AN ANALYTIC PROOF OF GUI–LI'S DIFFERENTIAL INEQUALITY

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Abstract. In a recent work due to C. Gui and Q. Li (*Math. Z.* **305** (2023) Art. 40), the following integral inequality
$$\int_0^{+\infty} e^{u(r)}dr \leq \pi$$
is proved for any radial C^2-solution u to the differential inequality
$$\Delta u + e^{2u} \leq 0 \quad \text{in } \mathbf{R}^2.$$
However, the argument provided in the paper is purely geometric. In this short note, we provide a purely analytic proof for the above inequality, hence partly answering Question 8.6 in the work of Gui and Li. In fact, we show that the inequality remains valid for any radial solution to the differential inequality in the punctured space $\mathbf{R}^2 \setminus \{0\}$. Comments on higher dimensional spaces are also made.

1. Introduction

Bishop's volume comparison theorem in differential geometry is a classical theorem that provides the estimate
$$\mathrm{vol}_g(M) \leq \mathrm{vol}(\mathbf{S}^n) \tag{1.1}$$
for any compact n-dimensional Riemannian manifold (M, g) whose the Ricci curvature satisfies
$$\mathrm{Ric}_g \geq (n-1)g;$$
see [Bis63], see also [Bes87, page 16]. Here and always, $\mathbf{S}^n$ denotes the unit n-sphere in $\mathbf{R}^{n+1}$ with the standard metric, namely the constant sectional curvature is 1. Without the compactness, (1.1) is not necessarily true. For e.g., with respect to the metric $g = e^{2u}\delta$, where δ denotes the standard Euclidean metric and u is any C^2-solution to
$$\Delta u + e^{2u} = 0 \quad \text{in } \mathbf{R}^2, \tag{1.2}$$
the volume of the conformal flat Riemannian manifold $(\mathbf{R}^2, g)$ enjoys either $\mathrm{vol}_g(\mathbf{R}^2) = 4\pi$ or $\mathrm{vol}_g(\mathbf{R}^2) = +\infty$. See [CL91, CW94] and related references. Geometrically, the Gaussian curvature of $(\mathbf{R}^2, g)$ is equal to 1 because
$$K_g = \Big(-\Delta u + K_\delta\Big)e^{-2u} = 1,$$
thanks to the flatness of $(\mathbf{R}^2, \delta)$; and as one is in two dimension, it is well-known that $\mathrm{Ric}_g = K_g g$. Thus, we have $\mathrm{Ric}_g = g = (2-1)g$. In the language of PDE, the above result simply reads
$$\int_{\mathbf{R}^2} e^{2u} dx_1 dx_2 \in \Big\{4\pi, +\infty\Big\}$$
with $x = (x_1, x_2)$. See Appendix B for an example of u so that the integral above is infinite.

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In a recent paper by Gui and Li, see [GL23], various geometric inequality on the conformal flat manifold $(\mathbf{R}^2, e^{2u}\delta)$ are derived, here instead of considering the PDE (1.2) the authors consider u being any C^2-solution to the following differential inequality

$$\Delta u + e^{2u} \leq 0 \quad \text{in } \mathbf{R}^2. \tag{1.3}$$

Among other results, Gui and Li show that if u is a C^2-radial solution to (1.3), then there holds

$$\text{vol}_g(\mathbf{R}^2) \leq 4\pi = \text{vol}(\mathbf{S}^2), \tag{1.4}$$

namely, Bishop's estimate (1.1) is true in this non-compact case. In other words, the alternative $\text{vol}_g(\mathbf{R}^2) = +\infty$ never occurs if u is a radial solution to the equation (1.2). Apart from the Gaussian curvature point of view, see [Bes87, Theorem 1.159(f) and section 1.119(a)], which reads

$$K_g = -(\Delta u)e^{-2u} \geq 1,$$

it is worth recalling that the presence of (1.3) is equivalent to saying that the Ricci curvature of $(\mathbf{R}^2, g)$ is bounded from below by 1, namely $\text{Ric}_g \geq g$, see [GL23, equation (6.6)] for even higher dimensions. So, the above result of Gui and Li extends Bishop's estimate to the case of non-compact manifolds in the radial setting. It turns out that limiting to the class of radial functions is necessary for (1.4) to hold because it is known that there is a solution u to (1.3) such that the conformal metric $e^{2u}\delta$ does not enjoy (1.4), in fact, $\text{vol}_g(\mathbf{R}^2)$ can be arbitrary large, see [Lyt23, Proposition 1.2].

To prove (1.4), Gui and Li first establish the following two-side inequalities

$$A(r)\Big(\int_{\mathbf{R}^2} e^{2u}dx - A(r)\Big) \leq l(r)^2 \leq 4\pi A(r) - A(r)^2 \tag{1.5}$$

where we denote by

$$A(r) = \int_{B_r} e^{2u}dx \quad \text{and} \quad l(r) = \int_{\partial B_r} e^u ds$$

the conformal perimeter and conformal volume of the ball B_r of radius $r > 0$ in $\mathbf{R}^2$, respectively. Assuming the finite of the integral $\int_{\mathbf{R}^2} e^{2u}dx$, which is also known as $\text{vol}_g(\mathbf{R}^2)$ or simply $A(+\infty)$, it now follows from (1.5) that

$$A(r)\Big(\int_{\mathbf{R}^2} e^{2u}dx - A(r)\Big) \leq 4\pi A(r) - A(r)^2$$

which, by canceling $A(r)$, then yields

$$\int_{\mathbf{R}^2} e^{2u}dx \leq 4\pi,$$

which is exactly the desired estimate (1.4). Interestingly, it is also shown in [GL23, page 20] that (1.4) can be derived from either of the two inequalities in (1.5). It is worth noting that the upper bound for l^2 in (1.5) can be though of as the reverse Alexandrov–Bol inequality, or the reverse isoperimetric inequality, because the classical Alexander–Bol inequality states

$$l(\partial\Omega)^2 \geq 4\pi A(\Omega) - A(\Omega)^2$$

for any Riemannian surface Ω with the Gaussian curvature ≤ 1, see [Ban76, Suz92, Top99]. (The two quantities $l(\partial\Omega)$ and $A(\Omega)$ are the conformal length of $\partial\Omega$ and conformal volume of Ω.) Hence, the lower bound for l^2 in (1.5) can be though of as an Alexandrov–Bol type inequality for higher Gaussian curvature since the Gaussian curvature of $(\mathbf{R}^2, e^{2u}\delta)$ is now ≥ 1. This can also be considered as the Lévy–Gromov isoperimetric inequality, see [Gro80].

To establish the lower bound for l^2 in (1.5), the authors exploit a sophisticated argument involving the well-known Heintze–Karcher inequality, see [HK78], and a key ordinary differential inequality, see (1.7), that we are about to describe. To be more precise, let u be any function satisfying

$$u'' + \frac{u'}{r} + e^{2u} \leq 0 \quad \text{for } r > 0, \tag{1.6}$$

then following interesting inequality

$$\int_0^{+\infty} e^{u(r)} dr \leq \pi \tag{1.7}$$

holds; see [GL23, Proposition 5.1]. The inequality (1.7) is sharp in the sense that the right hand side π cannot be replaced by any smaller number. This can be easily verified by testing the function

$$u_{\text{reg}}(r) = \log \frac{2}{1+r^2} \quad \text{for } r \geq 0.$$

This function u_{reg} is of class $C^2(\mathbf{R})$ and fulfills

$$u''_{\text{reg}} + \frac{u'_{\text{reg}}}{r} + e^{2u_{\text{reg}}} = 0.$$

In addition, there holds

$$\int_0^{+\infty} e^{u_{\text{reg}}(r)} dr = \pi.$$

We note that the integral in (1.7) can be made arbitrarily close to 0. This can be verified by using the function $\log(2\varepsilon/(1+r^2))$ with $\varepsilon \in (0,1)$.

The geometric meaning of (1.7) is that the conformal distance

$$\text{dist}_g(0,x) \leq \pi \quad \text{for any } x \in \mathbf{R}^2.$$

In particular, by the triangle inequality, the conformal diameter of $(\mathbf{R}^2, g)$ fulfills

$$\text{diam}_g(\mathbf{R}^2) \leq 2\pi. \tag{1.8}$$

In fact, it is proved in [GL23, Theorem 1.5 and Proposition 6.1] that $\text{diam}_g(\mathbf{R}^n) \leq \pi$ which is sharp. This is closely related to the Bonnet–Myers theorem for complete manifolds, see [Mye41]. However, our situation is different because $(\mathbf{R}^2, g)$ is no longer complete, see Lemma A.1. It is worth noting that without the symmetry of u, generally we do not expect (1.8). Again, we refer the reader to [GL23, Theorem 1.4] for a sufficient condition.

It is worth emphasizing that (1.7) is also related to the powerful sphere covering inequality and its dual and singular discovered in [GM18], in [BGJM19], and in [GHM20]. These inequalities are stated for domains of the Gaussian curvature ≤ 1. However, with help of the first inequality in (1.5), a reverse sphere covering inequality and its dual in the radial setting are obtained; see [GL23, Theorems 1.9 and 1.10].

In [GL23], the key inequality (1.7) was proved by geometric argument which mimics the idea of the proof of the Bonnet–Myers theorem. This procedure is possible, although $(\mathbf{R}^2, g)$ is not necessary complete, due to the radial symmetry of the conformal factor in the underlying metric g. The motivation of writing this note comes from a comment in [GL23], which shows the lack of an analytic argument for the proof of (1.7). In this note, on one hand, we indeed provide an analytic proof for (1.7), on the other hand, we slightly improve (1.7) for a larger class of functions satisfying (1.6).

Toward a possible generalization of (1.7), we observe from the geometric proof given in [GL23] that the function u satisfying (1.6) needs to be defined everywhere in $[0, +\infty)$. However, it turns out that (1.7) remains valid for any function u defined in $(0, +\infty)$. As the main finding of our note, let us state this as a theorem.

Theorem 1.1. *Let $u : (0,+\infty) \to \mathbf{R}$ be any C^2-function satisfying (1.6), namely*

$$u'' + \frac{u'}{r} + e^{2u} \leq 0 \quad \text{for } r > 0,$$

then the inequality (1.7) holds, namely

$$0 < \int_0^{+\infty} e^{u(r)} dr \leq \pi.$$

Moreover, the range $(0,\pi]$ in the inequality is optimal in the sense that the integral $\int_0^{+\infty} e^{u(r)} dr$ can be any number in $(0,\pi]$.

First we note that the upper bound π in the inequality is sharp, which can be easily verified by testing the function

$$u_{\text{sing}}(r) = \log 2 - \log\Big(r^{1-\sqrt{2}/2}(2 + r^{\sqrt{2}})\Big) \quad \text{with } r > 0.$$

This function is obviously singular at 0 and still solves

$$u''_{\text{sing}} + \frac{u'_{\text{sing}}}{r} + e^{2u_{\text{sing}}} = 0 \quad \text{for } r > 0,$$

but certainly we have

$$\int_0^{+\infty} e^{u_{\text{sing}}(r)} dr = \pi.$$

Now we let $\varepsilon \in (0,1)$ be arbitrary. By considering the function

$$u_0(r) = \log\varepsilon + u_{\text{sing}}(r) = \log(2\varepsilon) - \log\Big(r^{1-\sqrt{2}/2}(2 + r^{\sqrt{2}})\Big) \quad \text{with } r > 0$$

we know that the above function u_0 is well-defined, thanks to $\varepsilon > 0$, and still satisfies (1.6) because

$$u''_0 + \frac{u'_0}{r} + e^{2u_0} = u''_{\text{sing}} + \frac{u'_{\text{sing}}}{r} + \varepsilon^2 e^{2u_{\text{sing}}} = (\varepsilon^2 - 1)e^{2u_{\text{sing}}} < 0,$$

thanks to $\varepsilon^2 < 1$. In addition, there holds

$$\int_0^{+\infty} e^{u_0(r)} dr = \varepsilon \int_0^{+\infty} e^{u_{\text{sing}}(r)} dr = \varepsilon\pi.$$

Hence, this shows that the integral $\int_0^{+\infty} e^{u(r)} dr$ can be any number in $(0,\pi]$.

Our last comment concerns the higher dimensions. Again it was proved in [GL23] that under the condition $\text{Ric}_g \geq (n-1)g$, we still have

$$\text{diam}_g(\mathbf{R}^n) \leq \pi$$

for any $n \geq 3$. If we regard the integral in (1.7) is the conformal distance between 0 and infinity, then we know that (1.7) remains true in $\mathbf{R}^n$ with $n \geq 3$. In other words, in the radial setting a suitable bound on the Ricci curvature is enough to gain (1.7). We shall revisit this further in section 3.

2. **Proof**

It suffices to provide an analytic proof of the inequality in Theorem 1.1. Let $u \in C^2(0,+\infty)$ be a non-trivial function satisfying (1.6). For clarity, we divide our proof into several steps as follows.

Step 1. (A simpler version of (1.6).) Since the function u is defined only in $(0,+\infty)$, it is more convenient to use the following change of variable

$$r = e^s \quad \text{with } s \in \mathbf{R},$$

or equivalently $s = \log r$ with $r > 0$. Then we define

$$v(s) = u(r) \quad \text{for } s \in \mathbf{R}.$$

Then $v \in C^2(\mathbf{R})$ and by direct calculation we easily get

$$u'(r) = v'(s)e^{-s} \quad \text{and} \quad u''(r) = (v''(s) - v'(s))e^{-2s}.$$

From this and (1.6) we arrive at

$$v'' + e^{2(v+s)} \le 0 \quad \text{in } \mathbf{R} \tag{2.1}$$

and a change of variable leads to

$$\int_0^{+\infty} e^{u(r)} dr = \int_{-\infty}^{+\infty} e^{v(s)+s} ds.$$

Now we define a function w given by

$$w(s) = v(s) + s.$$

Obviously, $w \in C^2(\mathbf{R})$ and from (2.1) we get

$$w'' + e^{2w} \le 0 \quad \text{in } \mathbf{R}. \tag{2.2}$$

Our aim is to show that

$$\int_{-\infty}^{+\infty} e^{w(s)} ds \le \pi.$$

Thanks to $w \in C^2(\mathbf{R})$ and because $w'' < 0$ in $\mathbf{R}$, see (2.2) above, we know that w' is strictly decreasing in $\mathbf{R}$. Hence, either w' has some zero or w' has a fixed sign, namely, one of the following three alternatives occurs:

- either w' is sign-changing (due to the strict monotonicity),
- or $w' > 0$ everywhere in $\mathbf{R}$,
- or $w' < 0$ everywhere in $\mathbf{R}$.

Step 2. (Assuming the function w' has a fixed sign.) As discussed earlier, if w' has a fixed sign, then either $w' > 0$ or $w' < 0$ everywhere. We show that this is not the case.

Substep 2.1. We first rule out the case $w' > 0$ everywhere in $\mathbf{R}$. Indeed, by contradiction we assume $w' > 0$ everywhere. Consequently, w is monotone increasing in $\mathbf{R}$, so is the function e^{2w}. In particular, there holds

$$e^{2w(s)} \ge e^{2w(0)} \quad \text{for all } s \ge 0.$$

Now making use of (2.2) and the above estimate gives

$$w'(s) - w'(0) = \int_0^s w''(\tau)d\tau \le -\int_0^s e^{2w(\tau)}d\tau \le -\int_0^s e^{2w(0)}d\tau = -e^{2w(0)}s$$

for all $s \ge 0$. By sending $s \nearrow +\infty$ we conclude that w' is negative somewhere. This is a contradiction, hence the alternative $w' > 0$ everywhere cannot occur.

Substep 2.2. Now we rule out the alternative $w' < 0$ everywhere, whose proof is almost similar to the proof presented in the preceding case. Indeed, by contradiction we suppose that $w' < 0$ everywhere, namely w is monotone decreasing in $\mathbf{R}$, so is the function e^{2w}. Hence,

$$e^{2w(s)} \ge e^{2w(0)} \quad \text{for all } s \le 0.$$

Now making use of (2.2) and the above estimate gives

$$w'(0) - w'(s) = \int_s^0 w''(\tau)d\tau \le -\int_s^0 e^{2w(\tau)}d\tau \le -\int_s^0 e^{2w(0)}d\tau = e^{2w(0)}s$$

for all $s \le 0$. In other words, we have

$$w'(s) \ge w'(0) - e^{2w(0)}s.$$

By sending $s \searrow -\infty$ we conclude that w' is positive somewhere. This is a contradiction, hence the alternative $w' < 0$ everywhere cannot occur.

Step 3. (Assuming the function w' is sign-changing.) From now on we focus on the remaining alternative, namely w' is sign-changing, namely there is some $s_0 \in \mathbf{R}$ such that

$$w'(s_0) = 0.$$

In fact, as $w'' < 0$ in $\mathbf{R}$, see (2.2), such a number s_0 is unique. Moreover, there holds

$$w'(s) > 0 > w'(t) \quad \text{for any } s < s_0 < t.$$

Denote

$$C = e^{w(s_0)} \quad \text{and} \quad h = w - w(s_0).$$

Clearly, h enjoys $h(s_0) = 0$, $h'(s_0) = 0$, and $h' < 0$ in $(s_0, +\infty)$. Moreover,

$$\int_0^{+\infty} e^{u(r)} dr = \int_{-\infty}^{+\infty} e^{w(s)} ds = C \int_{-\infty}^{+\infty} e^{h(s)} ds$$

and we also have

$$h'' + C^2 e^{2h} \le 0 \quad \text{in } \mathbf{R}. \tag{2.3}$$

In the next two steps we show that

$$C \int_{s_0}^{+\infty} e^{h(s)} ds \le \frac{\pi}{2} \tag{2.4}$$

and that

$$C \int_{-\infty}^{s_0} e^{h(s)} ds \le \frac{\pi}{2}. \tag{2.5}$$

Once we have the above two estimates, the proof follows.

Step 4. (The integral $\int_{s_0}^{+\infty}$.) Now we show that (2.4) holds. Thanks to $h' < 0$ on $[s_0, +\infty)$ by multiplying both sides of (2.4) by h' and integrating over $[s_0, s]$ we arrive at

$$\int_{s_0}^{s} h'(t)\Big(h''(t) + C^2 e^{2h(t)}\Big) dt \ge 0,$$

which then yields

$$\big(h'(s)\big)^2 + C^2\Big(e^{2h(s)} - 1\Big) \ge 0 \quad \text{in } [s_0, +\infty), \tag{2.6}$$

thanks to $h(s_0) = 0$. As before, since $h' < 0$ in $[s_0, +\infty)$ and $h(s_0) = 0$, we deduce that $h < 0$ in $(s_0, +\infty)$. Thus, the function e^h is monotone decreasing in $(s_0, +\infty)$ and $e^{h(s)} \in (0, 1]$ for any $s \ge s_0$. Therefore, one can define a function $y : [s_0, +\infty) \to [0, \pi/2)$ as follows

$$y(s) = \arccos(e^{h(s)}).$$

Obviously, $y(s_0) = 0$ and

$$e^{h(s)} = \cos(y(s)) \quad \text{for } s \ge s_0,$$

namely $h(s) = \log \cos y(s)$. From this we obtain

$$h' = -\frac{\sin y}{\cos y} y',$$

which, in particular, gives $y' > 0$ everywhere in $(s_0, +\infty)$. We now come back to (2.6) to get

$$\Big(-\frac{\sin y}{\cos y} y'\Big)^2 + C^2\Big((\cos y)^2 - 1\Big) \ge 0 \quad \text{in } [s_0, +\infty),$$

thanks to $e^{2h} = (\cos y)^2$. Thus, resolving the above inequality gives

$$y'(s) \ge C|\cos y(s)| \ge C \cos y(s) \quad \text{for any } s > s_0,$$

thanks to $y' > 0$. We are now in position to obtain the following estimate

$$C\int_{s_0}^{+\infty} e^{h(s)}ds = C\int_{s_0}^{+\infty}\cos(y(s))ds \le \int_{s_0}^{+\infty} y'(s)ds \le \Big(\frac{\pi}{2} - y(s_0)\Big) = \frac{\pi}{2},$$

which yields the desired estimate (2.4).

Step 5. (The integral $\int_{-\infty}^{s_0}$.) Now, by a similar argument we show that (2.5) holds. Indeed, now multiply both sides of (2.4) by h' and integrate over $[s, s_0]$ with arbitrary $s < s_0$ to get

$$\int_s^{s_0} h'(t)\Big(h''(t) + C^2 e^{2h(t)}\Big)dt \le 0,$$

thanks to $h' > 0$ on $(-\infty, s_0]$. Hence, instead of (2.6) one should have

$$-\Big(h'(s)\Big)^2 + C^2\Big(1 - e^{2h(s)}\Big) \le 0 \quad \text{in } (-\infty, s_0], \tag{2.7}$$

but this is still nothing but (2.6), however, on $(-\infty, s_0]$. Now as $h' > 0$ in $(-\infty, s_0)$ and $h(s_0) = 0$, we get $h < 0$ in $(-\infty, s_0)$. Hence, $0 < e^{h(s)} \le 1$ for any $s \le s_0$. From this one can find a function $z : (-\infty, s_0] \to [0, \pi/2)$ such that

$$e^{h(s)} = \cos(z(s)) \quad \text{for } s \le s_0.$$

Arguing similarly, we arrive at

$$h' = -\frac{\sin z}{\cos z} z',$$

which gives $z' > 0$ everywhere in $(-\infty, s_0)$ and $z(s_0) = 0$. Coming back to (2.7) we show have

$$\Big(-\frac{\sin z}{\cos z} z'\Big)^2 + C^2\Big((\cos z)^2 - 1\Big) \ge 0 \quad \text{in } (-\infty, s_0],$$

thanks to $e^{2h} = (\cos z)^2$. Thus, as $z' \ge 0$, resolving the above inequality gives

$$z'(s) \ge C|\cos z(s)| \ge C\cos z(s) \quad \text{for any } s < s_0.$$

Hence, we estimate the integral in (2.5). Clearly, we have

$$C\int_{-\infty}^{s_0} e^{h(s)}ds = C\int_{-\infty}^{s_0}\cos(z(s))ds \le \int_{-\infty}^{s_0} z'(s)ds \le \Big(z(s_0) - \Big(-\frac{\pi}{2}\Big)\Big) = \frac{\pi}{2},$$

which yields the estimate (2.5) we need.

Step 6. (Completing the proof.) Finally, combing the two estimates (2.4) and (2.5) yields

$$C\int_{-\infty}^{+\infty} e^{h(s)}ds = C\int_0^{+\infty} e^{h(s)}ds + C\int_{-\infty}^0 e^{h(s)}ds \le \pi$$

which is our desired estimate. The optimality of this inequality can be verified easily by making use of the function u_{sing}.

Remark 2.1. In Step 2 of the above proof, we essentially show that the differential inequality (2.2), namely $w'' + e^{2w} \le 0$ in $\mathbf{R}$, does not admit solution w whose w' has a sign. In terms of u, if w' had a sign, then $\int_0^{+\infty} e^u dr$ would be divergent. Indeed, let us consider the case $w' < 0$ everywhere. Then we clearly have $u'(r) < 1/r$ in $(0, +\infty)$. A simple integration shows that the function $r \mapsto u(r) + \log r$ is monotone decreasing in $(0, +\infty)$. In particular, there holds

$$u(r) + \log r \ge u(1) + \log 1 = u(1) \tag{2.8}$$

for all $0 < r \le 1$. Thus, $u(r) \ge u(1) + \log(1/r)$ for $r \in (0, 1]$. Hence

$$\int_0^1 e^u dr \ge e^{u(1)}\int_0^1 \frac{dr}{r} = +\infty.$$

If $w' > 0$ everywhere, then (2.8) holds for all $r \geq 1$. Thus, we should arrive at

$$\int_1^{+\infty} e^u dr \geq e^{u(1)} \int_1^{+\infty} \frac{dr}{r} = +\infty.$$

3. Some further remarks

Let us discuss in this section the higher dimensional cases. First, we consider (1.6) in $\mathbf{R}^n$ with $n \geq 3$ although it has no geometric background. Previously, we regard (1.6) as the radial version of the equation $\Delta u + e^{2u} \leq 0$ in $\mathbf{R}^2$. In $\mathbf{R}^n$ with $n \geq 3$, a similar radial version reads as follows

$$u'' + \frac{n-1}{r} u' + e^{2u} \leq 0 \quad \text{for } r > 0. \tag{3.1}$$

Hence, it is natural to ask whether or not (1.7) remains true for any function u satisfying (3.1). The answer, unfortunately, is no. A simple counter-example in $\mathbf{R}^3$ is a modification of $u_{\rm reg}$ given by

$$u_1(r) = \log \frac{11}{10} + u_{\rm reg} = \log\Big(\frac{22}{10}\frac{1}{1+r^2}\Big) \quad \text{for } r \geq 0.$$

Then in $\mathbf{R}^3$, (3.1) is true because

$$u_1'' + \frac{2}{r} u_1' + e^{2u_1} = -\frac{50r^2 + 29}{25(1+r^2)^2} < 0 \quad \text{for any } r > 0.$$

However,

$$\int_0^{+\infty} e^{u_1(r)} dr = \frac{11}{10} \int_0^{+\infty} e^{u_1(r)} dr = \frac{11\pi}{10} > \pi.$$

However, we can formulate the following question.

Question 1. *Does there exist any constant $C > 0$ such that*

$$\int_0^{+\infty} e^{u(r)} dr \leq C$$

for any function $u : [0, +\infty) \to \mathbf{R}$ satisfying (3.1)? If the answer is yes, what is the sharp constant?

Unfortunately, the answer is still no. Indeed, for arbitrary $\varepsilon > 0$ let us consider

$$u_2(r) = \log\Big(\frac{1}{1+r}\Big) \quad \text{for } r \geq 0.$$

Then

$$u_2'' + \frac{n-1}{r} u_2' + e^{2u_2} = -\frac{n-1+(n-3)r}{(1+r)^2 r} < 0 \quad \text{for any } r > 0$$

as $n \geq 3$ and

$$\int_0^{+\infty} e^{u_2(r)} dr = \int_0^{+\infty} \frac{dr}{1+r} = +\infty.$$

Let us now discuss the case of scalar curvature R_g of $(\mathbf{R}^n, g)$ with $n \geq 3$. Then, under the conformal change

$$g = u^{\frac{4}{n-2}} \delta.$$

we know that R_g enjoys

$$R_g = \Big(-\frac{4(n-1)}{n-2} \Delta u + R_\delta\Big) u^{-\frac{n+2}{n-2}} = -\frac{4(n-1)}{n-2} (\Delta u) u^{-\frac{n+2}{n-2}};$$

see [Bes87, Corollary 1.161]. If we take the trace of the both sides of $\mathrm{Ric}_g \geq (n-1)g$, then we arrive at $R_g \geq n(n-1)$, which leads us to

$$-\Delta u \geq \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}.$$

Then we can ask the following question.

Question 2. *Does there exist any constant $C > 0$ such that*

$$\int_0^{+\infty} u(r)^{\frac{2}{n-2}} dr \leq C$$

for any function $u : [0, +\infty) \to \mathbf{R}$ satisfying

$$u'' + \frac{n-1}{r} u' + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \leq 0$$

for $r > 0$ with $n \geq 3$? If the answer is yes, what is the sharp constant?

Note that if we choose

$$u_{\mathrm{bub}}(r) = \Big(\frac{2}{1+r^2} \Big)^{\frac{n-2}{2}} \quad \text{with } r \geq 0,$$

which is just the standard bubble, then

$$u''_{\mathrm{bub}} + \frac{n-1}{r} u'_{\mathrm{bub}} + \frac{n(n-2)}{4} u_{\mathrm{bub}}^{\frac{n+2}{n-2}} = -2^{\frac{n-2}{2}} n(n-2) \Big(\frac{1}{1+r^2} \Big)^{\frac{n+2}{2}} + \frac{n(n-2)}{4} u_{\mathrm{bub}}^{\frac{n+2}{n-2}} = 0.$$

In this case, we easily get

$$\int_0^{+\infty} u_{\mathrm{bub}}^{2/(n-2)} dr = \pi.$$

In view of Bray's football theorem, see [Bra97], which still involves suitable smallness of the Ricci curvature, Question 2 is not expected to be true. This is indeed the case if u is singular at 0. For e.g., one can consider the following very slow decay function

$$u_3(r) = \Big(\frac{n-2}{n} \Big)^{\frac{n-2}{4}} r^{-\frac{n-2}{2}} \quad \text{with } r > 0.$$

Obviously,

$$u''_3 + \frac{n-1}{r} u'_3 + \frac{n(n-2)}{4} u_3^{\frac{n+2}{n-2}} = -\Big(\frac{n-2}{n} \Big)^{\frac{n-2}{4}} \frac{(n-2)^2}{4} r^{-\frac{n+2}{2}} + \frac{n(n-2)}{4} \Big(\frac{n-2}{n} \Big)^{\frac{n+2}{4}} r^{-\frac{n+2}{2}} = 0$$

and

$$\int_0^{+\infty} u_3^{2/(n-2)} dr = \sqrt{\frac{n-2}{n}} \int_0^{+\infty} \frac{dr}{r} = +\infty.$$

However, even with functions regular at 0, the answer to Question 2 is still no. Indeed, for $\varepsilon > 0$ to be determined later, let us consider

$$u_4(r) = \varepsilon (1+r)^{-\frac{n-2}{2}} \quad \text{with } r > 0.$$

Then

$$\begin{aligned} u''_4 + \frac{n-1}{r} u'_4 + \frac{n(n-2)}{4} u_4^{\frac{n+2}{n-2}} &= \varepsilon \frac{n(n-2)}{4} \Big(-\frac{n-2}{n} - \frac{2(n-1)}{nr} + \varepsilon^{\frac{4}{n-2}} \Big) (1+r)^{-\frac{n+2}{2}} \\ &\leq \varepsilon \frac{n(n-2)}{4} \Big(-\frac{n-2}{n} + \varepsilon^{\frac{4}{n-2}} \Big) (1+r)^{-\frac{n+2}{2}}. \end{aligned}$$

Keep in mind that $n \geq 3$. Hence, if we choose $\varepsilon > 0$ in such a way that

$$\varepsilon^{\frac{4}{n-2}} \leq \frac{n-2}{n}$$

and fix it, then

$$u_4'' + \frac{n-1}{r}u_4' + \frac{n(n-2)}{4}u_4^{\frac{n+2}{n-2}} \leq 0.$$

However,

$$\int_0^{+\infty} u(r)^{\frac{2}{n-2}}dr = \varepsilon^{\frac{2}{n-2}} \int_0^{+\infty} \frac{dr}{1+r} = +\infty.$$

(Apparently, the condition $n \geq 3$ plays an important role in the above construction, otherwise one cannot select $\varepsilon > 0$.)

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Appendix A. Completeness of conformal metrics

We list in this appendix a simple but useful criteria for the completeness of conformal metrics $e^{2u}\delta$ on $\mathbf{R}^2$. See [**GAR08**, Appendix A] for a similar result.

Lemma A.1. *Let $u : [0, +\infty) \to \mathbf{R}$ be a C^2-function. Then the conformal metric g on $\mathbf{R}^2$, defined by*

$$g(x) = e^{2u(|x|)}\delta(x) \quad \textit{for } x \in \mathbf{R}^2,$$

is complete if, and only if,

$$\int_0^{+\infty} e^{u(r)}dr = +\infty.$$

Proof. For the necessity, fix any $z_0 \in \mathbf{R}^2 \setminus \{0\}$, and consider the curve

$$\gamma : t \to \frac{tz_0}{|z_0|} \quad \text{for } t \in \mathbf{R},$$

which is simply a ray passing through the origin (at $t = 0$) and the point z_0 (at $t = |z_0|$). This is a divergent curve in M, see [**Car92**, page 153]. Indeed, take any compact subset $K \subset \mathbf{R}^2$, then there is some $R > 0$ such that $K \subset B_R$. Then

$$\gamma(t) \notin B_R \quad \text{for all } t \geq R.$$

Keep in mind that the length of γ is

$$2\int_0^{+\infty} \sqrt{g(\gamma'(t), \gamma'(t))}dt = 2\int_0^{+\infty} e^{u(r)}dr.$$

Thus, the completeness of $(\mathbf{R}^2, g)$ implies

$$\int_0^{+\infty} e^{u(r)}dr = +\infty.$$

For the sufficiency, let $\gamma : \mathbf{R} \to \mathbf{R}^2$ be a maximally extended geodesic curve in $(\mathbf{R}^2, g)$ parametrized over $\mathbf{R}$. Then, there holds

$$\lim_{t \to \pm\infty} |\gamma(t)| = +\infty.$$

Clearly, γ has infinite length if $\text{dist}_g(0, \gamma(t))$ becomes unbounded as $t \to \pm\infty$. And this is true because

$$\text{dist}_g(0, \gamma(t)) = \int_0^{\gamma(t)} e^{u(r)}dr$$

by definition. □

In view of Lemma A.1 above, if u is any solution to (1.6), then by (1.7) we know that $(R^2, e^{2u}\delta)$ is incomplete.

Appendix B. **Example of a conformal metric whose volume is infinity**

In this appendix, we provide a precise example of a conformal metric u whose volume $\mathrm{vol}_{e^{2u}\delta}(\mathbf{R}^2) = +\infty$. It seems that such an example is known among experts, but we cannot find any reference for it. So we decide to write it down for convenience.

The conformal metric we present here actually belongs to a larger class of solutions due to Gui and Li, see [**GL23**, equation (1.4)]. Indeed, let

$$u_{\mathrm{sol}}(x_1, x_2) = \log\Big(\frac{2e^{x_1}}{1+e^{2x_1}}\Big) \quad \text{in } \mathbf{R}^2,$$

which corresponds to [**GL23**, equation (1.4)] with $t = 0$. By direct verification, u_{sol} solves

$$\Delta u_{\mathrm{sol}} + e^{2u_{\mathrm{sol}}} = 0 \quad \text{in } \mathbf{R}^2.$$

In fact, u_{sol} can be rewritten as

$$u_{\mathrm{sol}}(x_1, x_2) = \log(\mathrm{sech}(x_1))$$

and notice that $\log(\mathrm{sech}(t))$ is a solution to the PDE in 1D, namely the following ODE

$$u'' + e^{2u} = 0 \quad \text{in } (0, +\infty).$$

This solution is bounded from above by 0, but does not decay to $-\infty$ at infinity. In fact, it is constant in the x_2-direction. See [**EGLX22**, Theorem 1.6] for further information on this special solution. Since u_{sol} does not depend on x_2, we immediately have

$$\int_{\mathbf{R}^2} e^{2u_{\mathrm{sol}}(x_1,x_2)} dx_1 dx_2 = +\infty$$

as claimed.

Remark B.1. Obviously, the function u_{sol} still solves

$$\Delta u_{\mathrm{sol}} + e^{2u_{\mathrm{sol}}} = 0 \quad \text{in } \mathbf{R}^n$$

for any $n \geq 3$. It is clearly non-radial and bounded from above by 0. It now follows from a non-existence result in [**EGLX22**, Lemma 4.1], without any calculation, that

$$\int_{\mathbf{R}^n} e^{2u_{\mathrm{sol}}(x)} dx = +\infty$$

with $n \geq 3$. This provides us an example in higher dimensional case.

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