 u_3|^2 + C_\nu\int_\Omega|u_3|^2,$$

yielding

$$\|\partial_t^2\vec{u}\|_{L^2} \lesssim \|\vec{u}\|_{L^2}. \tag{3.28}$$

Next, for $\alpha \in \mathbb{N}^3$ with $|\alpha| \leqslant s$, we have that

$$\partial^\alpha(\rho_0\partial_t^2\vec{u}) + \nabla\partial_t\partial^\alpha p - \mu\Delta\partial_t\partial^\alpha\vec{u} = g\partial^\alpha(\rho_0' u_3\vec{e}_3). \tag{3.29}$$

That implies
$$\int_\Omega \rho_0|\partial^\alpha\partial_t^2\vec{u}|^2+\mu\int_\Omega|\nabla\partial^\alpha\vec{u}|^2=\int_\Omega g\partial^\alpha(\rho_0'u_3)\partial^\alpha\partial_t^2u_3+\int_\Omega(\rho_0\partial^\alpha\partial_t^2\vec{u}-\partial^\alpha(\rho_0\partial_t^2\vec{u}))\cdot\partial^\alpha\partial_t^2\vec{u}.$$
Note that
$$\partial^\alpha(\rho_0\partial_t^2\vec{u})-\rho_0\partial^\alpha\partial_t^2\vec{u}=\sum_{0\neq\gamma\leqslant\alpha}\partial^\gamma\rho_0\partial^{\alpha-\gamma}\partial_t^2\vec{u}.$$
Thus, using (2.17) and Young's inequality, we have
$$\int_\Omega(\rho_0\partial^\alpha\partial_t^2\vec{u}-\partial^\alpha(\rho_0\partial_t^2u))\cdot\partial^\alpha\partial_t^2\vec{u}\lesssim\|\partial_t^2\vec{u}\|_{H^{s-1}}\|\partial_t^2\vec{u}\|_{H^s}\leqslant\nu\|\partial_t^2\vec{u}\|_{H^s}^2+C_\nu\|\partial_t^2\vec{u}\|_{L^2}^2,$$
for any $\nu>0$. Together with (3.28), we have
$$\begin{aligned}\int_\Omega\rho_0|\partial^\alpha\partial_t^2\vec{u}|^2&\leqslant\int_\Omega g\partial^\alpha(\rho_0'u_3)\partial^\alpha\partial_t^2u_3+\int_\Omega(\rho_0\partial^\alpha\partial_t^2\vec{u}-\partial^\alpha(\rho_0\partial_t^2\vec{u}))\cdot\partial^\alpha\partial_t^2\vec{u}\\&\leqslant 2\nu\|\partial_t^2\vec{u}\|_{H^s}^2+C_\nu(\|u_3\|_{H^s}^2+\|\partial_t^2\vec{u}\|_{L^2})\\&\leqslant 2\nu\|\partial_t^2\vec{u}\|_{H^s}^2+C_\nu\|\vec{u}\|_{H^s}^2.\end{aligned}$$
Summing over α, the resulting inequality implies
$$\rho_-\|\partial_t^2\vec{u}\|_{H^s}^2\leqslant C_s\nu\|\partial_t^2\vec{u}\|_{H^s}^2+C_\nu\|\vec{u}\|_{H^s}^2.$$
Let $\nu>0$ be small enough, we obtain (3.26). □

We are in position to obtain the exponential growth of the linearized equations (2.2).

Proof of Theorem 2.2. Let $a_s:=\|\sigma(0)\|_{H^s}+\|\vec{u}(0)\|_{H^{s+1}}$, we prove the inequality (2.11) by the induction on s.

In case $s=0$, we multiply by $\partial_t\vec{u}$ on both sides of (3.27) to obtain
$$\frac{1}{2}\frac{d}{dt}\int_\Omega\rho_0|\partial_t\vec{u}|^2=\int_\Omega\rho_0\partial_t\vec{u}\cdot\partial_t^2\vec{u}=-\int_\Omega|\nabla\partial_t\vec{u}|^2+\int_\Omega g\rho_0'u_3\partial_tu_3,$$
which implies
$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega\rho_0|\partial_t\vec{u}|^2-g\int_\Omega\rho_0'|u_3|^2\Big)+\int_\Omega|\nabla\partial_t\vec{u}|^2=0.$$
Integrating from 0 to t and using (3.22), we have
$$\begin{aligned}\int_\Omega\rho_0|\partial_t\vec{u}(t)|^2+\int_0^t\int_\Omega\mu|\nabla\partial_t\vec{u}(\tau)|^2d\tau&\leqslant\int_\Omega\rho_0|\partial_t\vec{u}(0)|^2-\int_\Omega g\rho_0'|u_3(0)|^2+\int_\Omega g\rho_0'|u_3(t)|^2\\&\leqslant C\|\sigma(0)\|_{L^2}^2+\Lambda^2\int_\Omega\rho_0|\vec{u}(t)|^2+\Lambda\int_\Omega\mu|\nabla\vec{u}(t)|^2.\end{aligned}\tag{3.30}$$

Using Cauchy-Schwarz's inequality, we have that
$$\begin{aligned}\int_\Omega\mu|\nabla\vec{u}(t)|^2&=\int_\Omega\mu|\nabla\vec{u}(0)|^2+2\int_0^t\int_\Omega\mu\nabla\vec{u}(\tau):\nabla\partial_t\vec{u}(\tau)d\tau\\&\leqslant\int_\Omega\mu|\nabla\vec{u}(0)|^2+\frac{1}{\Lambda}\int_0^t\int_\Omega\mu|\nabla\partial_t\vec{u}(\tau)|^2d\tau+\Lambda\int_0^t\int_\Omega\mu|\nabla\vec{u}(\tau)|^2d\tau,\end{aligned}\tag{3.31}$$
and that
$$\frac{d}{dt}\int_\Omega\rho_0|\vec{u}|^2\leqslant\frac{1}{\Lambda}\int_\Omega\rho_0|\partial_t\vec{u}|^2+\Lambda\int_\Omega\rho_0|\vec{u}|^2.\tag{3.32}$$

The three inequalities (3.30), (3.31) and (3.32) imply that

$$\frac{d}{dt}\int_\Omega \rho_0|\vec{u}(t)|^2+\int_\Omega \mu|\nabla\vec{u}(t)|^2\leqslant y_0+2\Lambda\int_\Omega \rho_0|\vec{u}(t)|^2+\Lambda\int_0^t\int_\Omega \mu|\nabla\vec{u}(\tau)|^2d\tau. \tag{3.33}$$

where

$$y_0=\frac{C}{\Lambda}\|\sigma(0)\|_{L^2}^2+\int_\Omega \mu|\nabla\vec{u}(0)|^2\lesssim a_0.$$

In view of Gronwall's inequality, we obtain from (3.33) that

$$\int_\Omega \rho_0|\vec{u}(t)|^2+\int_0^t\int_\Omega \mu|\nabla\vec{u}(\tau)|^2d\tau\leqslant e^{2\Lambda t}\int_\Omega \rho_0|\vec{u}(0)|^2+\frac{y_0}{2\Lambda}(e^{2\Lambda t}-1). \tag{3.34}$$

Because of (3.34), we observe

$$\|\vec{u}(t)\|_{L^2}^2+\int_0^t\|\nabla\vec{u}(\tau)\|_{L^2}^2d\tau\lesssim(\|\sigma(0)\|_{L^2}^2+\|\vec{u}(0)\|_{H^1}^2)e^{2\Lambda t}. \tag{3.35}$$

Now, we obtain from $(2.2)_1$ that

$$\frac{d}{dt}\|\sigma\|_{L^2}\leqslant\|\partial_t\sigma\|_{L^2}\lesssim\|\vec{u}\|_{L^2}.$$

That implies

$$\|\sigma(t)\|_{L^2}\leqslant\|\sigma(0)\|_{L^2}+\int_0^t\|\vec{u}(\tau)\|_{L^2}d\tau\leqslant C_{0,\Lambda}e^{\Lambda t}.$$

The inequality $(2.11)_{s=0}$ follows from the resulting inequality and (3.35).

Suppose that we have

$$\|\sigma(t)\|_{H^{s-1}}+\|\vec{u}(t)\|_{H^{s-1}}\leqslant C_{s-1,\Lambda}a_{s-1}e^{\Lambda_{s-1}t}. \tag{3.36}$$

For $\alpha\in\mathbb{N}^3$ with $|\alpha|\leqslant s$, using (3.29), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial^\alpha\partial_t\vec{u}|^2+\mu\int_\Omega|\nabla\partial^\alpha\partial_t\vec{u}|^2=\int_\Omega g\partial^\alpha(\rho_0'u_3)\partial^\alpha\partial_tu_3+\int_\Omega(\rho_0\partial^\alpha\partial_t^2\vec{u}-\partial^\alpha(\rho_0\partial_t^2\vec{u}))\cdot\partial^\alpha\partial_t\vec{u}.$$

From this, we further obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial^\alpha\partial_t\vec{u}|^2-\int_\Omega g\rho_0'|\partial^\alpha u_3|^2\Big)+\mu\int_\Omega|\nabla\partial^\alpha\partial_t\vec{u}|^2\\&=\sum_{0\neq\gamma\leqslant\alpha}\int_\Omega g\partial^\gamma\rho_0'\partial^{\alpha-\gamma}u_3\partial^\alpha\partial_tu_3-\sum_{0\neq\gamma\leqslant\alpha}\int_\Omega(\partial^\gamma\rho_0\partial^{\alpha-\gamma}\partial_t^2\vec{u})\cdot\partial^\alpha\partial_t\vec{u}\\&\lesssim(\|u_3\|_{H^{s-1}}+\|\partial_t^2\vec{u}\|_{H^{s-1}})\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}.\end{aligned}$$

We use Young's inequality and (3.26) to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial^\alpha\partial_t\vec{u}|^2-\int_\Omega g\rho_0'|\partial^\alpha u_3|^2\Big)+\mu\int_\Omega|\nabla\partial^\alpha\partial_t\vec{u}|^2\\&\leqslant\zeta\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}^2+C_\zeta(\|u_3\|_{H^{s-1}}^2+\|\partial_t^2\vec{u}\|_{H^{s-1}}^2)\\&\leqslant\zeta\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}^2+C_\zeta\|\vec{u}\|_{H^{s-1}}^2.\end{aligned}$$

Together with (3.22), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}^2-g\int_\Omega\rho_0'|\partial^\alpha u_3|^2\Big)+2\mu\|\nabla\partial^\alpha\partial_t\vec{u}\|_{L^2}^2\\&\leqslant 2\zeta\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}^2+C_\zeta a_{s-1}e^{2\Lambda_{s-1}t}.\end{aligned}$$

By multiplying the resulting inequality by $e^{-2\zeta t}$, we get

$$\frac{d}{dt}\left(e^{-2\zeta t}\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}^2\right) + 2\mu e^{-2\zeta t}\|\nabla\partial^\alpha\partial_t\vec{u}\|_{L^2}^2$$
$$\leqslant \frac{d}{dt}\left(e^{-2\zeta t}\int_\Omega g\rho_0'|\partial^\alpha u_3|^2\right) + 2\zeta e^{-2\zeta t}\int_\Omega g\rho_0'|\partial^\alpha u_3|^2 + C_\zeta a_{s-1}^2 e^{2(\Lambda_{s-1}-\zeta)t}$$

Integrating from 0 to t and using (3.22) yields

$$e^{-2\zeta t}\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}(t)\|_{L^2}^2 + 2\mu\int_0^t e^{-2\zeta s}\|\nabla\partial^\alpha\partial_t\vec{u}(s)\|_{L^2}^2 ds$$
$$\leqslant Ca_s^2 + e^{-2\zeta t}\int_\Omega g\rho_0'|\partial^\alpha u_3(t)|^2 + 2\zeta\int_0^t e^{-2\zeta s}\int_\Omega g\rho_0'|\partial^\alpha u_3(s)|^2 ds$$
$$+ Ca_{s-1}^2\int_0^t e^{2(\Lambda_{s-1}-\zeta)s}ds.$$

Note that $\text{div}\partial^\alpha u = 0$, we apply Lemma 3.2 to further get that

$$\begin{aligned}&\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}(t)\|_{L^2}^2 + 2\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\partial_t\vec{u}(\tau)\|_{L^2}^2 d\tau\\ &\leqslant Ca_s^2 e^{2\zeta t} + \Lambda^2\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2 + \Lambda\mu\|\nabla\partial^\alpha\vec{u}(t)\|_{L^2}^2\\ &\quad + 2\zeta\int_0^t e^{2\zeta(t-\tau)}\left(\Lambda^2\|\sqrt{\rho_0}\partial^\alpha\vec{u}(\tau)\|_{L^2}^2 + \Lambda\mu\|\nabla\partial^\alpha\vec{u}(s)\|_{L^2}^2\right)d\tau + Ca_{s-1}^2 e^{2\Lambda_{s-1}t}.\end{aligned} \tag{3.37}$$

Mimicking (3.31) and (3.32), we have

$$\frac{d}{dt}\|\sqrt{\rho_0}\partial^\alpha\vec{u}\|_{L^2}^2 \leqslant \frac{1}{\Lambda}\|\sqrt{\rho_0}\partial^\alpha\partial_t\vec{u}\|_{L^2}^2 + \Lambda\|\sqrt{\rho_0}\partial^\alpha\vec{u}\|_{L^2}, \tag{3.38}$$

and

$$\begin{aligned}\|\nabla\partial^\alpha\vec{u}(t)\|_{L^2}^2 &= \|\nabla\partial^\alpha\vec{u}(0)\|^2 + 2\int_0^t\int_\Omega \nabla\partial^\alpha\vec{u}(\tau) : \nabla\partial_t\partial^\alpha\vec{u}(\tau)d\tau\\ &\leqslant \|\nabla\partial^\alpha\vec{u}(0)\|^2 + \frac{1}{\Lambda}\int_0^t \|\nabla\partial^\alpha\partial_t\vec{u}(\tau)\|_{L^2}^2 ds + \Lambda\int_0^t \|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2 ds\\ &\leqslant \|\vec{u}(0)\|_{H^{s+1}}^2 + \frac{1}{\Lambda}\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\partial_t\vec{u}(\tau)\|_{L^2}^2 ds + \Lambda\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(s)\|_{L^2}^2 d\tau.\end{aligned} \tag{3.39}$$

Combining (3.37), (3.38) and (3.39) gives us that

$$\frac{d}{dt}\left(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2 + \mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2 d\tau\right)$$
$$= \frac{d}{dt}\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2 + \mu\|\nabla\partial^\alpha\vec{u}(t)\|_{L^2}^2 + 2\zeta\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2 d\tau$$
$$\leqslant Ca_s^2(e^{2\zeta t} + e^{2\Lambda_{s-1}t}) + 2\Lambda\left(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2 + \mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2 d\tau\right)$$
$$+ 2\zeta\int_0^t e^{2\zeta(t-\tau)}\left(\Lambda\|\sqrt{\rho_0}\partial^\alpha\vec{u}(\tau)\|_{L^2}^2 + 2\mu\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2\right)d\tau.$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\Big(e^{-2(\Lambda_{s-1}+\zeta)t}\Big(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2+\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\Big)\Big)\\&\leqslant Ca_s^2(e^{-2\zeta t}+e^{-2\Lambda_{s-1}t})+2(\Lambda-\Lambda_{s-1}-\zeta)e^{-2(\Lambda_{s-1}+\zeta)t}\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2\\&\quad+2(\Lambda+\zeta-\Lambda_{s-1})\mu e^{-2(\Lambda_{s-1}+\zeta)t}\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\\&\quad+2\zeta\Lambda e^{-2(\Lambda_{s-1}+\zeta)t}\int_0^t e^{2\zeta(t-\tau)}\|\sqrt{\rho_0}\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau.\end{aligned}$$

Define

$$y(t)=\sup_{0\leqslant\theta\leqslant t}\Big(e^{-2(\Lambda_{s-1}+\zeta+\sqrt{\zeta})\theta}\Big(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(\theta)\|_{L^2}^2+\mu\int_0^\theta e^{2\zeta(\theta-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\Big)\Big).$$

Note that for any $\theta\in[0,t]$,

$$\|\sqrt{\rho_0}\partial^\alpha\vec{u}(\theta)\|_{L^2}^2+\mu\int_0^\theta e^{2\zeta(\theta-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\leqslant e^{2(\Lambda_{s-1}+\zeta+\sqrt{\zeta})\theta}y(t),$$

yielding that

$$\int_0^t e^{2\zeta(t-\tau)}\|\sqrt{\rho_0}\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\leqslant\frac{1}{2(\Lambda_{s-1}+\sqrt{\zeta})}e^{2(\Lambda_{s-1}+\zeta+\sqrt{\zeta})t}y(t).$$

Note also that $\Lambda\leqslant\Lambda_{s-1}$, we thus deduce

$$\begin{aligned}&\frac{d}{dt}\Big(e^{-2(\Lambda_{s-1}+\zeta)t}\Big(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2+\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\Big)\Big)\\&\leqslant Ca_s^2(e^{-2\zeta t}+e^{-2\Lambda_{s-1}t})+2\zeta\Big(\frac{\Lambda}{2(\Lambda_{s-1}+\sqrt{\zeta})}+1\Big)e^{2\sqrt{\zeta}t}y(t)\\&\leqslant Ca_s^2(e^{-2\zeta t}+e^{-2\Lambda_{s-1}t})+3\zeta e^{2\sqrt{\zeta}t}y(t).\end{aligned}$$

We integrate the resulting inequality from 0 to t to obtain

$$\begin{aligned}&e^{-2(\Lambda_{s-1}+\zeta)t}\Big(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2+\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\Big)\\&\leqslant Ca_s^2\Big(\frac{1}{\zeta}+\frac{1}{\Lambda_{s-1}}\Big)+\frac{3}{2}\sqrt{\zeta}e^{2\sqrt{\zeta}t}y(t).\end{aligned}$$

That implies

$$\begin{aligned}&e^{-2(\Lambda_{s-1}+\zeta+\sqrt{\zeta})t}\Big(\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2+\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\Big)\\&\leqslant Ca_s^2\Big(\frac{1}{\zeta}+\frac{1}{\Lambda_{s-1}}\Big)+\frac{3}{2}\sqrt{\zeta}y(t).\end{aligned}$$

Taking the supremum, we have

$$y(t)\leqslant Ca_s^2\Big(\frac{1}{\zeta}+\frac{1}{\Lambda_{s-1}}\Big)+\frac{3}{2}\sqrt{\zeta}y(t).$$

Since ζ is sufficiently small, we have $y(t)$ is bounded for all time, i.e. for all $t>0$,

$$y(t)\leqslant 2Ca_s^2\Big(\frac{1}{\zeta}+\frac{1}{\Lambda_{s-1}}\Big).$$

As a result, we have

$$\|\sqrt{\rho_0}\partial^\alpha\vec{u}(t)\|_{L^2}^2+\mu\int_0^t e^{2\zeta(t-\tau)}\|\nabla\partial^\alpha\vec{u}(\tau)\|_{L^2}^2d\tau\leqslant C_{s,\zeta,\Lambda}a_s^2e^{2(\Lambda_{s-1}+\zeta+\sqrt{\zeta})t}.$$

Summing over α, we obtain

$$\|\vec{u}(t)\|_{H^s}^2 + \mu \int_0^t e^{2\zeta(t-\tau)} \|\nabla \vec{u}(\tau)\|_{H^s}^2 d\tau \leqslant C_{s,\zeta,\Lambda} a_s^2 e^{2\Lambda_s t}.$$

Similarly, we can obtain an estimate for $\|\sigma(t)\|_{H^s}$ and thus deduce our desired inequality (2.11). □

4. Nonlinear instability

4.1. **A high-order of approximate solution.** In the first part, we construct approximate solutions to the nonlinear equations (2.1), whose H^s-norm enjoys the controllability in a short time by induction. Second, we define the difference between the exact solutions and the approximate one (see (4.8)) and derive its error estimate. Combining those estimates, we deduce the existence of escaping time to obtain the nonlinear instability. We recall (2.15),

$$(\sigma_{(\mathsf{N})}, \vec{u}_{(\mathsf{N})}, p_{(\mathsf{N})})(t,x) := \sum_{j=1}^{\mathsf{N}} \mathsf{c}_j (\sigma_j, \vec{u}_j, p_j)(t,x),$$

which is a to the linearized equations (2.2). Using $(\sigma_{(\mathsf{N})}, \vec{u}_{(\mathsf{N})}, p_{(\mathsf{N})})$, we closely follow Grenier's idea to construct approximate solutions to the nonlinear equations (2.1) in the following lemma. To this purpose, we define

$$G_{\mathsf{N}}(t) = \sum_{j=1}^{\mathsf{M}} |\mathsf{c}_j| e^{\lambda_j t} + \sum_{j=\mathsf{M}+1}^{\mathsf{N}} |\mathsf{c}_j| e^{\frac{\Lambda}{2} t} = \sum_{j=j_m}^{\mathsf{M}} |\mathsf{c}_j| e^{\lambda_j t} + \sum_{j=\mathsf{M}+1}^{\mathsf{N}} |\mathsf{c}_j| e^{\frac{\Lambda}{2} t},$$

and

$$F_{\mathsf{N}}(t) = \sum_{j=1}^{\mathsf{N}} |\mathsf{c}_j| e^{\lambda_j t} = \sum_{j=j_m}^{\mathsf{N}} |\mathsf{c}_j| e^{\lambda_j t}.$$

Obviously we have $G_{\mathsf{N}}(t) \geqslant F_{\mathsf{N}}(t)$ for all $t \geqslant 0$. Let $0 < \epsilon_0 \ll 1$, there is a unique T^δ such that $\delta F_{\mathsf{N}}(T^\delta) = \epsilon_0$.

Lemma 4.1. *Let $0 < \delta \ll 1$ and $n \in \mathbb{N}$, there is an approximate solution to the nonlinear equations (2.1), of the form*

$$\begin{aligned} \sigma^a(t,x) &= \sum_{j=1}^n \delta^j \sigma^{\langle j\rangle}(t,x), \\ \vec{u}^a(t,x) &= \sum_{j=1}^n \delta^j \vec{u}^{\langle j\rangle}(t,x), \\ p^a(t,x) &= \sum_{j=1}^n \delta^j p^{\langle j\rangle}(t,x), \end{aligned} \tag{4.1}$$

satisfying

$$\begin{cases} \partial_t \sigma^a + \nabla(\rho_0 + \sigma^a) \cdot \vec{u}^a = R_n^a, \\ (\rho_0 + \sigma^a)(\partial_t \vec{u}^a + u^a \cdot \nabla \vec{u}^a) + \nabla p^a - \mu \Delta \vec{u}^a - g\sigma^a e_3 = \vec{S}_n^a, \\ div\ \vec{u}^{\langle j\rangle} = 0 \quad (1 \leqslant j \leqslant n). \end{cases} \tag{4.2}$$

Moreover, for any integer $s \geqslant 0$, if $0 \leqslant t \leqslant T^\delta$, then the j-th order coefficients $\sigma^{\langle j\rangle}(t,x), \vec{u}^{\langle j\rangle}(t,x)$ and $p^{\langle j\rangle}(t,x)$ for $1 \leqslant j \leqslant n$ satisfy

$$\|(\sigma^{\langle j\rangle}, \vec{u}^{\langle j\rangle}, p^{\langle j\rangle})(t)\|_{H^s} \leqslant C_{j,n} G_{\mathsf{N}}(jt), \tag{4.3}$$

and the $(n+1)$-th order remainders R_n^a and S_n^a satisfy

$$\|(R_n^a, \vec{S}_n^a)(t)\|_{H^s} \leqslant C_{s,n} \delta^{n+1} G_{\mathsf{N}}((n+1)t). \tag{4.4}$$

Proof. We use the induction on n to prove Lemma 4.1. For $n = 1$, we choose the normal mode solutions to the linearized equations

$$(\sigma^{\langle 1\rangle}, \vec{u}^{\langle 1\rangle}, p^{\langle 1\rangle})(t,x) = (\sigma_{(\mathsf{N})}, \vec{u}_{(\mathsf{N})}, p_{(\mathsf{N})})(t,x).$$

Obviously, the functions $\sigma^{\langle 1\rangle}$, $u^{\langle 1\rangle}$ and $p^{\langle 1\rangle}$ satisfy (4.3). Substituting $(\sigma^{\langle 1\rangle}, \vec{u}^{\langle 1\rangle}, p^{\langle 1\rangle})$ into the left hand side of (2.1), we obtain

$$\begin{aligned} R_1^a &= \nabla(\delta\sigma^{\langle 1\rangle})(\delta\vec{u}^{\langle 1\rangle}),\\ \vec{S}_1^a &= (\delta\sigma^{\langle 1\rangle})(\delta\partial_t\vec{u}^{\langle 1\rangle}) + (\rho_0 + \delta\sigma^{\langle 1\rangle})(\delta\vec{u}^{\langle 1\rangle})\cdot\nabla(\delta\vec{u}^{\langle 1\rangle}). \end{aligned}$$

Since $(\sigma^{\langle 1\rangle}, \vec{u}^{\langle 1\rangle}, p^{\langle 1\rangle}) \in H^s$ for any $s \geqslant 0$, we have that R_1^a and $\vec{S}_1^a$ satisfy (4.4).

Assume that we have constructed $(\sigma^{\langle j\rangle}, \vec{u}^{\langle j\rangle}, p^{\langle j\rangle})$ as well as R_j^a, S_j^a which satisfy (4.3)-(4.4) for $j < n$. We now construct $(\sigma^{\langle j+1\rangle}, \vec{u}^{\langle j+1\rangle}, p^{\langle j+1\rangle})$ as well as $R_{j+1}^a, \vec{S}_{j+1}^a$. Let

$$\sigma^{\langle j\rangle} = \sum_{h=1}^{j}\delta^h\sigma^{\langle h\rangle}, \quad \vec{u}^{\langle j\rangle} = \sum_{h=1}^{j}\delta^h\vec{u}^{\langle h\rangle}, \quad p^{\langle j\rangle} = \sum_{h=1}^{j}\delta^h p^{\langle h\rangle}.$$

Substituting $(\sigma^{\langle j\rangle}, \vec{u}^{\langle j\rangle}, p^{\langle j\rangle})$ into the left hand side of (4.2), we obtain the nonlinear parts

$$\begin{aligned} f_{j+1}(\delta) &= \nabla\sigma^{\langle j\rangle}\cdot\vec{u}^{\langle j\rangle},\\ r_{j+1}(\delta) &= \sigma^{\langle j\rangle}\partial_t\vec{u}^{\langle j\rangle} + (\rho_0 + \sigma^{\langle j\rangle})\vec{u}^{\langle j\rangle}\cdot\nabla\vec{u}^{\langle j\rangle}. \end{aligned}$$

For $0 \leqslant t \leqslant T^\delta$, we now expand $f_{j+1}(\delta)$ and $r_{j+1}(\delta)$ in terms of δ around $\delta = 0$. The coefficients of the $(j+1)$-th order term are $\frac{f_{j+1}^{(j+1)}(0)}{(j+1)!}$ and $\frac{r_{j+1}^{(j+1)}(0)}{(j+1)!}$, that are functions on (t,x). On the other hand, notice that for $0 \leqslant t \leqslant T^\delta$,

$$\begin{aligned} \frac{f_{j+1}^{(j+1)}(0)}{(j+1)!} &= \sum_{j_1+j_2=j+1} A_{j_1,j_2}\nabla\sigma^{\langle j_1\rangle}\cdot\nabla\vec{u}^{\langle j_2\rangle},\\ \frac{r_{j+1}^{(j+1)}(0)}{(j+1)!} &= \sum_{j_1+j_2=j+1} B_{j_1,j_2}\sigma^{\langle j_1\rangle}\partial_t\vec{u}^{\langle j_2\rangle} + \sum_{j_1+j_2=j+1} C_{j_1,j_2}\rho_0\vec{u}^{\langle j_1\rangle}\cdot\nabla\vec{u}^{\langle j_2\rangle}\\ &\quad + \sum_{j_1+j_2+j_3=j+1} D_{j_1,j_2,j_3}\sigma^{\langle j_1\rangle}\vec{u}^{\langle j_2\rangle}\cdot\nabla\vec{u}^{\langle j_3\rangle}, \end{aligned} \tag{4.5}$$

where $A_{j_1,j_2}, B_{j_1,j_2}, C_{j_1,j_2}$ and D_{j_1,j_2,j_3} depend on ρ_0 and physical parameters. By the induction hypothesis (4.3) for $(\sigma^{\langle h\rangle}, u^{\langle h\rangle}, p^{\langle h\rangle})$ $(1 \leqslant h \leqslant j)$, we obtain, for every $s \geqslant 0$,

$$\begin{aligned} \left\|\frac{f_{j+1}^{(j+1)}(0)}{(j+1)!}\right\|_{H^s} + \left\|\frac{r_{j+1}^{(j+1)}(0)}{(j+1)!}\right\|_{H^s} &\leqslant C_{j,m}(G_{\mathsf{N}}(j_1t)G_{\mathsf{N}}(j_2t) + G_{\mathsf{N}}(j_1t)G_{\mathsf{N}}(j_2t)G_{\mathsf{N}}(j_3t))\\ &\leqslant C_{j,m}G_{\mathsf{N}}((j+1)t). \end{aligned} \tag{4.6}$$

We now define the $(j+1)$-th order coefficients $\sigma^{\langle j+1\rangle}$, $\vec{u}^{\langle j+1\rangle}$ and $p^{\langle j+1\rangle}$ as solutions of the following inhomogeneous linear system:

$$\begin{cases} \partial_t\sigma^{\langle j+1\rangle} + \nabla\rho_0\cdot\vec{u}^{\langle j+1\rangle} = -\frac{f_{j+1}^{(j+1)}(0)}{(j+1)!},\\ \rho_0\partial_t\vec{u}^{\langle j+1\rangle} + \nabla p^{\langle j+1\rangle} - \mu\Delta\vec{u}^{\langle j+1\rangle} - g\sigma^{\langle j+1\rangle}e_3 = \frac{r_{j+1}^{(j+1)}(0)}{(j+1)!}, \end{cases}$$

with initial data $\sigma^{\langle j+1\rangle}(0,x)=0, \vec{u}^{\langle j+1\rangle}(0,x)=\vec{0}$. It follows from Proposition 2.2 and Duhamel's principle that

$$\begin{aligned}\|\vec{u}^{\langle j+1\rangle}(t)\|_{H^s} &\leqslant C_{j+1,n}\int_0^t e^{\Lambda_s(t-\tau)}\Big\|\frac{f_{j+1}^{(j+1)}(0)}{(j+1)!}(0)\Big\|_{H^s}d\tau\\ &\leqslant C_{j+1,n}\int_0^t e^{\Lambda_s(t-\tau)}G_{\mathsf{N}}((j+1)\tau)d\tau\\ &\leqslant C_{j+1,n}e^{\Lambda_s t}\Big(\sum_{h=1}^{\mathsf{M}}|\mathsf{c}_k|\int_0^t e^{((j+1)\lambda_k-\Lambda_s)\tau}d\tau+\sum_{h=\mathsf{M}+1}^{\mathsf{N}}|\mathsf{c}_k|\int_0^t e^{((j+1)\frac{\Lambda_s}{2}-\Lambda_s)\tau}d\tau\Big).\end{aligned}\tag{4.7}$$

Here we have used (4.6) and the definition of G_{N} in the last inequality. As $1\leqslant h\leqslant \mathsf{M}$, we have from (2.12) that

$$(j+1)\lambda_h\geqslant 2\lambda_h>\Lambda_s,$$

that implies

$$\int_0^t e^{((j+1)\lambda_h-\Lambda_s)\tau}d\tau=\frac{e^{((j+1)\lambda_h-\Lambda_s)t}-1}{(j+1)\lambda_k-\Lambda_h}\leqslant\frac{e^{((j+1)\lambda_h-\Lambda_s)t}}{(j+1)\lambda_h-\Lambda_s}.$$

Plugging the resulting inequality into (4.7), we have

$$\|\vec{u}^{\langle j+1\rangle}(t)\|_{H^s}\leqslant C_{j+1,n}\int_0^t e^{\Lambda_s(t-\tau)}G_{\mathsf{N}}((j+1)\tau)d\tau\leqslant C_{j+1,n}G_{\mathsf{N}}((j+1)t).$$

The same argument gives us the bound of $\|\sigma^{\langle j+1\rangle}(t)\|_{H^s}$ and $\|p^{\langle j+1\rangle}(t)\|_{H^s}$ to deduce (4.3).

Once, we have $(\sigma^{\langle j\rangle},\vec{u}^{\langle j\rangle},p^{\langle j\rangle})$ for $1\leqslant j\leqslant n$, let

$$\sigma^a=\sum_{j=1}^n\delta^j\sigma^{\langle j\rangle},\quad \vec{u}^a=\sum_{j=1}^n\delta^j\vec{u}^{\langle j\rangle},\quad p^a=\sum_{j=1}^n\delta^j p^{\langle j\rangle}.$$

They satisfy

$$\begin{aligned}\partial_t\sigma^a+\nabla\rho_0\cdot\vec{u}^a&=-\sum_{j=1}^m\frac{\delta^{j+1}f_{j+1}^{(j+1)}(0)}{(j+1)!},\\ \rho_0\partial_t\vec{u}^a+\nabla p^a-\mu\Delta\vec{u}^a-g\sigma^a\vec{e}_3&=-\sum_{j=1}^m\frac{\delta^{j+1}r_{j+1}^{(j+1)}(0)}{(j+1)!},\end{aligned}$$

Let

$$f(\delta)=\nabla\sigma^a\cdot\vec{u}^a,\quad r(\delta)=\sigma^a\partial_t\vec{u}^a+\sigma^a\vec{u}^a\cdot\nabla\vec{u}^a,$$

we thus have $(\sigma^a,\vec{u}^a,p^a)$ is a solution of (4.2) with

$$R_n^a=-\sum_{j=1}^n\frac{\delta^{j+1}f^{(j+1)}(0)}{(j+1)!}+f(\delta),\quad \vec{S}_n^a=-\sum_{j=1}^n\frac{\delta^{j+1}r^{(j+1)}(0)}{(j+1)!}+r(\delta).$$

We now prove that R_n^a and $\vec{S}_n^a$ satisfy (4.4) to complete the proof. Since $f(\delta)$ and $r(\delta)$ are quadratic in terms of $(\sigma^a,\vec{u}^a)$, we have that their $(j+1)$-th order terms are the same as those ones of $f_{j+1}(\delta)$ and $r_{j+1}(\delta)$ for all $1\leqslant j+1\leqslant n$, respectively. Consequently, R_n^a and $\vec{S}_n^a$ have the form

$$\sum_{h\geqslant n+1}C((\sigma^{\langle j\rangle},\vec{u}^{\langle j\rangle},p^{\langle j\rangle})_{1\leqslant j\leqslant n})\delta^h.$$

Mimicking the proof of (4.6), we obtain (4.4). Lemma 4.1 is proven. □

4.2. **The difference functions.** Let us recall

$$(\sigma^d, \vec{u}^d, p^d) = (\sigma^\delta, \vec{u}^\delta, p^\delta) - (\sigma^a, \vec{u}^a, p^a)$$

satisfy the nonlinear equations in Ω,

$$\begin{cases} \partial_t \sigma^d + \nabla(\rho_0 + \sigma^a) \cdot \vec{u}^d + \nabla \sigma^d \cdot \vec{u}^\delta = -R_n^a, \\ (\rho_0 + \sigma^a)\partial_t \vec{u}^d + \nabla p^d - \mu \Delta \vec{u}^d - g\sigma^d e_3 \\ \qquad = -(\rho_0 + \sigma^a)(\vec{u}^\delta \cdot \nabla \vec{u}^d + \vec{u}^d \cdot \nabla \vec{u}^a) - \sigma^d(\partial_t \vec{u}^a + \vec{u}^\delta \cdot \nabla \vec{u}^\delta) - \vec{S}_n^a, \\ \operatorname{div} \vec{u}^d = 0. \end{cases} \tag{4.8}$$

We will derive the following inequality in this section.

Proposition 4.1. *Let $s \geqslant 3$ and $(\sigma^a, \vec{u}^a, q^a, R_n^a, \vec{S}_n^a)(t,x) \in L^\infty_{loc}(H^s)$ as in Lemma 4.1. Assume that $\|\sigma^\delta\|_{H^4} \leqslant \frac{1}{2}\rho_-$ and $\|\sigma^a\|_{H^4} \leqslant \frac{1}{2}\rho_-$ There exists a universal constant $C_0 > 0$ such that the following inequality holds*

$$\frac{d}{dt}\big(\|\sigma^d\|_{H^4}^2 + \|\vec{u}^d\|_{H^3}^2\big) + \|\nabla \vec{u}^d\|_{H^3}^2 \leqslant C_0 \begin{pmatrix} 1 + \|\partial_t \sigma^a\|_{H^2} + \|\partial_t \vec{u}^a\|_{H^2}^2 \\ + \|\sigma^a\|_{H^5}^2 + \|\vec{u}^a\|_{H^4}^4 + \|\vec{u}^d\|_{H^3}^4 \end{pmatrix} (\|\sigma^d\|_{H^4}^2 + \|\vec{u}^d\|_{H^3}^2) \\ + C_0\big(\|R_n^a\|_{H^4}^2 + \|\vec{S}_n^a\|_{H^3}^2\big). \tag{4.9}$$

To prove Proposition 4.1, we state the following lemma.

Lemma 4.2. *With the same assumption as in Proposition 4.1, the following inequality holds*

$$\|\partial_t \vec{u}^d\|_{H^2}^2 \lesssim \|\nabla \vec{u}^d\|_{H^3}^2 + \big(1 + \|\partial_t \vec{u}^a\|_{H^2}^2 + \|\vec{u}^a\|_{H^3}^4 + \|\vec{u}^d\|_{H^3}^4\big)(\|\vec{u}^d\|_{H^3}^2 + \|\sigma^d\|_{H^4}^2) + \|\vec{S}_n^a\|_{H^2}. \tag{4.10}$$

Proof. Multiplying $(4.8)_2$ by $\partial_t \vec{u}^d$ and using $\operatorname{div}\partial_t \vec{u}^d = 0$, we obtain

$$\int (\rho_0 + \sigma^a)|\partial_t \vec{u}^d|^2 = \int_\Omega (\mu \Delta \vec{u}^d + g\sigma^d e_3) \cdot \partial_t \vec{u}^d - \int_\Omega (\rho_0 + \sigma^a)(\vec{u}^\delta \cdot \nabla \vec{u}^d + \vec{u}^d \cdot \nabla \vec{u}^a) \cdot \partial_t \vec{u}^d \\ - \int_\Omega \sigma^d(\partial_t \vec{u}^a + \vec{u}^\delta \cdot \nabla \vec{u}^\delta) \cdot \partial_t \vec{u}^d - \int_\Omega \vec{S}_n^a \cdot \partial_t \vec{u}^d.$$

Using Sobolev embedding and Cauchy-Schwarz's inequality, we get further

$$\begin{aligned} \|\partial_t \vec{u}^d\|_{L^2}^2 &\lesssim \|\sqrt{\rho_0 + \sigma^a}\partial_t \vec{u}^d\|_{L^2}^2 \\ &\lesssim (\|\vec{u}^d\|_{H^2} + \|\sigma^d\|_{L^2})\|\partial_t \vec{u}^d\|_{L^2} + (\|\partial_t \vec{u}^a\|_{L^2} + \|\vec{u}^\delta\|_{H^2}\|\nabla \vec{u}^\delta\|_{L^2})\|\sigma^d\|_{H^2}\|\partial_t \vec{u}^d\|_{L^2} \\ &\quad + (1 + \|\sigma^a\|_{H^2})\big(\|\vec{u}^\delta\|_{L^4}\|\nabla \vec{u}^d\|_{L^4} + \|\vec{u}^d\|_{L^4}\|\nabla \vec{u}^a\|_{L^4}\big)\|\partial_t \vec{u}^d\|_{L^2} + \|S_n^a\|_{L^2}\|\partial_t \vec{u}^d\|_{L^2}. \end{aligned}$$

That implies

$$\begin{aligned} \|\partial_t \vec{u}^d\|_{L^2} \lesssim\ & \|\vec{u}^d\|_{H^2} + (\|\vec{u}^d\|_{H^1} + \|\vec{u}^a\|_{H^2})\|\vec{u}^d\|_{H^2} \\ & + (\|\partial_t \vec{u}^a\|_{L^2} + \|\vec{u}^a\|_{H^2}^2 + \|\vec{u}^d\|_{H^2}^2)\|\sigma^d\|_{H^2} + \|\vec{S}_n^a\|_{L^2}. \end{aligned}$$

Applying ∂_i (resp. ∂_{ij}^2) to $(4.8)_2$, multiplying the resulting equation by $\partial_i \vec{u}^d$ (resp. ∂_{ij}^2) and mimicking the above arguments, we obtain

$$\begin{aligned} \|\partial_t \vec{u}^d\|_{H^2} &\lesssim \|\vec{u}^d\|_{H^4} + (\|\vec{u}^d\|_{H^3} + \|\vec{u}^a\|_{H^3})\|\vec{u}^d\|_{H^3} \\ &\qquad + \big(1 + \|\partial_t \vec{u}^a\|_{H^2} + \|\vec{u}^\delta\|_{H^3}\|\vec{u}^\delta\|_{H^2}\big)\|\sigma^d\|_{H^4} + \|\vec{S}_n^a\|_{H^2} \\ &\lesssim \|\vec{u}^d\|_{H^4} + (\|\vec{u}^d\|_{H^3} + \|\vec{u}^a\|_{H^3})\|\vec{u}^d\|_{H^3} \\ &\qquad + \big(1 + \|\partial_t \vec{u}^a\|_{H^2} + \|\vec{u}^d\|_{H^3}^2 + \|\vec{u}^a\|_{H^3}^2\big)\|\sigma^d\|_{H^4} + \|\vec{S}_n^a\|_{H^2}. \end{aligned}$$

Applying Cauchy-Schwarz's inequality, we have

$$\|\partial_t\vec{u}^d\|_{H^2}^2 \lesssim \|\vec{u}^d\|_{H^4}^2 + \begin{pmatrix} 1+\|\vec{u}^a\|_{H^3}^2+\|\partial_t\vec{u}^a\|_{H^2}^2 \\ +\|\vec{u}^d\|_{H^3}^2+\|\vec{u}^a\|_{H^3}^4+\|\vec{u}^d\|_{H^3}^4 \end{pmatrix} (\|\vec{u}^d\|_{H^3}^2+\|\sigma^d\|_{H^4}^2)+\|\vec{S}_n^a\|_{H^2}.$$

The inequality (4.10) thus follows, Lemma 4.2 is proven. □

We now prove Proposition 4.1.

Proof. Letting $\alpha \in \mathbb{N}^3$ with $|\alpha| \leqslant 4$ and taking the α-derivative of $(4.8)_1$, we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sigma^d\|_{H^4}^2 &= -\int_\Omega \partial^\alpha(\nabla\sigma^d\cdot\vec{u}^\delta)\cdot\partial^\alpha\sigma^d - \int_\Omega \partial^\alpha(\nabla(\rho_0+\sigma^a)\vec{u}^d)\partial^\alpha\sigma^d - \int \partial^\alpha R_n^a\partial^\alpha\sigma^d \\
&= -\int_\Omega \Big(\partial^\alpha(\nabla\sigma^d\cdot\vec{u}^\delta) - \vec{u}^\delta\partial^\alpha\nabla\sigma^d\Big)\partial^\alpha\sigma^d - \int_\Omega \partial^\alpha(\nabla\sigma^a\cdot\vec{u}^d)\partial^\alpha\sigma^d \\
&\quad - \int_\Omega \big(\partial^\alpha(\nabla\rho_0\cdot\vec{u}^d)+\partial^\alpha R_n^a\big)\partial^\alpha\sigma^d.
\end{aligned} \tag{4.11}$$

Using Gagliardo-Nirenberg's inequality, we bound the first integral in the right hand side of (4.11) as

$$\begin{aligned}
\int_\Omega \Big(\partial^\alpha(\nabla\sigma^d\cdot\vec{u}^\delta) - u^\delta\partial^\alpha\nabla\sigma^d\Big)\partial^\alpha\sigma^d &\lesssim \|\sigma^d\|_{H^4}\Big(\|\nabla\sigma^d\|_\infty\|\vec{u}^\delta\|_{H^4}+\|\nabla\vec{u}^\delta\|_\infty\|\nabla\sigma^d\|_{H^3}\Big) \\
&\lesssim \|\sigma^d\|_{H^4}^2(\|\vec{u}^d\|_{H^4}+\|\vec{u}^a\|_{H^4}).
\end{aligned} \tag{4.12}$$

For the second integral in the right hand side of (4.11), we use Sobolev embedding to get

$$\begin{aligned}
&\int_\Omega \partial^\alpha(\nabla\sigma^a\cdot\vec{u}^d)\partial^\alpha\sigma^d \\
&\lesssim \|\nabla\sigma^a\|_{L^4}\|\vec{u}^d\|_{L^4}\|\sigma^d\|_{L^2} + \big(\|\nabla^2\sigma^a\|_{L^4}\|\nabla\vec{u}^d\|_{L^4}+\|\nabla\sigma^a\|_{L^\infty}\|\nabla^2\vec{u}^d\|_{L^2}\big)\|\nabla^2\sigma^d\|_{L^2} \\
&\quad + \big(\|\nabla^3\sigma^a\|_{L^4}\|\nabla\vec{u}^d\|_{L^\infty}+\|\nabla^2\sigma^a\|_{L^4}\|\nabla^2\vec{u}^d\|_{L^4}+\|\nabla\sigma^a\|_{L^\infty}\|\nabla^3\vec{u}^d\|_{L^2}\big)\|\nabla^3\sigma^d\|_{L^2} \\
&\quad + \begin{pmatrix} \|\nabla^5\sigma^a\|_{L^2}\|\vec{u}^d\|_{L^\infty}+\|\nabla^4\sigma^a\|_{L^2}\|\nabla\vec{u}^d\|_{L^\infty}+\|\nabla^3\sigma^a\|_{L^4}\|\nabla^2\vec{u}^d\|_{L^4} \\ +\|\nabla^2\sigma^a\|_{L^4}\|\nabla^3\vec{u}^d\|_{L^4}+\|\nabla\sigma^a\|_{L^\infty}\|\nabla^4\vec{u}^d\|_{L^2} \end{pmatrix}\|\nabla^4\sigma^d\|_{L^2} \\
&\lesssim \|\sigma^a\|_{H^5}\|\vec{u}^d\|_{H^4}\|\sigma^d\|_{H^4}.
\end{aligned} \tag{4.13}$$

For the last integral in the right hand side of (4.11), we estimate

$$\int_\Omega \big(\partial^\alpha(\nabla\rho_0\cdot\vec{u}^d)+\partial^\alpha R_n^a\big)\partial^\alpha\sigma^d \lesssim (\|\vec{u}^d\|_{H^4}+\|R_n^a\|_{H^4})\|\sigma^d\|_{H^4}. \tag{4.14}$$

In view of (4.12), (4.13) and (4.14), we get

$$\frac{d}{dt}\|\sigma^d\|_{H^4}^2 \lesssim \|\sigma^d\|_{H^4}^2(\|\vec{u}^d\|_{H^4}+\|\vec{u}^a\|_{H^4}) + \|\sigma^a\|_{H^5}\|\vec{u}^d\|_{H^4}\|\sigma^d\|_{H^4} + (\|\vec{u}^d\|_{H^4}+\|R_n^a\|_{H^4})\|\sigma^d\|_{H^4}.$$

Using Young's inequality, we obtain further

$$\frac{d}{dt}\|\sigma^d\|_{H^4}^2 \leqslant \nu\|\nabla\vec{u}^d\|_{H^3}^2 + C_\nu\big(1+\|\vec{u}^a\|_{H^4}+\|\sigma^a\|_{H^5}^2+\|\sigma^d\|_{H^4}^2\big)(\|\sigma^d\|_{H^4}^2+\|\vec{u}^d\|_{H^3}^2)+\|R_n^a\|_{H^4}^2. \tag{4.15}$$

We proceed to derive energy estimates for the difference of velocity. Applying ∂^α with $|\alpha| \leqslant 3$ to $(4.8)_2$, we have

$$\begin{aligned}(\rho_0+\sigma^a)\partial_t\partial^\alpha\vec{u}^d =& -\sum_{0\neq\beta\leqslant\alpha}\partial^\beta(\rho_0+\sigma^a)\partial_t\partial^{\alpha-\beta}\vec{u}^d - \nabla\partial^\alpha p^d + \mu\Delta\partial^\alpha\vec{u}^d + g\partial^\alpha\sigma^d\vec{e}_3 - \partial^\alpha\vec{S}^a_n \\ &- \partial^\alpha((\rho_0+\sigma^a)(\vec{u}^\delta\cdot\nabla\vec{u}^d + \vec{u}^d\cdot\nabla\vec{u}^a)) - \partial^\alpha(\sigma^d(\partial_t\vec{u}^a + \vec{u}^\delta\cdot\nabla\vec{u}^\delta)) \\ =& -\sum_{0\neq\beta\leqslant\alpha}\partial^\beta(\rho_0+\sigma^a)\partial_t\partial^{\alpha-\beta}\vec{u}^d - \nabla\partial^\alpha p^d + \mu\Delta\partial^\alpha\vec{u}^d + g\partial^\alpha\sigma^d\vec{e}_3 - \partial^\alpha\vec{S}^a_n \\ &- \partial^\alpha((\rho_0+\sigma^a)(\vec{u}^\delta\cdot\nabla\vec{u}^d + \vec{u}^d\cdot\nabla\vec{u}^a)) - \partial^\alpha(\sigma^d(\partial_t\vec{u}^a + \vec{u}^\delta\cdot\nabla\vec{u}^\delta)).\end{aligned}$$

Multiplying the resulting identity by $\partial^\alpha\vec{u}^d$ to get that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_0+\sigma^a}\partial^\alpha\vec{u}^d\|^2_{L^2} + \mu\int_\Omega|\nabla\partial^\alpha\vec{u}^d|^2 \\ &= \frac{1}{2}\int_\Omega\partial_t\sigma^a|\partial^\alpha\vec{u}^d|^2 - \sum_{0\neq\beta\leqslant\alpha}\int_\Omega\partial^\beta(\rho_0+\sigma^a)\partial_t\partial^{\alpha-\beta}u^d\cdot\partial^\alpha\vec{u}^d + g\int_\Omega\partial^\alpha\sigma^d\partial^\alpha u^d_3 - \int_\Omega\partial^\alpha\vec{S}^a_n\cdot\partial^\alpha\vec{u}^d \\ &\quad - \int_\Omega\partial^\alpha((\rho_0+\sigma^a)(\vec{u}^\delta\cdot\nabla\vec{u}^d + \vec{u}^d\cdot\nabla\vec{u}^a))\cdot\partial^\alpha\vec{u}^d - \int\partial^\alpha(\sigma^d(\partial_t\vec{u}^a + \vec{u}^\delta\cdot\nabla\vec{u}^\delta))\cdot\partial^\alpha\vec{u}^d.\end{aligned} \tag{4.16}$$

Now, we estimate each integrals in the right hand side of (4.16). It can be seen that

$$\begin{aligned}&\int_\Omega\partial_t\sigma^a|\partial^\alpha\vec{u}^d|^2 + g\int_\Omega\partial^\alpha\sigma^d\partial^\alpha u^d_3 - \int_\Omega\partial^\alpha\vec{S}^a_n\cdot\partial^\alpha u^d \\ &\quad\lesssim (1+\|\partial_t\sigma^a\|_{H^2})\|\vec{u}^d\|^2_{H^3} + \|\sigma^d\|^2_{H^3} + \|\vec{S}^a_n\|^2_{H^3}.\end{aligned} \tag{4.17}$$

Next, we estimate the second integral as follows

$$\begin{aligned}\sum_{0\neq\beta\leqslant\alpha}\int_\Omega\partial^\beta(\rho_0+\sigma^a)\partial_t\partial^{\alpha-\beta}\vec{u}^d\cdot\partial^\alpha\vec{u}^d &\lesssim \sum_{0\neq\beta\leqslant\alpha}(1+\|\partial^\beta\sigma^a\|_{H^2})\|\partial_t\partial^{\alpha-\beta}\vec{u}^d\|_{L^2}\|\partial^\alpha\vec{u}^d\|_{L^2} \\ &\quad + (1+\|\partial^\alpha\sigma^a\|_{L^2})\|\partial_t\vec{u}^d\|_{H^2}\|\partial^\alpha\vec{u}^d\|_{L^2} \\ &\lesssim (1+\|\sigma^a\|_{H^4})\|\partial_t\vec{u}^d\|_{H^2}\|\vec{u}^d\|_{H^3}.\end{aligned}$$

Using Young's inequality and thanks to (4.10), we get further

$$\begin{aligned}&\sum_{0\neq\beta\leqslant\alpha}\int_\Omega\partial^\beta(\rho_0+\sigma^a)\partial_t\partial^{\alpha-\beta}\vec{u}^d\cdot\partial^\alpha\vec{u}^d \\ &\leqslant \nu\|\nabla\vec{u}^d\|^2_{H^3} + C_\nu\big(1+\|\partial_t\vec{u}^a\|^2_{H^2} + \|\vec{u}^a\|^4_{H^3} + \|\vec{u}^d\|^4_{H^3}\big)(\|\vec{u}^d\|^2_{H^3} + \|\sigma^d\|^2_{H^4}) + C_\nu\|\vec{S}^a_n\|_{H^2}.\end{aligned} \tag{4.18}$$

Next, we use Gagliardo-Nirenberg's inequality and Sobolev embedding to have

$$\begin{aligned}&\int_\Omega\partial^\alpha((\rho_0+\sigma^a)\vec{u}^\delta\cdot\nabla\vec{u}^d)\cdot\partial^\alpha\vec{u}^d \\ &\quad= \int_\Omega\big[\partial^\alpha((\rho_0+\sigma^a)\vec{u}^\delta\cdot\nabla\vec{u}^d) - (\rho_0+\sigma^a)\vec{u}^\delta\cdot\nabla\partial^\alpha\vec{u}^d\big]\cdot\partial^\alpha\vec{u}^d \\ &\quad\lesssim \|\vec{u}^d\|_{H^3}\big(\|(\rho_0+\sigma^a)\vec{u}^\delta\|_{H^3}\|\nabla\vec{u}^d\|_{L^\infty} + \|\nabla((\rho_0+\sigma^a)\vec{u}^\delta)\|_{L^\infty}\|\nabla\vec{u}^d\|_{H^2}\big) \\ &\quad\lesssim \|\vec{u}^d\|^2_{H^3}(1+\|\sigma^a\|_{H^3})(\|\vec{u}^d\|_{H^3} + \|\vec{u}^a\|_{H^3}).\end{aligned}$$

Arguing similarly to (4.13), we have

$$\int_\Omega\partial^\alpha((\rho_0+\sigma^a)\vec{u}^d\cdot\nabla\vec{u}^a)\cdot\partial^\alpha\vec{u}^d \lesssim (1+\|\sigma^a\|_{H^3})\|\vec{u}^a\|_{H^4}\|\vec{u}^d\|^2_{H^3}.$$

Hence, the fifth integral will be bounded as follows

$$\int_\Omega \partial^\alpha((\rho_0+\sigma^a)(\vec{u}^\delta\cdot\nabla\vec{u}^d+\vec{u}^d\cdot\nabla\vec{u}^a))\cdot\partial^\alpha\vec{u}^d \lesssim (\|\vec{u}^d\|_{H^3}+\|\vec{u}^a\|_{H^4})\|\vec{u}^d\|_{H^3}^2. \tag{4.19}$$

Similarly, for the sixth integral, we have

$$\int_\Omega \partial^\alpha(\sigma^d\vec{u}^\delta\cdot\nabla\vec{u}^d)\cdot\partial^\alpha\vec{u}^d \lesssim \|\vec{u}^d\|_{H^3}^2\|\sigma^d\|_{H^3}(\|\vec{u}^d\|_{H^3}+\|\vec{u}^a\|_{H^3}),$$

and

$$\int_\Omega \partial^\alpha(\sigma^d\vec{u}^\delta\cdot\nabla\vec{u}^a)\cdot\partial^\alpha\vec{u}^d \lesssim \|\vec{u}^a\|_{H^4}(\|\vec{u}^d\|_{H^3}+\|\vec{u}^a\|_{H^3})\|\vec{u}^d\|_{H^3}\|\sigma^d\|_{H^3}.$$

These inequalities imply

$$\int_\Omega \partial^\alpha(\sigma^d(\partial_t\vec{u}^a+\vec{u}^\delta\cdot\nabla\vec{u}^\delta))\cdot\partial^\alpha\vec{u}^d \lesssim (1+\|\vec{u}^a\|_{H^4}^2+\|\vec{u}^d\|_{H^3}^4)(\|\vec{u}^d\|_{H^3}^2+\|\sigma^d\|_{H^3}^2). \tag{4.20}$$

Combining (4.15), (4.17), (4.18) and (4.19) and (4.20) gives us that

$$\begin{aligned}
&\frac{d}{dt}\big(\|\sigma^d\|_{H^4}^2+\|\vec{u}^d\|_{H^3}^2\big)+\|\nabla\vec{u}^d\|_{H^3}^2\\
&\leqslant C\nu\|\nabla\vec{u}^d\|_{H^3}^2+C_\nu\begin{pmatrix}1+\|\partial_t\sigma^a\|_{H^2}+\|\partial_t\vec{u}^a\|_{H^2}^2\\ +\|\sigma^a\|_{H^5}^2+\|\vec{u}^a\|_{H^4}^4+\|\vec{u}^d\|_{H^3}^4\end{pmatrix}(\|\sigma^d\|_{H^4}^2+\|\vec{u}^d\|_{H^3}^2)\\
&\qquad +C_\nu\big(\|\vec{S}_n^a\|_{H^3}^2+\|R_n^a\|_{H^4}^2\big).
\end{aligned} \tag{4.21}$$

Let ν be sufficiently small, we deduce (4.9). Proof of Lemma 4.1 is finished. □

4.3. **Nonlinear instability.** Based on the approximate solution $(\sigma^a,\vec{u}^a,q^a)$ in Lemma 4.1, for which n will be chosen later (see (4.27)), we now construct a family of solutions $(\sigma^\delta,\vec{u}^\delta)$ to (2.1) which are nonlinearly unstable. For any $\delta>0$, we define $(\sigma^\delta,\vec{u}^\delta)$ to be the unique solution of (2.1) with initial data $(\sigma^a(0),\vec{u}^a(0))$. We want to bound $(\sigma^d,\vec{u}^d)=(\sigma^\delta-\sigma^a,\vec{u}^\delta-\vec{u}^a)$ which is a solution of (4.8) satisfying $(\sigma^d(0),\vec{u}^d(0))=(0,\vec{0})$.

In what follows, the constants C_i are universal ones depending only on physical parameters, M,N and $\mathsf{c}_j (j\geqslant 1)$, being refered later.

Proof of Theorem 2.3. First, we begin the proof with the inequality

$$\|\vec{u}_{(\mathsf{N})}(t)\|_{L^2}\geqslant C_1F_{\mathsf{N}}(t)\quad \text{for all } t\geqslant 0. \tag{4.22}$$

Indeed, from (2.15), we obtain that

$$\|\vec{u}_{(\mathsf{N})}(t)\|_{L^2}^2=\sum_{i=j_m}^{\mathsf{N}}\mathsf{c}_i^2e^{2\lambda_it}\|\vec{u}_i\|_{L^2}^2+2\sum_{j_m\leqslant i<j\leqslant\mathsf{N}}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega\vec{u}_i\cdot\vec{u}_j. \tag{4.23}$$

It can be seen that

$$\begin{aligned}
\|\vec{u}_{(\mathsf{N})}(t)\|_{L^2}^2\geqslant &\sum_{j=j_m}^{\mathsf{N}}\mathsf{c}_j^2e^{2\lambda_jt}\|\vec{u}_j\|_{L^2}^2+2\sum_{j_m+1\leqslant i<j\leqslant\mathsf{N}}\mathsf{c}_i\mathsf{c}_je^{(\lambda_i+\lambda_j)t}\int_\Omega\vec{u}_i\cdot\vec{u}_j\\
&-|\mathsf{c}_{j_m}|\|\vec{u}_{j_m}\|_{L^2}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2}\Big)e^{(\lambda_{j_m}+\lambda_{j_m+1})t}.
\end{aligned}$$

By Cauchy-Schwarz's inequality, we obtain

$$2\sum_{j_m+1\leqslant i<j\leqslant \mathsf{N}} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega \vec{u}_i\cdot\vec{u}_j \geqslant -2\sum_{j_m+1\leqslant i<j\leqslant \mathsf{N}} |\mathsf{c}_i||\mathsf{c}_j|e^{(\lambda_i+\lambda_j)t}\|\vec{u}_i\|_{L^2}\|\vec{u}_j\|_{L^2}$$
$$\geqslant -e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2}\Big)^2.$$

This yields

$$\|\vec{u}_{(\mathsf{N})}(t)\|_{L^2}^2 \geqslant \sum_{j=j_m}^{\mathsf{N}} \mathsf{c}_j^2 e^{2\lambda_j t}\|\vec{u}_j\|_{L^2}^2 - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2}\Big)^2$$
$$-|\mathsf{c}_{j_m}|e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|\vec{u}_{j_m}\|_{L^2}\Big(\sum_{j=j_m+1}^{\mathsf{N}}|\mathsf{c}_j|\|\vec{u}_j\|_{L^2}\Big).$$

Due to the assumption (2.14), we deduce that

$$\|\vec{u}_{(\mathsf{N})}(t)\|_{L^2}^2 \geqslant \sum_{j=j_m}^{\mathsf{N}} \mathsf{c}_j^2 e^{2\lambda_j t}\|\vec{u}_j\|_{L^2}^2 - \frac{1}{4}\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|\vec{u}_{j_m}\|_{L^2}^2$$
$$-\frac{1}{2}\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|\vec{u}_{j_m}\|_{L^2}^2.$$

This yields

$$\|\vec{u}_{(\mathsf{N})}(t)\|_{L^2}^2 \geqslant \mathsf{c}_{j_m}^2\Big(e^{2\lambda_{j_m}t}-\frac{1}{2}e^{(\lambda_{j_m}+\lambda_{j_m+1})t}-\frac{1}{4}e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big)\|\vec{u}_{j_m}\|_{L^2}^2$$
$$+\sum_{j=j_m+1}^{\mathsf{N}}\mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|_{L^2}^2.$$

Notice that for all $t\geqslant 0$,

$$e^{2\lambda_{j_m}t}-\frac{1}{2}e^{(\lambda_{j_m}+\lambda_{j_m+1})t}-\frac{1}{4}e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\geqslant \frac{1}{4}e^{2\lambda_{j_m}t}.$$

Hence, we have (4.22).

Second, we estimate the existence time interval for $(\sigma^\delta,\vec{u}^\delta)$. Let ω be a small positive constant which assures the local-in-time existence. Let $T^\star$ (depending on δ) be the first time t such that

$$\begin{aligned}&\text{either } \|\partial_t\sigma^a(t)\|_{H^2}+\|\partial_t\vec{u}^a(t)\|_{H^2}+\|\sigma^a(t)\|_{H^5}+\|\vec{u}^a(t)\|_{H^4}=\frac{\omega}{2}\\ &\text{or } \|\sigma^d(t)\|_{H^4}+\|\vec{u}^d(t)\|_{H^3}=\frac{\omega}{2}.\end{aligned}\tag{4.24}$$

Note that $\sigma^d(0)=0$, $\vec{u}^d(0)=\vec{0}$ and that

$$\|\sigma^a(0)\|_{H^s}+\|\vec{u}^a(0)\|_{H^s}=O(\delta),$$

so $T^\star$ is well-defined for sufficiently small δ.

We now prove that $T^\delta \leqslant T^\star$. Suppose that $T^\delta > T^\star$, on one hand, we deduce from the construction of approximate solutions in Lemma 4.1 that

$$\begin{aligned}
\|\partial_t\sigma^a(t)\|_{H^2} + \|\partial_t\vec{u}^a(t)\|_{H^2} + \|\sigma^a(t)\|_{H^5} + \|\vec{u}^a(t)\|_{H^4} &\leqslant \sum_{k=1}^n \widetilde{C_{k,n}}\delta^k(\|\sigma^{\langle k\rangle}\|_{H^5} + \|\vec{u}^{\langle k\rangle}\|_{H^4}) \\
&\leqslant \sum_{k=1}^n \widehat{C_{k,n}}\delta^k G_{\mathsf{N}}(kt) \\
&\leqslant C\sum_{k=1}^n (\delta G_{\mathsf{N}}(t))^k \\
&\leqslant C\sum_{k=1}^n \mathsf{N}^k\varepsilon_0^k \leqslant \frac{\omega}{4}.
\end{aligned} \tag{4.25}$$

On the other hand, it follows from the definition of $T^\star$ (4.24) and the inequality (4.9) that

$$\frac{d}{dt}\big(\|\sigma^d\|_{H^4}^2 + \|\vec{u}^d\|_{H^3}^2\big) \leqslant C_0\Big(1+\frac{\omega}{2}+\frac{\omega^2}{2}+\frac{\omega^4}{8}\Big)\big(\|\sigma^d\|_{H^4}^2 + \|\vec{u}^d\|_{H^3}^2\big) + C_0\big(\|R_n^a\|_{H^4}^2 + \|\vec{S}_n^a\|_{H^3}^2\big).$$

Owing to (4.4), we further get

$$\frac{d}{dt}\big(\|\sigma^d\|_{H^4}^2 + \|\vec{u}^d\|_{H^3}^2\big) \leqslant C_2(1+\omega)^4\big(\|\sigma^d\|_{H^4}^2 + \|\vec{u}^d\|_{H^3}^2\big) + C_2\delta^{2(n+1)}G_{\mathsf{N}}(2(n+1)t). \tag{4.26}$$

Choosing n sufficiently large such that

$$C_2(\omega+1)^4 < (n+1)\Lambda < 2(n+1)\lambda_{\mathsf{M}}. \tag{4.27}$$

Hence, applying Gronwall's inequality to (4.26), we have

$$\begin{aligned}
&\|\sigma^d(t)\|_{H^4}^2 + \|\vec{u}^d(t)\|_{H^3}^2 \\
&\leqslant C_2\delta^{2(n+1)}\int_0^t e^{-C_2(1+\omega)^4(t-\tau)}\Big(\sum_{j=1}^{\mathsf{M}}|\mathsf{c}_j|e^{2(n+1)\lambda_j\tau} + \sum_{j=\mathsf{M}+1}^{\mathsf{N}}|\mathsf{c}_j|e^{(n+1)\Lambda\tau}\Big)d\tau \\
&\leqslant C_3\delta^{2(n+1)}\Big(\sum_{j=1}^{\mathsf{M}}|\mathsf{c}_j|e^{2(n+1)\lambda_j t} + \sum_{j=\mathsf{M}+1}^{\mathsf{N}}|\mathsf{c}_j|e^{(n+1)\Lambda t}\Big) \\
&\leqslant C_4\mathsf{N}\varepsilon_0^{2(n+1)}.
\end{aligned}$$

That implies

$$\|\sigma^d(t)\|_{H^4} + \|\vec{u}^d(t)\|_{H^3} \leqslant \sqrt{C_4\mathsf{N}}\varepsilon_0^{n+1} \leqslant \frac{\omega}{4}. \tag{4.28}$$

The two inequalities (4.25) and (4.28) imply a contradiction to the definition of $T^\star$ (4.24). Hence, we have $T^\delta \leqslant T^\star$.

Once we have that $T^\delta \leqslant T^\star$, we conclude the nonlinear instability. Choosing $t = T^\delta$, it thus follows from (4.1), (4.3) and (4.22) that

$$\begin{aligned}\|\vec{u}^a(T^\delta)\|_{L^2} &\geqslant \delta\|\vec{u}_{(\mathsf{N})}(T^\delta)\|_{L^2} - \sum_{k=2}^{n} \delta^k \|\vec{u}^{\langle k\rangle}(T^\delta)\|_{L^2} \\ &\geqslant C_1\delta F_{\mathsf{N}}(T^\delta) - \sum_{k=2}^{n} C_{k,n}\delta^k G_{\mathsf{N}}(kT^\delta) \\ &\geqslant C_1\delta F_{\mathsf{N}}(T^\delta) - \sum_{k=2}^{n} C_{k,n}(\delta G_{\mathsf{N}}(T^\delta))^k \\ &\geqslant C_1\varepsilon_0 - \sum_{k=2}^{m} C_{k,n}\mathsf{N}^k\varepsilon_0^k \\ &\geqslant \frac{C_1\varepsilon_0}{2}.\end{aligned} \tag{4.29}$$

Thanks to (4.29), we let $t = T^\delta$ in (4.28) to deduce

$$\|\vec{u}^\delta(T^\delta)\|_{L^2} \geqslant \|\vec{u}^a(T^\delta)\|_{L^2} - \|\vec{u}^d(T^\delta)\|_{L^2} \geqslant \frac{C_1\varepsilon_0}{2} - \sqrt{C_4\mathsf{N}\varepsilon_0^{n+1}} \geqslant \frac{C_1\epsilon_0}{4}. \tag{4.30}$$

The inequality (2.16) is proven. This ends the proof of Theorem 2.3. □

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