EXT-GROUPS, DOLD k-INVARIANT AND FORMALITY

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ABSTRACT. The aim of this paper is to study the formality of a complex in an abelian category by using k-invariants of Dold. Our main result is a criterion for the exact functor of abelian categories (enough projective and injective objects) which induces monomorphisms on Ext-groups.

CONTENTS

INTRODUCTION

In this paper, we study the formality of a chain complex in an abelian category by using the k-invariants defined by Dold [Dol60]. Let $\mathcal{A}$ be abelian category with enough projectives and injectives. A chain complex C in $\mathcal{A}$ is called *formal* if it is quasi-isomorphic to a complex whose differentials vanish. Denote by $H(C)$ the complex whose differentials vanish and $H(C)_i = H_i(C)$ for all i. Then, C is formal if and only if C is quasi-isomorphic to $H(C)$. Since we are interested in the complexes up to quasi-isomorphism, naturally, we consider the derived category. Denote by $\mathbf{D}^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. Then, a complex $C \in \mathbf{D}^b(\mathcal{A})$ is formal if and only if C is isomorphic to $H(C)$, in the category $\mathbf{D}^b(\mathcal{A})$. Recall the following classical result (see Corollary 2.4)

Theorem. *Let $\mathcal{A}$ be an abelian category. Suppose that the category $\mathcal{A}$ is hereditary, i.e., we have $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = 0$ for $i \geq 2$ and $A, B \in \mathcal{A}$. Then, each complex $C \in \mathbf{D}^b(\mathcal{A})$ is formal.*

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Our main theorem, Theorem 2.3, can be considered as a relative version of the above theorem.

Theorem (Theorem 2.3)**.** *Let $\mathcal{A}, \mathcal{B}$ be two abelian categories with enough projectives and injectives. Let $\phi : \mathcal{A} \to \mathcal{B}$ be an exact functor. The two following conditions on the functor ϕ are equivalent:*

(1) *The complex $C \in \mathbf{D}^b(\mathcal{A})$ is formal if the complex $\phi(C) \in \mathbf{D}^b(\mathcal{B})$ is formal.*
(2) *For $i \geq 2$, the maps $\mathrm{Ext}^i_{\mathcal{A}}(A, B) \to \mathrm{Ext}^i_{\mathcal{B}}(\phi(A), \phi(B))$ induced by ϕ are monomorphisms.*

A consequence of this main theorem is Theorem 3.2 which is a multivariable version of a result of W. van der Kallen [vdK15]. We note that the proof of van der Kallen in the one-variable case used spectral sequence. Another application of the main theorem is Proposition 3.3, which gives us a relation between the formality of a complex of strict polynomial functors and parametrized functors.

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1. Postnikov tower and k-invariant

In this section, we recall the Postnikov tower of a complex in an abelian category. Then, the k-invariant of Dold [Dol60] will be presented by using the language of derived category and its triangulated structure.

Fix abelian categories with enough projectives and injectives.

1.1. **Postnikov tower of a complex.** A morphism $f : C \to D$ of chain complexes is called *quasi-isomorphism* if the maps $H_n(f) : H_n(C) \to H_n(D)$ are isomorphisms. It follows from the definition that a complex C is acyclic if and only if the morphism $0 \to C$ is a quasi-isomorphism. We see that an isomorphism is a quasi-isomorphism and the composite of two quasi-isomorphisms is also a quasi-isomorphism. However, in general the existence of quasi-isomorphism $C \to D$ does not deduce the existence of a quasi-isomorphism $D \to C$.

Let two chain complexes C, D. We say that C is quasi-isomorphic to D if there are chain complexes $C = C_1, C_2, \ldots, C_{n-1}, C_n = D$ such that for all $1 \leq i \leq n-1$, there is a quasi-isomorphism $C_i \to C_{i+1}$ or $C_{i+1} \to C_i$. Then, the binary relation "quasi-isomorphic" is an equivalence relation in the class of chain complexes.

For an integer k and a complex C, we define a new complex $C[k]$ by setting $C[k]_n = C_{n+k}$ and $d_n^{C[k]} = (-1)^k d_{n+k}^C$.

We define *truncation functors* , see for example [Wei94, 1.2.7] or [KS06, Definition 12.3.1]. If C is a chain complex and n is an integer, denote by $\tau_{\geq n}(C)$ the subcomplex of C defined by

$$(\tau_{\geq n}(C))_i = \begin{cases} 0 & \text{if } i < n, \\ \ker\left(d_n^C\right) & \text{if } i = n, \\ C_i & \text{if } i > n. \end{cases}$$

The complex $\tau_{\geq n}(C)$ is called the *truncation of C below n*. The quotient complex $\tau_{\leq n}(C) := C/\tau_{\geq n+1}(C)$ is called the *truncation of C above n*. We obtain the functors $\tau_{\geq n} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}_{+}(\mathcal{A})$ and $\tau_{\leq n} : \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}_{-}(\mathcal{A})$. We see that if $n \leq m$ then $\tau_{\leq n} \circ \tau_{\leq m} = \tau_{\leq n}$. The proof of the following proposition is straightforward.

Proposition 1.1. *Let C be a chain complex.*

(1) *The natural morphism $H_i(\tau_{\geq n}(C)) \to H_i(C)$ is an isomorphism for $i \geq n$ and $H_i(\tau_{\geq n}(C)) = 0$ for $i < n$.*
   In particular, if $H_i(C) = 0$ for $i < n$ then the inclusion $\tau_{\geq n}(C) \to C$ is a quasi-isomorphism.
(2) *The natural morphism $H_i(C) \to H_i(\tau_{\leq n}(C))$ is an isomorphism for $i \leq n$ and $H_i(\tau_{\leq n}(C)) = 0$ for $i > n$.*
   In particular, if $H_i(C) = 0$ for $i > n$ then the projection $p_n : C \to \tau_{\leq n}(C)$ is a quasi-isomorphism.

Corollary 1.2. *Assume $f : C \to D$ is a morphism such that $H_i(f)$ is an isomorphism for $i < n$ and $H_i(D) = 0$ for $i \geq n$ then D is quasi-isomorphic to $\tau_{\leq n}(C)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & D \\ {\scriptstyle p_n^C}\downarrow & & \downarrow{\scriptstyle p_n^D} \\ \tau_{\leq n}(C) & \xrightarrow{\tau_{\leq n}(f)} & \tau_{\leq n}(D). \end{array}$$

Since $H_i(D) = 0$ for $i > n$, by Proposition 1.1, p_n^D is a quasi-isomorphism. Thus, it remains to show that $\tau_{\leq n}(f)$ is a quasi-isomorphism. For $i > n$, since $H_i(\tau_{\leq n}(C)) = 0 = H_i(\tau_{\leq n}(D))$, $H_i(f)$ is an isomorphism. We next suppose that $i \leq n$. From the above diagram, we have $H_i(\tau_{\leq n}(f)) \circ H_i(p_n^C) = H_i(p_n^D) \circ H_i(f)$. By Proposition 1.1, $H_i(p_n^C)$ and $H_i(p_n^C)$ are isomorphisms. Moreover, by hypotheses, $H_i(f)$ is an isomorphism. Hence, $H_i(\tau_{\leq n}(f))$ is an isomorphism. □

We also have a similar result to the above corollary for the functor $\tau_{\geq n}$.

By definition, for a chain complex C and an integer n, there is an exact sequence of complexes

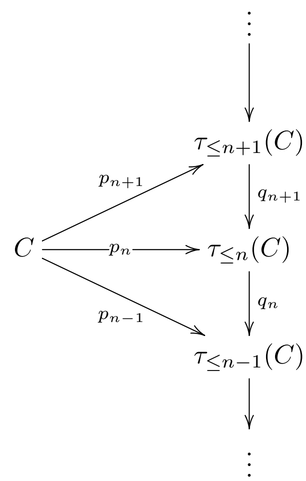
$$0 \to \tau_{\geq n+1}(C) \to C \xrightarrow{p_n} \tau_{\leq n}(C) \to 0.$$

Applying this short exact sequence for the complex $\tau_{\leq n}(C)$, there is a short exact sequence of complexes for each integer n

$$0 \to \tau_{\geq n}(\tau_{\leq n}(C)) \to \tau_{\leq n}(C) \xrightarrow{q_n} \tau_{\leq n-1}(C) \to 0 \tag{1}$$

where q_n is the composite morphism $\tau_{\leq n}(C) \xrightarrow{p_{n-1}} \tau_{\leq n-1}(\tau_{\leq n}(C)) \xrightarrow{\equiv} \tau_{\leq n-1}(C)$.

The Postnikov tower for a complex $C \in \mathbf{Ch}(\mathcal{A})$ is the commutative diagram:



By Proposition 1.1, the map $p_n : C \to \tau_{\leq n}(C)$ induces an isomorphism on i^{th} homology for $i \leq n$ and $H_i(\tau_{\leq n}(C)) = 0$ for $i > n$. Otherwise, by definition, $\ker(q_n) = \tau_{\geq n}(\tau_{\leq n}(C))$ is a subquotient of C with

$$(\tau_{\geq n}(\tau_{\leq n}(C)))_i = \begin{cases} \dfrac{C_{n+1}}{\ker\left(d^C_{n+1}\right)} & \text{if } i = n+1, \\ \ker\left(d^C_n\right) & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

We have a natural morphism $\tau_{\geq n}(\tau_{\leq n}(C)) \to H_n(C)[-n]$, which is a quasi-isomorphism. In general, the complex $\tau_{\leq n}(C)$ is not defined (up to quasi-isomorphism) from $\tau_{\leq n-1}(C)$ and $H_n(C)$.

1.2. **The derived category.** We are only interested in the complexes up to quasi-isomorphism. The quasi-isomorphisms are not isomorphisms, but one would like to consider them to be isomorphisms. Naturally, we consider the derived category. Bounded derived category of an abelian category together with its triangulated structure defined by A. Grothendieck and J-L. Verdier will be recalled briefly here.

Definition 1.3. Let $\mathcal{A}$ be an abelian category. The *bounded derived category of* $\mathcal{A}$, denoted by $\mathbf{D}^b(\mathcal{A})$, is defined to be the localization of category $\mathbf{K}^{-,b}(\mathcal{A})$ at the collection of quasi-isomorphisms, where $\mathbf{K}^{-,b}(\mathcal{A})$ ...

We call a *triangle* of category $\mathbf{D}^b(\mathcal{A})$ a diagram of type

$$X \to Y \to Z \to X[-1].$$

Let $X \to Y \to Z \to X[-1]$ and $X' \to Y' \to Z' \to X'[-1]$ be two triangles. A morphism of triangles is a commutative diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[-1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[-1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & X'[-1] \end{array}$$

Let $f : X \to Y$ be a morphism in $\mathbf{Ch}(\mathcal{A})$. We have a short exact sequence

$$0 \to Y \to \mathrm{Cone}(f) \to X[-1] \to 0$$

in $\mathbf{Ch}(\mathcal{A})$ where $\mathrm{Cone}(f)$ is the mapping cone of the morphism f, i.e., $\mathrm{Cone}(f)^n = Y^n \oplus X^{n+1}$ and $d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^{n+1}_X \end{pmatrix}$.

Definition 1.4. A triangle of $\mathbf{D}^b(\mathcal{A})$ is called *distinguished* if it is isomorphic to a triangle of form

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[-1].$$

Theorem 1.5 ([Wei94, Collaries 10.2.5, 10.4.3])**.** *Let $\mathcal{A}$ be an abelian category. The category $\mathbf{D}^b(\mathcal{A})$ equipped the distinguished triangles of the above definition is a triangulated category, i.e., it satisfies the following properties*

(TR0): *Each triangle of $\mathbf{D}^b(\mathcal{A})$ isomorphic to a distinguished triangle is a distinguished triangle.*

(TR1): *For each object X of $\mathbf{D}^b(\mathcal{A})$, the triangle $X \xrightarrow{\mathrm{Id}_X} X \to 0 \to X[-1]$ is distinguished.*

(TR2): *Each morphism $u : X \to Y$ of $\mathbf{D}^b(\mathcal{A})$ is contained in a distinguished triangle $X \xrightarrow{u} Y \to Z \to X[-1]$.*

(TR3): *A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[-1]$ of $\mathbf{D}^b(\mathcal{A})$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[-1] \xrightarrow{-u[-1]} Y[-1]$ is distinguished.*

(TR4): *For each pair of distinguished triangles:*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[-1], \qquad X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[-1]$$

and all morphisms $f : X \to X', g : Y \to Y'$ such that $g \circ u = u' \circ f$, there is a morphism $h : Z \to Z'$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[-1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[-1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & X'[-1] \end{array}$$

We don't need the octahedral axiom [Ver96, TRIV, page 94] in the sequel.

Proposition 1.6. *Let*

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence in $\mathbf{Ch}^{-,b}(\mathcal{A})$. There is a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to X[-1]$$

in $\mathbf{D}^b(\mathcal{A})$, and Z is isomorphic to $\mathrm{Cone}(f)$ in $\mathbf{D}^b(\mathcal{A})$.

1.3. **Dold k-invariants.** Recall that for each complex C and an integer n, there is a short exact sequence of complexes (1), moreover, the complex $\tau_{\geq n}(\tau_{\leq n}(C))$ is quasi-isomorphic to $H_n(C)[-n]$.

Proposition 1.7. *There is only one morphism* $\mathrm{k}^n(C) \in \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\tau_{\leq n-1}(C), H_n(C)[-n-1])$ *such that the below triangle is distinguished in* $\mathbf{D}^b(\mathcal{A})$

$$H_n(C)[-n] \to \tau_{\leq n}(C) \xrightarrow{q_n} \tau_{\leq n-1}(C) \xrightarrow{\mathrm{k}^n(C)} H_n(C)[-n-1]. \tag{2}$$

Moreover, k^n *is a natural transformation between the functors* $\tau_{\leq n-1}$ *and* $[-n-1]\circ H_n$.

Proof. Applying Proposition 1.6 for the short exact sequence (1), remark that $\tau_{\geq n}(\tau_{\leq n}(C)) \simeq H_n(C)[-n]$ in the category $\mathbf{D}^b(\mathcal{A})$, there is a morphism $\mathrm{k}^n(C) : \tau_{\leq n-1}(C) \to H_n(C)[-n-1]$ in the category $\mathbf{D}^b(\mathcal{A})$ such that we have the distinguished triangle (2). □

Definition 1.8. The morphism $\mathrm{k}^n(C) : \tau_{\leq n-1}(C) \to H_n(C)[-n-1]$ in the category $\mathbf{D}^b(\mathcal{A})$ is called n^{th} k-*invariant* of C.

Since

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\tau_{\leq n-1}(C), H_n(C)[-n-1]) = \mathrm{Ext}^{n+1}(\tau_{\leq n-1}(C), H_n(C))$$

the n^{th} k-invariant $\mathrm{k}^n(C)$ is an element of $\mathrm{Ext}^{n+1}(\tau_{\leq n-1}(C), H_n(C))$. By **(TR4)**, the complex $\tau_{\leq n}(C) \in \mathbf{D}^b(\mathcal{A})$ is well defined up to quasi-isomorphism from the n^{th} k-invariant $\mathrm{k}^n(C)$, but this quasi-isomorphism is not in general canonical. These k-invariants specify how to construct a complex $C \in \mathbf{D}^b(\mathcal{A})$ inductively from the homologies of C.

Note that if C is quasi-isomorphic to D then $\mathrm{k}^n(C) = \mathrm{k}^n(D)$ for all n.

Lemma 1.9. *Let* $\mathcal{A}, \mathcal{B}$ *be two abelian categories. Let* $\phi : \mathcal{A} \to \mathcal{B}$ *be an exact functor and* C *be an object of* $\mathbf{D}^b(\mathcal{A})$. *Then we have* $\phi(\mathrm{k}^n(C)) = \mathrm{k}^n(\phi(C))$.

Proof. By definition, there is a distinguished triangle in the category $\mathbf{D}^b(\mathcal{A})$

$$H_n(C)[-n] \to \tau_{\leq n}(C) \xrightarrow{q_n} \tau_{\leq n-1}(C) \xrightarrow{\mathrm{k}^n(C)} H_n(C)[-n-1].$$

Since the functor ϕ is exact, we have $H_n(\phi(C)) \simeq \phi(H_n(C))$ and $\tau_{\leq n}(\phi(C)) \simeq \phi(\tau_{\leq n}(C))$. By applying the exact functor ϕ, we obtain a below distinguished triangle in $\mathbf{D}^b(\mathcal{B})$

$$H_n(\phi(C))[-n] \to \tau_{\leq n}(\phi(C)) \xrightarrow{q_n} \tau_{\leq n-1}(\phi(C)) \xrightarrow{\phi(\mathrm{k}^n(C))} H_n(C)[-n-1].$$

Then we have $\mathrm{k}^n(\phi(C)) = \phi(\mathrm{k}^n(C))$. □

2. Formality and k-invariant

2.1. **Formality.** A complex is called *formal* if it is quasi-isomorphic to a complex whose differentials vanish. Denote by $H(C)$ the complex whose differentials vanish and $H(C)_i = H_i(C)$ for all i. Then, C is formal if and only if C is quasi-isomorphic to $H(C)$.

Acyclic complex is formal. A complex C is formal if $H_i(C) = 0$ for $i \neq n$, for some fixed n. If C_i or C_{i+1} vanishes for each i, then C is formal.

Proposition 2.1. *Let* $C \in \mathbf{D}^b(\mathcal{A})$ *and* n *be an integer. The complex* $\tau_{\leq n}(C)$ *is formal if and only if the complex* $\tau_{\leq n-1}(C)$ *is formal and the* k-*invariant* $\mathrm{k}^n(C) = 0$.

Proof. (⇒). Since the complex $\tau_{\leq n}(C)$ is formal, there exists an isomorphism $f : \tau_{\leq n}(C) \to H(\tau_{\leq n}(C)) = \tau_{\leq n}(H(C))$ in $\mathbf{D}^b(\mathcal{A})$. By applying the functor $\tau_{\leq n-1}$, we obtain an isomorphism

$$\tau_{\leq n-1}(f) : \tau_{\leq n-1}(C) \simeq \tau_{\leq n-1}(\tau_{\leq n}(H(C))) = \tau_{\leq n-1}(H(C)).$$

Then the complex $\tau_{\leq n-1}(C)$ is formal. The n^{th} k-invariant $\mathrm{k}^n(C)$ of the complex C is null based on the commutativity of the following diagram in the derived category $\mathbf{D}^b(\mathcal{A})$:

$$\begin{array}{ccccccc} H_n(C)[-n] & \longrightarrow & \tau_{\leq n}(C) & \longrightarrow & \tau_{\leq n-1}(C) & \xrightarrow{\mathrm{k}^n(C)} & H_n(C)[-n-1] \\ \| & & f\downarrow\simeq & & \tau_{\leq n-1}(f)\downarrow\simeq & & \| \\ H_n(C)[-n] & \longrightarrow & \tau_{\leq n}(H(C)) & \xrightarrow[q_n]{} & \tau_{\leq n-1}(H(C)) & \xrightarrow{0} & H_n(C)[-n-1]. \end{array}$$

(⇐). Since $\tau_{\leq n-1}(C)$ is formal, there is an isomorphism $g : \tau_{\leq n-1}(C) \xrightarrow{\simeq} \tau_{\leq n-1}(H(C))$ in $\mathbf{D}^b(\mathcal{A})$. Moreover, since $\mathrm{k}^n(C) = 0$, the following diagram is commutative where the two lines are distinguished triangles in the category $\mathbf{D}^b(\mathcal{A})$:

$$\begin{array}{ccccccc} \tau_{\leq n-1}(C)[-1] & \xrightarrow{0} & H_n(C)[-n] & \longrightarrow & \tau_{\leq n}(C) & \longrightarrow & \tau_{\leq n-1}(C) \\ \downarrow g[-1] & & \| & & & & \downarrow g \\ \tau_{\leq n-1}(H(C))[-1] & \xrightarrow{0} & H_n(C)[-n] & \longrightarrow & \tau_{\leq n}(H(C)) & \xrightarrow{q_n} & \tau_{\leq n-1}(H(C)). \end{array}$$

By **(TR4)**, there is an isomorphism $\tau_{\leq n}(C) \simeq \tau_{\leq n}(H(C))$ in the category $\mathbf{D}^b(\mathcal{A})$. By definition, the complex $\tau_{\leq n}(C)$ is formal. □

Proposition 2.2. *Let $C \in \mathbf{D}^b(\mathcal{A})$. The complex C is formal if and only if $\mathrm{k}^n(C) = 0$ for all n.*

Proof. (⇒). Let n be an integer. Since the complex C is formal, the complexes $\tau_{\leq n}(C)$ are also formal. By Proposition 2.1, the k-invariants $\mathrm{k}^n(C)$ are null.

(⇐). Since C is an object of $\mathbf{D}^b(\mathcal{A})$, its homology is bounded. Then there is a strictly positive integer m such that $p_m : C \to \tau_{\leq m}(C)$ is an isomorphism in $\mathbf{D}^b(\mathcal{A})$ and that the complex $\tau_{\leq -m}(C)$ is acyclic. Then, the complex $\tau_{\leq -m}(C)$ is formal. Moreover, by hypothesis, $\mathrm{k}^n(C) = 0$ for all n. By using Proposition 2.1 and an induction, the complex $\tau_{\leq m}(C)$ is formal. Hence, the complex C is formal. □

2.2. **Main result.** We can now formulate our main result.

Theorem 2.3. *Let $\mathcal{A}, \mathcal{B}$ be two abelian categories. Let $\phi : \mathcal{A} \to \mathcal{B}$ be an exact functor. The two following conditions on the functor ϕ are equivalent:*

1. *The complex $C \in \mathbf{D}^b(\mathcal{A})$ is formal if the complex $\phi(C) \in \mathbf{D}^b(\mathcal{B})$ is formal.*
2. *For $i \geq 2$, the maps $\mathrm{Ext}^i_{\mathcal{A}}(A, B) \to \mathrm{Ext}^i_{\mathcal{B}}(\phi(A), \phi(B))$ induced by ϕ are monomorphisms.*

As an application of this theorem, we obtain a multivariable version of a result of W. van der Kallen [vdK15] in the following section.

Before proving Theorem 2.3, we remark that, under the hypothesis of the theorem, the complex $\phi(C)$ is formal if the complex C is too. Indeed, since the complex C is formal, $C \simeq H(C)$. We obtain an isomorphism $\phi(C) \simeq \phi(H(C))$. Otherwise, it follows from the exactness of ϕ that the homology of the complex $\phi(C)$ is equal to $\phi(H(C))$. Then, we have an isomorphism $\phi(C) \simeq H(\phi(C))$ that we wish.

Proof of Theorem 2.3. (2) ⇒ (1). Since C is an object of $\mathbf{D}^b(\mathcal{A})$, there are integers i_1, i_2 such that $H_n(C) = 0$ for $n \leq i_1$ or $n \geq i_2$. Therefore, the morphism $p_n : C \to \tau_{\leq n}(C)$ is an isomorphism if $n \geq i_2$ and the complex $\tau_{\leq n}(C)$ is acyclic if $n \leq i_1$. In particular, the complex $\tau_{\leq n}(C)$ is formal if $n \leq i_1$, and if the complex $\tau_{\leq n}(C)$ is formal for some $n \geq i_2$ then the complex C is formal. It suffices to show that the complex $\tau_{\leq n}(C)$ is formal if the complex $\tau_{\leq n-1}(C)$ is formal.

We assume that $\tau_{\leq n-1}(C)$ is formal with $n-1 \geq i_1$. Then, there is an isomorphism $\tau_{\leq n-1}(C) \simeq \bigoplus_{i=i_1}^{n-1} H_i(C)[-i]$ in the category $\mathbf{D}^b(\mathcal{A})$. Since the functor ϕ is exact, we have an isomorphism $\tau_{\leq n-1}(\phi(C)) \simeq \bigoplus_{i=i_1}^{n-1} H_i(\phi(C))[-i]$ in the category $\mathbf{D}^b(\mathcal{B})$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(\tau_{\leq n-1}(C), H_n(C)[-n-1]) & \xrightarrow{\phi} & \mathrm{Hom}_{\mathbf{D}^b(\mathcal{B})}(\tau_{\leq n-1}(\phi(C)), H_n(\phi(C))[-n-1]) \\ \downarrow \simeq & & \downarrow \simeq \\ \bigoplus_{i=i_1}^{n-1} \mathrm{Ext}_{\mathcal{A}}^{n+1-i}(H_i(C), H_n(C)) & \xrightarrow{\phi_*} & \bigoplus_{i=i_1}^{n-1} \mathrm{Ext}_{\mathcal{B}}^{n+1-i}(H_i(\phi(C)), H_n(\phi(C))). \end{array}$$

By Assertion (2), the map ϕ_* is a monomorphism. Otherwise, since the complex $\phi(C)$ is formal, by Proposition 2.2, we have $\mathrm{k}^n(\phi(C)) = 0$. By combining with Lemma 1.9, we then have $\mathrm{k}^n(C) = 0$. By Proposition 2.1, the complex $\tau_{\leq n}(C)$ is formal.

(1) ⇒ (2). Let $e \in \mathrm{Ext}_{\mathcal{A}}^i(A, B)$ such that $\phi_*(e) \in \mathrm{Ext}_{\mathcal{B}}^i(\phi(A), \phi(B))$ is null. Since $\mathrm{Ext}_{\mathcal{A}}^i(A, B) \simeq \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(A[0], B[-i])$, we can see e as a morphism $A[0] \to B[-i]$ in $\mathbf{D}^b(\mathcal{A})$. Then there is a distinguished triangle in the category $\mathbf{D}^b(\mathcal{A})$

$$B[-i+1] \to C \to A[0] \xrightarrow{e} B[i].$$

Since the functor ϕ is exact, we obtain a distinguished triangle in $\mathbf{D}^b(\mathcal{B})$

$$\phi(B)[-i+1] \to \phi(C) \to \phi(A)[0] \xrightarrow{\phi_*(e)} \phi(B)[i].$$

Since $\phi_*(e) = 0$, the complex $\phi(C)$ is formal. By Assertion (1), the complex C is also formal. Then the morphism e is null. □

The following corollary is a direct consequence of Theorem 2.3. This result is classic and we can see it in [KS06, Corollary 13.1.20] another proof of this result. Moreover, Corollary 2.4 explains the reason that we have the hypothesis "$i \geq 2$" in Assertion (2) of Theorem 2.3.

Corollary 2.4. *Let $\mathcal{A}$ be an abelian category. Suppose that the category $\mathcal{A}$ is hereditary i.e., we have $\mathrm{Ext}_{\mathcal{A}}^i(A, B) = 0$ for $i \geq 2$ and $A, B \in \mathcal{A}$. Let $C \in \mathbf{D}^b(\mathcal{A})$. There exists an isomorphism in $\mathbf{D}^b(\mathcal{A})$:*

$$C \simeq \bigoplus_{i \in \mathbb{Z}} H_i(C)[-i]. \tag{3}$$

Proof. We define an endofunctor $\phi : \mathcal{A} \to \mathcal{A}$ by $\phi(A) = 0$ for all $A \in \mathcal{A}$. It is evident that ϕ is an exact functor and that the complex $\phi(C)$ is formal. Moreover, since the category is hereditary, the maps $\mathrm{Ext}_{\mathcal{A}}^i(A, B) \to \mathrm{Ext}_{\mathcal{A}}^i(\phi A, \phi B)$ induced by ϕ are monomorphisms for $i \geq 2$. By Theorem 2.3, the complex C is formal. Then the complex C is isomorphic to its homology, this finishes the proof. □

3. Application for strict polynomial functors

Fix a field $\Bbbk$ of positive characteristic p. Let $\mathcal{V}_\Bbbk$ denote the category of the $\Bbbk$-vector spaces of finite dimension and the linear maps. For each natural number d, $\Gamma^d\mathcal{V}_\Bbbk$ is the Schur category whose objects are those of the category $\mathcal{V}_\Bbbk$ and whose morphisms are given by $\mathrm{Hom}_{\Gamma^d\mathcal{V}_\Bbbk}(V,W) := \Gamma^d\,\mathrm{Hom}(V,W)$. We know that $\Gamma^d\mathcal{V}_\Bbbk$ is a $\Bbbk$-linear category.

For each n-tuple $\mathbf{d} = (d_1,\dots,d_n)$ of natural numbers, define the category $\Gamma^{\mathbf{d}}\mathcal{V}_\Bbbk$ the tensor product $\bigotimes_{i=1}^n \Gamma^{d_i}\mathcal{V}_\Bbbk$. This is a $\Bbbk$-linear category.

Definition 3.1. The category of the $\Bbbk$-linear functors from $\Gamma^{\mathbf{d}}\mathcal{V}_\Bbbk$ to $\mathcal{V}_\Bbbk$ is called the *category of homogeneous strict polynomial functors* of degree $\mathbf{d}$, denoted by $\mathcal{P}_{\mathbf{d}}$.

Then, $\mathcal{P}_{\mathbf{d}}$ is a $\Bbbk$-linear abelian category. For more details we refer the reader to [FS97],[SFB97] [Tou10], [Tou14] and the references given there.

For a tuple of natural numbers $\mathbf{m} = (m_1,\dots,m_n)$, denote by $\mathrm{GL}_{\mathbf{m},\Bbbk}$ the product of the group schemes $\mathrm{GL}_{m_i,\Bbbk}$ for $i = 1,\dots,n$ and by $\Bbbk^{\mathbf{m}}$ the object $(\Bbbk^{m_1},\dots,\Bbbk^{m_n})$. If $F \in \mathcal{P}_{\mathbf{d}}$, the vector space $F(\Bbbk^{\mathbf{m}})$ is canonically endowed with an action of $\mathrm{GL}_{\mathbf{m},\Bbbk}$. Moreover, A. Touzé proved [Tou10, Lemma 2.3] that the evaluation map

$$\mathrm{Ext}^*_{\mathcal{P}_{\mathbf{d}}}(F,G) \to \mathrm{Ext}^*_{\mathrm{GL}_{\mathbf{m},\Bbbk}}(F(\Bbbk^{\mathbf{m}}), G(\Bbbk^{\mathbf{m}})).$$

is an isomorphism if $\mathbf{m} \geq \mathbf{d}$, i.e., $m_i \geq d_i, i = 1,\dots,n$. This allows us to perform Ext-computations in the category of strict polynomial functor, in which computations are easier.

For each $F \in \mathcal{P}_{\mathbf{d}}$, denote by $F^{(r)} \in \mathcal{P}_{p^r\mathbf{d}}$ the r^{th} Frobenius twist of F, see [FS97], [SFB97]. If M is a $\mathrm{GL}_{\mathbf{m},\Bbbk}$-module, denote by $M^{[r]}$ the r^{th} Frobenius twist of M, see [Jan03, Section I.9.10]. There is an isomorphism $F^{(r)}(\Bbbk^{\mathbf{m}}) \simeq F(\Bbbk^{\mathbf{m}})^{[r]}$ natural in F.

Theorem 3.2. *Let $C \in \mathbf{Ch}(\mathcal{P}_{\mathbf{d}})$ be a finite complex and r be a natural number. The complex C is formal if the complex $C^{(r)}$ is formal.*

Proof. Denote by $\phi : \mathcal{P}_{\mathbf{d}} \to \mathcal{P}_{p^r\mathbf{d}}$ the functor $F \mapsto F^{(r)}$. This functor is exact.

Let $\mathbf{m} = (m_1,\dots,m_n)$ be a tuple of natural numbers such that $m_i \geq d_i$ for $i = 1,\dots,n$. The following diagram is commutative

$$\begin{array}{ccc}
\mathrm{Ext}^*_{\mathcal{P}_{\mathbf{d}}}(F,G) & \xrightarrow{\quad (\mathbf{??}) \quad} & \mathrm{Ext}^*_{\mathcal{P}_{p^r\mathbf{d}}}\left(F^{(r)}, G^{(r)}\right) \\
\downarrow{\scriptstyle (\dagger_1)} & & \downarrow{\scriptstyle (\dagger_2)} \\
\mathrm{Ext}^*_{\mathrm{GL}_{\mathbf{m},\Bbbk}}(F(\Bbbk^{\mathbf{m}}), G(\Bbbk^{\mathbf{m}})) & \xrightarrow{\quad (\dagger_3) \quad} & \mathrm{Ext}^*_{\mathrm{GL}_{\mathbf{m},\Bbbk}}\left(F(\Bbbk^{\mathbf{m}})^{[r]}, G(\Bbbk^{\mathbf{m}})^{[r]}\right).
\end{array}$$

where the vertical arrows $(\dagger_1),(\dagger_2)$ are induced by the evaluation functor on $\Bbbk^{\mathbf{m}}$. Since $\mathbf{m} \geq \mathbf{d}$, the maps $(\dagger_1),(\dagger_2)$ are isomorphisms. Moreover, by [Jan03, Section II.10.16], the map $(\dagger_3)$ is also an monomorphism. Thus, the map (**??**) is an monomorphisms. ☐

The one variable version of this theorem was proved by W. van der Kallen [vdK15]. This is an important step to study the effect of the Frobenius twist on the Ext-groups in the category of strict polynomial functors, i.e., to study Ext-groups of form $\mathrm{Ext}^*_{\mathcal{P}_{p^r\mathbf{d}}}\left(F, G^{(r)}\right)$, see [Tou13], [Cha15].

As another application of our main theorem 2.3, we give a relation between the formality of a complex of strict polynomial functors and parametrized functors.

Proposition 3.3. *Let $C \in \mathbf{Ch}(\mathcal{P}_{\mathbf{d}})$ be a finite complex and r be a natural number. The three following assertions are equivalent.*

(1) *The complex C is formal.*
(2) *The complex $C_{\mathbf{V}}$ is formal for all n-tuples $\mathbf{V}$ of objects of $\mathcal{V}_{\Bbbk}$.*
(3) *There is a n-tuple $\mathbf{V}$ of objects of $\mathcal{V}_{\Bbbk}$ such that $V_i \neq 0$ for all i and the complex $C_{\mathbf{V}}$ is formal.*

Proof. The implication (2) $\Rightarrow$ (3) is obvious. Since the functor $(-)_{\mathbf{V}}$ is exact, we deduce (1) $\Rightarrow$ (2). It remains to show (3) $\Rightarrow$ (1). Let $\mathbf{V}$ be a n-tuple of objects of $\mathcal{V}_{\Bbbk}$ such that $V_i \neq 0$ for all i. Since V_i is non-null, $\Bbbk$ is a direct factor of V_i. Then, the functor F is a direct factor of the functor $F_{\mathbf{V}}$ for all strict polynomial functors F. Thus, the maps $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(n)}(F, G) \to \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(n)}(F_{\mathbf{V}}, G_{\mathbf{V}})$ are always monomorphisms. By Theorem 2.3, the complex C is formal. □

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