Weighted sampling recovery of functions with mixed smoothness

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Abstract

We study sparse-grid linear sampling algorithms and their optimality for approximate recovery of functions with mixed smoothness on $\mathbb{R}^d$ from a set of n their sampled values in two different settings: (i) functions to be recovered are in weighted Sobolev spaces $W^r_{p,w}(\mathbb{R}^d)$ of mixed smoothness and the approximation error is measured by the norm of the weighted Lebesgue space $L_{q,w}(\mathbb{R}^d)$, and (ii) functions to be recovered are in Sobolev spaces with measure $W^r_p(\mathbb{R}^d;\mu_w)$ of mixed smoothness and the approximation error is measured by the norm of the Lebesgue space with measure $L_q(\mathbb{R}^d;\mu_w)$. Here, the function w, a tensor-product Freud-type weight is the weight in the setting (i), and the density function of the measure μ_w in the setting (ii). The optimality of linear sampling algorithms is investigated in terms of the relevant sampling n-widths. We construct sparse-grid linear sampling algorithms which are completely different for the settings (i) and (ii) and which give upper bounds of the corresponding sampling n-widths. We prove that in the one-dimensional case, these algorithms realize the right convergence rate of the sampling widths. In the setting (ii) for the high dimensional case ($d \geq 2$), we also achieve the right convergence rate of the sampling n-widths for $1 \leq q \leq 2 \leq p \leq \infty$ through a non-constructive method.

Keywords and Phrases: Linear sampling recovery; Sampling widths; Weighted Sobolev space of mixed smoothness; Sobolev space with measure of mixed smoothness; Sparse Smolyak grids; Sparse hyperbolic cross grids; Convergence rate.

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1 Introduction

We begin with definitions of weighted function spaces. Let

$$w(\boldsymbol{x}) := w_{\lambda,\tau,\eta,a,b}(\boldsymbol{x}) := \bigotimes_{i=1}^{d} w(x_i), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{1.1}$$

be the tensor product of d copies of the generating univariate Freud-type weight

$$w(x) := |x|^{\tau}(1+|x|)^{\eta} \exp\left(-a|x|^{\lambda} + b\right), \quad \lambda > 1, \ a > 0, \ \tau, \eta \geq 0, \ b \in \mathbb{R}. \tag{1.2}$$

The most important parameters in the weight w are λ and τ. The parameter b which produces only a positive constant in the weight w is introduced for a certain normalization for instance, for the standard Gaussian weight which is one of the most important weights. The weight w has a singularity at 0 if $\tau > 0$. In what follows, we fix the weight w and hence the parameters $\lambda, \tau, \eta, a, b$.

Let $1 \leq q < \infty$ and Ω be a Lebesgue measurable set on $\mathbb{R}^d$. We denote by $L_{q,w}(\Omega)$ the weighted Lebesgue space of all measurable functions f on Ω such that the norm

$$\|f\|_{L_{q,w}(\Omega)} := \left(\int_{\Omega} |f(\boldsymbol{x})w(\boldsymbol{x})|^q \mathrm{d}\boldsymbol{x} \right)^{1/q} \tag{1.3}$$

is finite.

Put $\mathbb{R}^d_\tau = \mathbb{R}^d$ if $\tau = 0$, and $\mathbb{R}^d_\tau = (\mathbb{R} \setminus \{0\})^d$ if $\tau > 0$, where τ is the parameter in the definition (1.2) of the generating univariate weight w. For $q = \infty$ and $\Omega = \mathbb{R}^d$, we define the space $L_{\infty,w}(\mathbb{R}^d) := C_w(\mathbb{R}^d)$ of all measurable functions on $\mathbb{R}^d$ such that

$$f \in C(\mathbb{R}^d_\tau), \ \lim_{|x_j| \to \infty} f(\boldsymbol{x})w(\boldsymbol{x}) = 0, \ j = 1, ..., d,$$

for $\tau = 0$, and

$$f \in C(\mathbb{R}^d_\tau), \ \lim_{|x_j| \to \infty} f(\boldsymbol{x})w(\boldsymbol{x}) = \lim_{x_j \to 0} f(\boldsymbol{x})w(\boldsymbol{x}) = 0, \ j = 1, ..., d,$$

for $\tau > 0$ The norm in $L_{\infty,w}(\mathbb{R}^d)$ is defined by

$$\|f\|_{L_{\infty,w}(\mathbb{R}^d)} := \sup_{\boldsymbol{x} \in \mathbb{R}^d_\tau} |f(\boldsymbol{x})w(\boldsymbol{x})|.$$

For $r \in \mathbb{N}$ and $1 \leq p \leq \infty$, the weighted Sobolev space $W^r_{p,w}(\Omega)$ of mixed smoothness r is defined as the normed space of all functions $f \in L_{p,w}(\Omega)$ such that the weak partial derivative $D^{\boldsymbol{k}} f$ belongs to $L_{p,w}(\Omega)$ for every $\boldsymbol{k} \in \mathbb{N}^d_0$ satisfying the inequality $|\boldsymbol{k}|_\infty \leq r$. The norm of a function f in this space is defined by

$$\|f\|_{W^r_{p,w}(\Omega)} := \left(\sum_{|\boldsymbol{k}|_\infty \leq r} \|D^{\boldsymbol{k}} f\|^p_{L_{p,w}(\Omega)} \right)^{1/p}. \tag{1.4}$$

Let γ be the standard d-dimensional Gaussian measure with the density function

$$v_{\rm g}(\boldsymbol{x}) := (2\pi)^{-d/2}\exp(-|\boldsymbol{x}|^2/2).$$

The well-known spaces $L_p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ which are used in many applications, are defined in the same way by replacing the norm (1.5) with the norm

$$\|f\|_{L_p(\Omega;\gamma)} := \left(\int_\Omega |f(\boldsymbol{x})|^p \gamma(\mathrm{d}\boldsymbol{x})\right)^{1/p} = \left(\int_\Omega |f(\boldsymbol{x})\,(v_{\rm g})^{1/p}\,(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x}\right)^{1/p}.$$

Thus, the spaces $L_p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ coincide with $L_{p,w}(\Omega)$ and $W^r_{p,w}(\Omega)$, where $w := (v_{\rm g})^{1/p}$ for a fixed $1 \le p < \infty$.

The spaces $L_p(\Omega;\gamma)$ and $W^r_p(\Omega;\gamma)$ with Gaussian measure can be generalized for any positive measure. Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable set. Let v be a nonzero nonnegative Lebesgue measurable function on Ω. Denote by μ_v the measure on Ω defined via the density function v, i.e., for every Lebesgue measurable set $A \subset \Omega$,

$$\mu_v(A) := \int_A v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

For $1 \le p \le \infty$, let $L_p(\Omega;\mu_v)$ be the space with measure μ_v of all Lebesgue measurable functions f on Ω such that the norm

$$\|f\|_{L_p(\Omega;\mu_v)} := \left(\int_\Omega |f(\boldsymbol{x})|^p \mu_v(\mathrm{d}\boldsymbol{x})\right)^{1/p} = \left(\int_\Omega |f(\boldsymbol{x})|^p v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right)^{1/p} \tag{1.5}$$

for $1 \le p < \infty$, and

$$\|f\|_{L_\infty(\Omega;\mu_v)} := \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega}|f(\boldsymbol{x})| \tag{1.6}$$

is finite, where ess sup is taken with respect to the measure μ_v. For $r \in \mathbb{N}$, the Sobolev spaces $W^r_p(\Omega;\mu_v)$ with measure μ_v and the classical Sobolev space $W^r_p(\Omega)$ are defined in the same way as in(1.4) by replacing $L_{p,w}(\Omega)$ with $L_p(\Omega;\mu)$ and $L_p(\Omega)$, respectively. If $1 \le p < \infty$ and $v = w$, the spaces $L_p(\Omega;\mu_w)$ and $W^r_p(\Omega;\mu_w)$ coincide with $L_{p,w^{1/p}}(\Omega)$ and $W^r_{p,w^{1/p}}(\Omega)$, respectively. Conversely, the spaces $L_{p,w}(\Omega)$ and $W^r_{p,w}(\Omega)$ coincide with $L_p(\Omega;\mu_{w^p})$ and $W^r_p(\Omega;\mu_{w^p})$, respectively. Notice that the functions $w^{1/p}$ and w^p are again a weight of the form (1.2).

In what follows, for the fixed weight w we use the abbreviations:

$$L_p(\Omega;\mu) := L_p(\Omega;\mu_w), \quad W^r_p(\Omega;\mu) := W^r_p(\Omega;\mu_w).$$

For a normed space X of functions on Ω, the boldface $\boldsymbol{X}$ denotes the unit ball in X.

Let us formulate a setting of linear sampling recovery problem. Let X be a normed space X of functions on Ω. Given sample points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_k \in \Omega$, we consider the approximate recovery of a continuous function f on Ω from their values $f(\boldsymbol{x}_1),\dots,f(\boldsymbol{x}_k)$ by a linear sampling algorithm S_k on Ω of the form

$$S_k(f) := \sum_{i=1}^k f(\boldsymbol{x}_i)h_i, \tag{1.7}$$

where $h_1, \dots, h_k$ are given continuous functions on Ω. For convenience, we assume that some of the sample points $\boldsymbol{x}_i$ may coincide. The approximation error is measured by the norm $\|f - S_k(f)\|_X$. Denote by $\mathcal{S}_n$ the family of all linear sampling algorithms S_k of the form (1.7) with $k \le n$. Let $F \subset X$ be a set of continuous functions on Ω. To study the optimality of linear sampling algorithms from $\mathcal{S}_n$ for F and their convergence rates we use the (linear) sampling n-width

$$\varrho_n(F, X) := \inf_{S_n \in \mathcal{S}_n} \sup_{f \in F} \|f - S_n(f)\|_X. \tag{1.8}$$

Notice that any function $f \in W^r_{p,w}(\mathbb{R}^d)$ is equivalent in the sense of the Lesbegue measure to a continuous (not necessarily bounded) function on $\mathbb{R}^d_\tau$ (see [6, Lemma 3.1] for $\tau = 0$; in the case $\tau > 0$, this equivalence can be proven similarly). Hence throughout the present paper, we always assume that $W^r_{p,w}(\mathbb{R}^d) \subset C_w(\mathbb{R}^d)$, and that $W^r_{p,w}(\mathbb{R}^d)$ coincides with $W^r_{p,w}(\mathbb{R}^d_\tau)$ in the sense of the Lebesgue measure. Therefore, linear sampling algorithms of the form (1.7) with $\Omega = \mathbb{R}^d_\tau$ are well-defined for functions $f \in W^r_{p,w}(\mathbb{R}^d)$. All this also holds true for the space with measure $W^r_p(\mathbb{R}^d; \mu_w)$.

There is a large number of works devoted to the problem of (unweighted) linear sampling recovery of functions having a mixed smoothness on a compact domain, see for survey and bibliography in [11], [21], [25], [7]. Smolyak sparse-grid sampling algorithms for compact domains (cube or torus) have been widely and efficiently used in both approximation theory and numerical analysis, especially, in sampling recovery for functions having a mixed smoothness. A huge body of papers investigated various aspects on approximation and numerical implementation by Smolyak sparse-grid sampling algorithms. The reader can consult [3], [11], [21], [25] for survey and bibliography.

The right convergence rate of sampling n-widths is one of central problems in sampling recovery of functions having a mixed smoothness. We refer the reader to [7, 11] for survey and bibliography of on the right convergence rate of the sampling n-widths $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)\big)$, where $\mathbb{T}^d$ denotes the d-dimensional torus. It is interesting to notice that the right convergence of these widths in the cases $1 < p < q \le 2$, $2 \le p < q < \infty$ and $p = 2,\ q = \infty$ can be achieved by Smolyak sparse-grid sampling algorithms which are asymptotically optimal, and besides of that so far we do not know any other type of asymptotically optimal algorithm. From results of [14] on inequality between sampling and Kolmogorov n-widths and known results on relevant Kolmogorov n-widths, one can deduce in a non-constructive way the right convergence rate of $\varrho_n(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d))$ for the case $1 < q \le 2 \le p \le \infty$ (see [7]).

The problem of optimal sampling recovery of functions on $\mathbb{R}^d$ equipped with Gaussian measure has been investigated in [10]. We proved in a non-constructive way the right convergence rate of the sampling n-widths

$$\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d; \gamma), L_2(\mathbb{R}^d; \gamma)\big) \asymp n^{-r} (\log n)^{r(d-1)} \tag{1.9}$$

for $2 < p \le \infty$, and

$$\varrho_n\big(\boldsymbol{W}^r_2(\mathbb{R}^d; \gamma), L_2(\mathbb{R}^d; \gamma)\big) \asymp n^{-r/2} (\log n)^{r(d-1)/2} \tag{1.10}$$

which is obtained by using inequalities between sampling widths and Kolmogorov widths and the right convergence rate of $d_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_q(\mathbb{R}^d;\gamma)\big)$ proven in the same paper [10].

In the present paper, we are interested in constructing sparse-grid linear sampling algorithms for approximate recovery of functions with mixed smoothness on $\mathbb{R}^d$ in the two different settings:

(i) functions to be recovered are in weighted Sobolev spaces $W^r_{p,w}(\mathbb{R}^d)$ of mixed smoothness and the approximation error is measured by the norm of the weighted Lebesgue space $L_{q,w}(\mathbb{R}^d)$; and

(ii) functions to be recovered are in the Sobolev spaces with measure $W^r_p(\mathbb{R}^d;\mu)$ of mixed smoothness and the approximation error is measured by the norm of the Lebesgue space with measure $L_q(\mathbb{R}^d;\mu)$.

The parameters p and q may be different and range in $[1,\infty]$. The asymptotic optimality of linear sampling algorithms is investigated in terms of the relevant sampling widths. We construct sparse-grid sampling algorithms which are completely different for the settings (i) and (ii), and which give upper bounds of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d))$ and $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu_w), L_q(\mathbb{R}^d;\mu_w)\big)$, respectively. In, the case $d=1$, these algorithms will realize the right convergence rate of the sampling widths. In particular, we develop a counterpart of Smolyak sparse-grid sampling algorithms for the setting (i), and extend the assembling method of quadrature in [10] for the setting (ii). We are also interested in the right convergence rate of the sampling n-widths of $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu_w), L_q(\mathbb{R}^d;\mu_w)\big)$ for $d\ge 2$ and $1\le q\le 2\le p\le\infty$.

We briefly describe the results of the present paper. Before doing this we emphasize the following. Both the settings (i) and (ii) are natural. The setting (i) comes from the classical theory of weighted approximation (for knowledge and bibliography see, e.g., [20], [17], [16]). The setting (ii) is related to many theoretical and applied topics especially, related to Gaussian measure γ and other probability measures. Below we will see that these settings lead to very different approximation results except the case $1\le p=q<\infty$ when they are coincide after re-notation.

Throughout the present paper, for given $p,q\in[1,\infty]$ and the parameter $\lambda>1$ in the definition (1.2) of the generating univariate weight w, we make use of the notations

$$r_\lambda := (1-1/\lambda)r;$$

$$\delta_{\lambda,p,q} := \begin{cases}(1-1/\lambda)(1/p-1/q) & \text{if } \ p\le q,\\ (1/\lambda)(1/q-1/p) & \text{if } \ p>q;\end{cases} \tag{1.11}$$

(with the convention $1/\infty := 0$) and

$$r_{\lambda,p,q} := r_\lambda - \delta_{\lambda,p,q}.$$

The setting (i). Let $1<p<\infty$, $1\le q\le\infty$ and $r_{\lambda,p,q}>0$. Then with some restrictions on the weight w and p, q, we prove that

$$\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)) \ll \begin{cases} n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)} & \text{if } \ p = q, \\ n^{-r_{\lambda,p,q}}(\log n)^{(r_{\lambda,p,q}+1/q)(d-1)} & \text{if } \ p \neq q < \infty, \\ n^{-r_{\lambda,p,q}}(\log n)^{(r_{\lambda,p,q}+1)(d-1)} & \text{if } \ p \neq q = \infty; \end{cases} \tag{1.12}$$

and

$$\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)) \gg \begin{cases} n^{-r_\lambda}(\log n)^{r_\lambda(d-1)} & \text{if } \ p = q, \\ n^{-r_{\lambda,p,q}}(\log n)^{r_{\lambda,p,q}(d-1)} & \text{if } \ p < q, \\ n^{-r_{\lambda,p,q}} & \text{if } \ p > q. \end{cases} \tag{1.13}$$

In the one-dimensional case, we prove the right convergence rate

$$\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})) \asymp n^{-r_{\lambda,p,q}}. \tag{1.14}$$

We also obtain upper and lower bound of $\varrho_n(\boldsymbol{W}^r_{\infty,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d))$ in the case when $1 \le q \le \infty$ and $r_{\lambda,\infty,q} > 0$:

$$n^{-r_{\lambda,\infty,q}} \ll \varrho_n\big(\boldsymbol{W}^r_{\infty,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \ll n^{-r_{\lambda,\infty,q}} \log n. \tag{1.15}$$

Some results on univariate weighted interpolation from [18] are employed in proving the upper bounds (1.12). The Smolyak sampling algorithms performing these upper bounds are constructed on sparse grids of sampled points which form *a step hyperbolic cross in the function domain* $\mathbb{R}^d$. It is remarkable to notice that these sparse grids are completely different from the classical Smolyak sparse grids for compact domains (see Figures 1 and 2). To prove the lower bounds in (1.13) and in (1.14) in the case $p < q$ we adopt a traditional technique to construct for arbitrary n sampled points a fooling function vanishing at these points. The lower bounds in (1.13) and in (1.14) in the case $p \ge q$ are derived from the lower bound of Kolmogorov widths, a Bernstein-type inequality by using a discretization technique.

The setting (ii). Let $1 \le q < p \le \infty$, $r > 1/p$ or $1 < q = p < \infty$, $r_\lambda > 0$. Then with some restrictions on the density function w and p, q, we prove that

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \ll \begin{cases} n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)} & \text{if } \ 1 < p = q < \infty, \\ n^{-r}(\log n)^{(r+1/2)(d-1)} & \text{if } \ 1 < q < p < \infty, \\ n^{-r}(\log n)^{(r+1)(d-1)} & \text{if either } q = 1 \text{ or } p = \infty, \end{cases} \tag{1.16}$$

and

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \gg \begin{cases} n^{-r_\lambda}(\log n)^{r_\lambda(d-1)} & \text{if } \ 1 < p = q < \infty, \\ n^{-r}(\log n)^{r(d-1)} & \text{if } \ 1 \le q < p \le \infty. \end{cases} \tag{1.17}$$

In the one-dimensional case, we prove the right convergence rate

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R};\mu), L_q(\mathbb{R};\mu)) \asymp \begin{cases} n^{-r} & \text{if } \ 1 \le q < p \le \infty \text{ and } r > 1/p; \\ n^{-r_\lambda} & \text{if } 1 < q = p < \infty \text{ and } r_\lambda > 0. \end{cases} \tag{1.18}$$

Notice that in the case $1 \leq p = q < \infty$, the results on the setting (ii) follow from the results on the setting (i) since as noticed above in this case they coincide. The sparse grid linear sampling algorithms performing the upper bounds (1.16) for the case $1 < q < p < \infty$ in the setting (ii) are constructed by *assembling* Smolyak linear sampling algorithms for the related Sobolev spaces on the unit d-cube to the integer-shifted d-cubes which cover $\mathbb{R}^d$. They are based on very sparse sample points contained in a d-ball of radius $C \log^{1/\lambda} n$ (see Figure 2), and completely different from the sample points in step hyperbolic cross for the setting (i). The lower bounds in (1.17) are derived from known lower bounds of Kolmogorov n-widths $d_n\big(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)\big)$.

The case $1 \leq p < q \leq \infty$ in the setting (ii) is excluded from consideration since in this case we do not have a continuous embedding of $W^r_p(\mathbb{R}^d;\mu)$ into $L_q(\mathbb{R}^d;\mu)$. The gap between the upper bounds (convergence rates) (1.12) and (1.16) and the lower bounds in (1.13) and (1.17) for $d \geq 2$ (and $d = 1$, $p = \infty$) is a logarithmic factor which may be different depending on various values of p and q. In the setting (i) for all p, q and in the setting (ii) for $p = q$, the main parameter $r_{\lambda,p,q}$ in the convergence rate strictly smaller than $r - (1/p - 1/q)_+$ which is the main parameter in the convergence rate of the respective unweighted sampling n-widths. While for $1 \leq q < p \leq \infty$, the main parameter r in the convergence rate in the setting (i) is the same as the main parameter in the convergence rate of the respective unweighted sampling n-widths (cf. [7, Theorem 2.9]).

In the present paper, extending the result (1.10) and technique in [10] (see also [7] for a particular case of w) to the sampling n-widths $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big)$, we prove in a non-constructive way the right convergence rate

$$\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big) \asymp n^{-r}(\log n)^{r(d-1)} \tag{1.19}$$

in the case $1 \leq q \leq 2 < p \leq \infty$, which coincides with the right convergence rate of the unweighted sampling n-widths $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)\big)$ in the same case (cf. [7, Theorem 2.17]). Extending the result (1.10) to the generating univariate Freud-type weight

$$w(x) := \exp\left(-ax^4 + b\right), \ a > 0, \ b \in \mathbb{R},$$

we prove the right convergence rate

$$\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp n^{-3r/4}(\log n)^{3r(d-1)/4}, \tag{1.20}$$

for $1 \leq q \leq 2$. A key role playing in the proof of this result is the RKHS structure of the Hilbert space $W^r_2(\mathbb{R}^d;\mu_w)$ which is derived from some old results [2], [15] on properties of the orthonormal polynomials associated with the weight w^2.

The paper is organized as follows. In Section 2, for the setting (i), we prove the right convergence rate of $\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R}))$ and construct asymptotically optimal linear sampling algorithms. In Section 3, for the setting (i), we prove upper and lower bounds of $\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d))$ for $d \geq 2$, and construct linear sampling algorithms which give the upper bound. In Section 4, for the setting (ii), we prove upper and lower bounds of $\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $d \in \mathbb{N}$, $1 \leq q \leq p \leq \infty$, construct linear sampling algorithms which give the upper bound for $d \geq 2$ and the right convergence rate for $d = 1$. We also prove the right convergence rate of the sampling widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $1 \le q \le 2 < p \le \infty$ and $1 \le q \le p = 2$.

Notation. Denote $\boldsymbol{x} =: (x_1, ..., x_d)$ for $\boldsymbol{x} \in \mathbb{R}^d$; $\boldsymbol{1} := (1, ..., 1) \in \mathbb{R}^d$; $|\boldsymbol{x}|_p := \left(\sum_{j=1}^d |x_j|^p\right)^{1/p}$ $(1 \le p < \infty)$ and $|\boldsymbol{x}|_\infty := \max_{1\le j\le d} |x_j|$ with the abbreviation: $|\boldsymbol{x}| := |\boldsymbol{x}|_2$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, the inequality $\boldsymbol{x} \le \boldsymbol{y}$ $(\boldsymbol{x} < \boldsymbol{y})$ means $x_i \le y_i$ $(x_i < y_i)$ for every $i = 1, ..., d$. For $x \in \mathbb{R}$, denote $\operatorname{sign}(x) := 1$ if $x \ge 0$, and $\operatorname{sign}(x) := -1$ if $x < 0$. We use letters C and K to denote general positive constants which may take different values. For the quantities $A_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ independent of $n, f, \boldsymbol{k}$ such that $A_n(f, \boldsymbol{k}) \le CB_n(f, \boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f, \boldsymbol{k}) \gg B_n(f, \boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f, \boldsymbol{k}) \asymp B_n(f, \boldsymbol{k})$ if $S_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k}) \ll S_n(f, \boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G. For a Banach space X, denote by the boldface $\boldsymbol{X}$ the unit ball in X.

2 Univariate sampling recovery in the setting (i)

In this section, for the setting (i) in the one-dimensional case, we prove the right convergence rate of sampling n-widths $\varrho_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ and construct asymptotically optimal linear sampling linear algorithms for $1 < p < \infty$ and $1 \le q \le \infty$. We also give some upper and lower bounds of $\varrho_n\big(\boldsymbol{W}^r_{\infty,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ for $1 \le q \le \infty$.

Let the univariate generalized Freud weight v be defined by

$$v(x) := |x|^\mu \exp\left(-2a|x|^\lambda + 2b\right), \quad \mu \ge -1, \tag{2.1}$$

where λ, a, b are the same as in the definition (1.2) of the weight w. Let $(p_m)_{m\in\mathbb{N}_0}$ be the sequence of orthonormal polynomials with respect to the weight v. For a given even number $m \in \mathbb{N}$, denote by $x_{m,k}$, $1 \le k \le m/2$, the positive zeros of p_m, and by $x_{m,-k} = -x_{m,k}$ the negative ones (since m is even, 0 is not a zero of p_m). These zeros are located as

$$-a_m + \frac{Ca_m}{m^{2/3}} < x_{m,-m/2} < \cdots < x_{m,-1} < x_{m,1} < \cdots < x_{m,m/2} < a_m - \frac{Ca_m}{m^{2/3}}, \tag{2.2}$$

with a positive constant C independent of m (see, e. g., [16, (4.1.32)]). Here a_m is the Mhaskar-Rakhmanov-Saff number which is

$$a_m := (C_\lambda m)^{1/\lambda} \asymp m^{1/\lambda}, \quad C_\lambda := \frac{2^{\lambda-1}\Gamma(\lambda/2)^2}{\Gamma(\lambda)}, \tag{2.3}$$

and Γ is the gamma function (see, e.g., [16, (4.1.4)]).

Throughout this paper, we fix a number ρ with $0 < \rho < 1$, and denote by $j(m)$ the smallest integer satisfying $x_{m,j(m)} \ge \rho a_m$. It is useful to remark that

$$x_{m,j(m)} \asymp m^{1/\lambda}, \ m \in \mathbb{N}_0; \quad \Delta_{m,k} \asymp \frac{a_m}{m} \asymp m^{1/\lambda-1}, \quad |k| \le j(m), \tag{2.4}$$

where $\Delta_{m,k} := x_{m,k} - x_{m,k-1}$ is the distance between consecutive zeros of the polynomial $p_m(w)$. The first relation follows from the definition, (2.3) and (2.2). For the second relations see [16, (4.1.47)]. From their proofs there, one can easily see that they are still hold true for the general case of the weight w. By (2.2) and (3.6), for m sufficiently large we have that

$$Cm \leq j(m) < m/2 \tag{2.5}$$

with a positive constant C depending on λ, a, b and ρ only.

For an even $m \in \mathbb{N}$, let $\mathcal{P}^*_{m+1}$ be the subspace of $\mathcal{P}_{m+1}$ defined by

$$\mathcal{P}^*_{m+1} := \{\varphi \in \mathcal{P}_{m+1} : \varphi(\pm a_m) = \varphi(x_{m,k}) = 0,\ |k| > j(m)\}. \tag{2.6}$$

Here $\mathcal{P}_m$ denotes the space of polynomials of degree at most m. For $|k| \leq j(m)$, we define

$$\ell_{m,k}(x) := \frac{p_m(x)}{p'(x_{m,k})(x - x_{m,k})} \frac{a_m - x^2}{a_m - x^2_{m,k}}. \tag{2.7}$$

The polynomials $\ell_{m,k}$ belong to $\mathcal{P}_{m+1}$. The dimension of the space $\mathcal{P}^*_{m+1}$ is $2j(m)$ and the polynomials $\{\ell_{m,k}\}_{|k|\leq j(m)}$ constitute a basic of the space $\mathcal{P}^*_{m+1}$, i.e., every polynomial $\varphi \in \mathcal{P}^*_{m+1}$ can be represented as

$$\varphi = \sum_{|k|\leq j(m)} \varphi(x_{m,k})\ell_{m,k}. \tag{2.8}$$

We extend the formula (2.8) to the functions f in $C_w(\mathbb{R})$ as the interpolation operator

$$I_m : C_w(\mathbb{R}) \to \mathcal{P}^*_{m+1}$$

by defining

$$I_m(f) := \sum_{|k|\leq j(m)} f(x_{m,k})\ell_{m,k}. \tag{2.9}$$

The operator I_m is a projector from $C_w(\mathbb{R})$ onto $\mathcal{P}^*_{m+1}$, and $I_m f$ belongs to $\mathcal{P}_{m+1}$. The polynomial $I_m f$ interpolates f at the zeros $x_{m,k}$ for $|k| \leq j(m)$, and vanishes at the points $\pm a_m$ and the zeros $x_{m,k}$ for $|k| > j(m)$. Also, the number $2j(m)$ of interpolation points in I_m is strictly smaller than m. However, due to (2.5) it has the convergence rate as $2j(m) \asymp m$ when m going to infinity. The space of polynomials $\mathcal{P}^*_{m+1}$ and the interpolation operator I_m have been introduced in [18].

Condition C. The number p with $1 < p < \infty$ and the numbers $\tau \geq 0$, $\eta \geq 0$, $\mu \geq -1$ associated with the weights w and v, satisfy

(i) $\tau + 1/p$ is not an integer;

(ii) $-1/p < \tau - \mu/2 < 1 - 1/p - \eta$.

Throughout this and the next sections, we assume without mention that Condition C holds for a given p with $1 < p < \infty$ in the assumptions of lemmata and theorems. For argument of necessity of this condition see [18].

The following two lemmata were proven in [18, Theorem 3.7, Lemma 3.3].

Lemma 2.1 *Let $1 < p < \infty$. Then for every $m \in \mathbb{N}$,*

$$\|f - I_m(f)\|_{L_{p,w}(\mathbb{R})} \le C m^{-r_\lambda} \|f\|_{W^r_{p,w}(\mathbb{R})} \quad \forall f \in W^r_{p,w}(\mathbb{R}). \tag{2.10}$$

Lemma 2.2 *Let $1 < p < \infty$. Then there holds the equivalence*

$$\|\varphi\|_{L_{p,w}(\mathbb{R})} \asymp \left(m^{1/\lambda - 1} \sum_{|k| \le j(m)} |\varphi(x_{m,k}) w(x_{m,k})|^p \right)^{1/p} \quad \forall \varphi \in \mathcal{P}^*_{m+1} \quad \forall m \in \mathbb{N}. \tag{2.11}$$

Lemma 2.3 *Let $1 \le p, q \le \infty$. Then we have the following.*

(i) *There holds the Bernstein-type inequality*

$$\|\varphi'\|_{L_{p,w}(\mathbb{R})} \ll m^{1-1/\lambda} \|\varphi\|_{L_{p,w}(\mathbb{R})} \quad \forall \varphi \in \mathcal{P}_m, \quad \forall m \in \mathbb{N}. \tag{2.12}$$

(ii) *For any fixed $\delta > 0$, there exists a constant $C_\delta > 0$ such that*

$$\|\varphi\|_{L_{p,w}(\mathbb{R})} \le C_\delta \|\varphi\|_{L_{p,w}(I^\delta_m)} \quad \forall \varphi \in \mathcal{P}_m, \quad \forall m \in \mathbb{N}, \tag{2.13}$$

where $I^\delta_m := [-a_m, -\delta a_m/m] \cup [\delta a_m/m, a_m]$

(iii) *For $1 \le p < q \le \infty$, there holds the Nikol'skii-type inequality*

$$\|\varphi\|_{L_{q,w}(\mathbb{R})} \ll m^{(1-1/\lambda)(1/p-1/q)} \|\varphi\|_{L_{p,w}(\mathbb{R})} \quad \forall \varphi \in \mathcal{P}_m, \quad \forall m \in \mathbb{N}. \tag{2.14}$$

(iv) *For $1 \le q < p \le \infty$, there holds the Nikol'skii-type inequality*

$$\|\varphi\|_{L_{q,w}(\mathbb{R})} \ll m^{(1/\lambda)(1/q-1/p)} \|\varphi\|_{L_{p,w}(\mathbb{R})}, \quad \forall \varphi \in \mathcal{P}_m, \ \forall m \in \mathbb{N}. \tag{2.15}$$

Proof. The claims (i)–(iii) were proven in [19, pp. 106, 107] (see also [16, Lemma 4.1.3]). The claim (iv) can be obtained from the claim (iii) and (2.3) by applying the Hölder inequality for $\nu = p/q > 1$ and $1/\nu + 1/\nu' = 1$. Indeed, for a fixed $\delta > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}} |\varphi(x) w(x)|^q \mathrm{d}x &\ll (2a_m(1 - \delta/m))^{1/\nu'} \left(\int_{I^\delta_m} |\varphi(x) w(x)|^{q\nu} \mathrm{d}x \right)^{1/\nu} \\ &\ll m^{q(1/\lambda)(1/q-1/p)} \left(\int_{\mathbb{R}} |\varphi(x) w(x)|^p \mathrm{d}x \right)^{q/p}. \end{aligned} \tag{2.16}$$

∎

For every $k \in \mathbb{N}_0$, let m_k be the largest number such that $m_k + 1 \le 2^k$. Then by (2.5) we have $2^k \asymp 2j(m_k) \le 2^k$. For the sequence of sampling operators $(I_{m_k})_{k \in \mathbb{N}_0}$ with $I_{m_k} \in \mathcal{S}_{2^k}$, from Lemma 2.1 with $s = 0$ it follows that

$$\big\| f - I_{m_k}(f) \big\|_{L_{p,w}(\mathbb{R}^d)} \le C 2^{-r_\lambda k} \|f\|_{W^r_{p,w}(\mathbb{R})}, \quad k \in \mathbb{N}_0, \quad f \in W^r_{p,w}(\mathbb{R}) \ \ (1 < p < \infty). \tag{2.17}$$

We define the one-dimensional operators for $k \in \mathbb{N}_0$

$$\Delta^I_k := I_{m_k} - I_{m_{k-1}}, \ k > 0, \quad \Delta^I_0 := I_1. \tag{2.18}$$

Lemma 2.4 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. Then we have that for every $f \in W^r_{p,w}(\mathbb{R})$,*

$$f = \sum_{k \in \mathbb{N}_0} \Delta^I_k(f) \tag{2.19}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R})$ of the series. Moreover,

$$\big\| \Delta^I_k(f) \big\|_{L_{q,w}(\mathbb{R})} \le C 2^{-r_{\lambda,p,q} k} \|f\|_{W^r_{p,w}(\mathbb{R})}, \quad k \in \mathbb{N}_0. \tag{2.20}$$

Proof. Let $f \in W^r_{p,w}(\mathbb{R})$. Since $\Delta^I_k f \in \mathcal{P}_{m_k+1}$ by the claims (iii) and (iv) of Lemma 2.3 we have that

$$\big\| \Delta^I_k(f) \big\|_{L_{q,w}(\mathbb{R})} \ll 2^{\delta_{\lambda,p,q} k} \big\| \Delta^I_k(f) \big\|_{L_{p,w}(\mathbb{R})}, \quad k \in \mathbb{N}_0. \tag{2.21}$$

By (2.17) and (2.18) we have that for every $f \in W^r_{p,w}(\mathbb{R})$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} \big\| \Delta^I_k(f) \big\|_{L_{p,w}(\mathbb{R})} &\le \big\| f - I_{2j(m_k)}(f) \big\|_{L_{p,w}(\mathbb{R})} + \big\| f - I_{2j(m_{k-1})}(f) \big\|_{L_{p,w}(\mathbb{R})} \\ &\ll 2^{-r_\lambda k} \|f\|_{W^r_{p,w}(\mathbb{R})}, \end{aligned}$$

which together with (2.21) proves (2.20) and hence the absolute convergence of the series in (2.19) follows. The equality in (2.19) is implied from (2.10) and the equality

$$I_{m_k} = \sum_{s \le k} \Delta^I_s. \tag{2.22}$$

□

We recall some well-known inequalities which are quite useful for lower estimation of sampling n-widths. From the definition (1.8) we have the following result. If F is a set of continuous functions on $\mathbb{R}^d$ and X is a normed space of functions on $\mathbb{R}^d$, then we have

$$\varrho_n(F, X) \ge \inf_{\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \subset \mathbb{R}^d} \ \sup_{f \in F: \ f(\boldsymbol{x}_i) = 0, \ i = 1, \ldots, n} \|f\|_X. \tag{2.23}$$

Let $n \in \mathbb{N}$ and let X be a Banach space and F a central symmetric compact set in X. Then the Kolmogorov n-width of F is defined by

$$d_n(F, X) := \inf_{L_n} \sup_{f \in F} \inf_{g \in L_n} \|f - g\|_X,$$

where the left-most infimum is taken over all subspaces L_n of dimension $\le n$ in X. If X is a normed space of functions on Ω and $F \subset X$ is a set of continuous functions on Ω, then from the definitions we have

$$\varrho_n(F, X) \ge d_n(F, X). \tag{2.24}$$

Theorem 2.5 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. For any $n \in \mathbb{N}$, let $k(n)$ be the largest integer such that $2^{k(n)} \le n$. Then the sampling operators $I_{m_{k(n)}} \in \mathcal{S}_n$, $n \in \mathbb{N}$, are asymptotically optimal for the sampling n-widths $\varrho_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ and*

$$\varrho_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \asymp \sup_{f \in \boldsymbol{W}^r_{p,w}(\mathbb{R})} \big\|f - I_{m_{k(n)}}(f)\big\|_{L_{q,w}(\mathbb{R})} \asymp n^{-r_{\lambda,p,q}}. \tag{2.25}$$

Proof. By using Lemma 2.4 we derive for $f \in \boldsymbol{W}^r_{p,w}(\mathbb{R})$ and $n \in \mathbb{N}_0$,

$$\begin{aligned}
\big\|f - I_{m_{k(n)}}(f)\big\|_{L_{q,w}(\mathbb{R})} &= \Bigg\|\sum_{k > k(n)} \Delta^I_k(f)\Bigg\|_{L_{q,w}(\mathbb{R})} \le \sum_{k > k(n)} \big\|\Delta^I_k(f)\big\|_{L_{q,w}(\mathbb{R})} \\
&\ll \sum_{k > k(n)} 2^{-r_{\lambda,p,q}k} \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \\
&\le 2^{-r_{\lambda,p,q}k(n)} \sum_{k > k(n)} 2^{-r_{\lambda,p,q}(k - k(n))} \ll n^{-r_{\lambda,p,q}},
\end{aligned} \tag{2.26}$$

which proves the upper bound in (2.25).

In order to prove the lower bound in the case $p < q$ in (2.25) we employ the inequality (2.23). Let $\{m_1, ..., m_n\} \subset \mathbb{R}$ be arbitrary n points. For a given $n \in \mathbb{N}$, we put $\delta = n^{1/\lambda - 1}$ and $t_j = \delta j$, $j \in \mathbb{N}_0$. Then there is $i \in \mathbb{N}$ with $n + 1 \le i \le 2n + 2$ such that the interval (t_{i-1}, t_i) does not contain any point from the set $\{m_1, ..., m_n\}$. Take a nonnegative function $\varphi \in C^\infty_0([0,1])$, $\varphi \ne 0$, and put

$$b_s := \|\varphi^{(s)}(y)\|_{L_p([0,1])}, \ \ s = 0, 1, ..., r.$$

Define the functions g and h on $\mathbb{R}$ by

$$g(x) := \begin{cases} \varphi(\delta^{-1}(x - t_{i-1})), & x \in (t_{i-1}, t_i), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$h(x) := (gw^{-1})(x).$$

Let us estimate the norm $\|h\|_{W^r_{p,w}(\mathbb{R})}$. For a given $k \in \mathbb{N}_0$ with $0 \le k \le r$, we have

$$h^{(k)} = (gw^{-1})^{(k)} = \sum_{s=0}^{k} \binom{k}{s} g^{(k-s)} (w^{-1})^{(s)}. \tag{2.27}$$

By a direct computation we find that for $x \in \mathbb{R}$,

$$(w^{-1})^{(s)}(x) = (w^{-1})(x)(\operatorname{sign}(x))^s \sum_{j=1}^{s} c_{s,j}(\lambda, a)|x|^{\lambda_{s,j}}, \tag{2.28}$$

where $\operatorname{sign}(x) := 1$ if $x \ge 0$, and $\operatorname{sign}(x) := -1$ if $x < 0$,

$$\lambda_{s,s} = s(\lambda - 1) > \lambda_{s,s-1} > \cdots > \lambda_{s,1} = \lambda - s, \tag{2.29}$$

and $c_{s,j}(\lambda, a)$ are polynomials in the variables λ and a of degree at most s with respect to each variable. Hence, we obtain

$$h^{(k)}(x)w(x) = \sum_{s=0}^{k} \binom{k}{s} g^{(k-s)}(x)(\operatorname{sign}(x))^s \sum_{j=1}^{s} c_{s,j}(\lambda, a)|x|^{\lambda_{s,j}} \tag{2.30}$$

which implies that

$$\int_{\mathbb{R}} |h^{(k)}w|^p(x)\mathrm{d}x \le C \max_{0\le s\le k} \max_{1\le j\le s} \int_{t_{i-1}}^{t_i} |x|^{p\lambda_{s,j}} |g^{(k-s)}(x)|^p \mathrm{d}x.$$

From (2.29), the inequality $n^{1/\lambda} \le x \le (2n+2)n^{1/\lambda - 1}$ for $x \in [t_{i-1}, t_i]$, and

$$\int_{t_{i-1}}^{t_i} |g^{(k-s)}(x)|^p \mathrm{d}x = b_{k-s}^p \delta^{-p(k-s-1/p)} = b_{k-s}^p n^{p(k-s-1/p)(1-1/\lambda)},$$

we derive

$$\begin{aligned}
\int_{\mathbb{R}} |h^{(k)}w|^p(x)\mathrm{d}x &\le C \max_{0\le s\le k} \int_{t_{i-1}}^{t_i} |x|^{p\lambda_{s,s}} |g^{(k-s)}(x)|^p \mathrm{d}x \\
&\le C \max_{0\le s\le k} \left(n^{1/\lambda}\right)^{ps(\lambda-1)} \int_{t_{i-1}}^{t_i} |g^{(k-s)}(x)|^p \mathrm{d}x \\
&\le C \max_{0\le s\le k} n^{ps(\lambda-1)/\lambda} n^{p(k-s-1/p)(1-1/\lambda)} \\
&= C n^{p(1-1/\lambda)(k-1/p)} \le C n^{p(1-1/\lambda)(r-1/p)}.
\end{aligned}$$

If we define

$$\bar{h} := C^{-1} n^{-(1-1/\lambda)(r-1/p)} h,$$

then $\bar{h} \in \boldsymbol{W}_{p,w}^r(\mathbb{R})$, $\operatorname{supp}(\bar{h}) \subset (t_{i-1}, t_i)$ and

$$\begin{aligned}
\int_{\mathbb{R}} |(\bar{h}w)(x)|^q \mathrm{d}x &= C^{-q} n^{-q(1-1/\lambda)(r-1/p)} \int_{t_{i-1}}^{t_i} |g(x)|^q \mathrm{d}x \\
&= C^{-q} n^{-q(1-1/\lambda)(r-1/p)} \delta \int_0^1 |\varphi(x)|^q \mathrm{d}x \gg n^{-q(1-1/\lambda)(r-1/p+1/q)}
\end{aligned}$$

Since the interval (t_{i-1}, t_i) does not contain any point from the set $\{m_1, ..., m_n\}$, we have $\bar{h}(m_k) = 0$, $k = 1, ..., n$. Hence, by the inequality (2.23),

$$\varrho_n\big(\boldsymbol{W}_{p,w}^r(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \ge \|\bar{h}\|_{L_{p,w}(\mathbb{R})} \gg n^{-r_{\lambda,p,q}}.$$

The lower bound in (2.25) in the case $p < q$ is proven.

We now prove the lower bound in (2.25) in the case $p \ge q$. By (2.24) we have

$$\varrho_n\big(\boldsymbol{W}_{p,w}^r(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \ge d_n\big(\boldsymbol{W}_{p,w}^r(\mathbb{R}), L_{q,w}(\mathbb{R})\big), \tag{2.31}$$

hence it is sufficient to show that for $p \geq q$,

$$d_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \gg n^{-r_{\lambda,p,q}}. \tag{2.32}$$

From Lemma 2.3(i) we derive the inequality

$$\|\varphi\|_{W^r_{p,w}(\mathbb{R})} \ll m^{r_\lambda}\|\varphi\|_{L_{p,w}(\mathbb{R})} \quad \forall \varphi \in \mathcal{P}_{m+1}, \ \ \forall m \in \mathbb{N}.$$

This implies the inclusion

$$Cm^{-r_\lambda}\mathcal{B}^*_{N(m),p} \subset \boldsymbol{W}^r_{p,w}(\mathbb{R})$$

for some constant $C > 0$ independent of m, where $N(m) := 2j(m)$ and

$$\mathcal{B}^*_{N(m),p} := \left\{\varphi \in \mathcal{P}^*_{m+1} : \|\varphi\|_{L_{p,w}(\mathbb{R})} \leq 1\right\}.$$

is the unit ball of $N(m)$-dimensional subspace $L^*_{N(m),p} := \mathcal{P}^*_{m+1} \cap L_{p,w}(\mathbb{R})$ in $L_{p,w}(\mathbb{R})$. Given $n \in \mathbb{N}$, let $N := N(\bar{m}) := \min\{N(m) : \ N(m) \geq 2n\}$. Due to (2.5) we have

$$2n \leq N \asymp \bar{m} \asymp n. \tag{2.33}$$

Hence,

$$d_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \geq d_n\big(C\bar{m}^{-r_\lambda}\mathcal{B}^*_{N,p}, L^*_{N,q}\big) \asymp n^{-r_\lambda} d_n\big(\mathcal{B}^*_{N,p}, L^*_{N,q}\big). \tag{2.34}$$

For $1 \leq \nu < \infty$ and $M \in \mathbb{N}$, denote by ℓ^M_ν the normed space of $\boldsymbol{x} = (x_j)^M_{j=1}$ with $x_j \in \mathbb{R}$, equipped with the norm

$$\|\boldsymbol{x}\|_{\ell^M_\nu} := \left(\sum_{j=1}^M |x_j|^\nu\right)^{1/\nu},$$

and by B^M_ν the unit ball in ℓ^M_ν. From Lemma 2.2 it follows that for $\nu = p, q$, there holds the norm equivalence

$$\|\varphi\|_{L^\nu_w(\mathbb{R})} \asymp m^{-(1-1/\lambda)/\nu}\|\boldsymbol{\varphi}\|_{\ell^N_\nu} \ \ \varphi \in \mathcal{P}^*_{m+1}, \ \ \forall m \in \mathbb{N}, \tag{2.35}$$

where $\boldsymbol{\varphi} := \big(\varphi(x_{m,k})w(x_{m,k})\big)_{|k|\leq j(\bar{m})}$. Hence by (2.33) and the well-known equality

$$d_n\big(B^N_p, \ell^N_q\big) = (N-n)^{1/q-1/p}, \quad N > n \ (p \geq q) \tag{2.36}$$

(see, e.g., [26, Page 232]) we deduce that

$$\begin{aligned} d_n\big(\mathcal{B}^*_{N,p}, L^*_{N,q}\big) &\asymp n^{(1-1/\lambda)(1/p-1/q)} d_n\big(B^N_p, \ell^N_q\big) \\ &= n^{(1-1/\lambda)(1/p-1/q)}(N-n)^{1/q-1/p} \asymp n^{\delta_{\lambda,p,q}}. \end{aligned}$$

This together with (2.31), (2.32) and (2.34) proves the lower bound in (2.25) in the case $p \geq q$. □

By using a similar technique with some modification, we now give upper and lower of $\varrho_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big)$ in the case when $p = \infty$ and $1 \leq q \leq \infty$ and when the generating weights w and v are of the form:

$$w(x) := \exp\left(-a|x|^\lambda + b\right), \quad \lambda > 1, \ a > 0, \ b \in \mathbb{R}, \tag{2.37}$$

and

$$v(x) := w^2(x) = \exp\left(-2a|x|^\lambda + 2b\right), \tag{2.38}$$

where λ, a, b are the same as in (2.37).

Lemma 2.6 *Let $1 \le q \le \infty$ and $r_{\lambda,\infty,q} > 0$. Let the generating weights w and v be given by (2.37) and (2.38). Then we have that for every $f \in W^r_{\infty,w}(\mathbb{R})$,*

$$f = \sum_{k \in \mathbb{N}_0} \Delta_k^I(f) \tag{2.39}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R})$ of the series. Moreover,

$$\left\| \Delta_k^I(f) \right\|_{L_{q,w}(\mathbb{R})} \ll 2^{-r_{\lambda,\infty,q} k} k \, \|f\|_{W^r_{\infty,w}(\mathbb{R})}, \quad k \in \mathbb{N}_0. \tag{2.40}$$

Proof. We first prove the inequality

$$\|f - I_m(f)\|_{L^\infty_w(\mathbb{R})} \ll m^{-r_\lambda} \log m \, \|f\|_{W^r_{\infty,w}(\mathbb{R})}, \quad f \in W^r_{\infty,w}(\mathbb{R}). \tag{2.41}$$

For $1 \le p \le \infty$ and $f \in L_{p,w}(\mathbb{R})$, we define

$$E_m(f)_{w,p} := \inf_{\varphi \in \mathcal{P}_m} \|f - \varphi\|_{L_{p,w}(\mathbb{R})}. \tag{2.42}$$

Then there holds the inequality [12]

$$E_m(f)_{w,p} \ll m^{-r_\lambda} \|f\|_{W^r_{p,w}(\mathbb{R})}, \quad f \in W^r_{p,w}(\mathbb{R}). \tag{2.43}$$

On the other hand, by [18, Theorem 3.2]

$$\|f - I_m(f)\|_{L^\infty_w(\mathbb{R})} \ll \log m \, E_{m'}(f)_{w,\infty} + e^{-cm} \|f\|_{L^\infty_w(\mathbb{R})}, \tag{2.44}$$

where $m' = \left\lfloor \left(\frac{\rho}{\rho+1}\right)^\lambda m \right\rfloor \asymp m$ and c is a positive constant independent of m and f. From the last two inequalities we easily deduce (2.41). Now by using (2.41), we can prove the lemma in the same way as the proof of Lemma 2.4. ∎

Theorem 2.7 *Let $1 \le q \le \infty$ and $r_{\lambda,\infty,q} > 0$. Let the generating weights w and v be given by (2.37) and (2.38). For any $n \in \mathbb{N}$, let $k(n)$ be the largest integer such that $2^{k(n)} \le n$. Then we have*

$$n^{-r_{\lambda,\infty,q}} \ll \varrho_n\big(\boldsymbol{W}^r_{\infty,w}(\mathbb{R}), L_{q,w}(\mathbb{R})\big) \le \sup_{f \in \boldsymbol{W}^r_{\infty,w}(\mathbb{R})} \left\| f - I_{m_{k(n)}}(f) \right\|_{L_{q,w}(\mathbb{R})} \ll n^{-r_{\lambda,\infty,q}} \log n. \tag{2.45}$$

Proof. The proof of the upper bound in (2.45) is similar to the proof of the upper bound in (2.25) of Theorem 2.5 with replacing Lemma 2.4 by Lemma 2.6. The lower bound in (2.45) can be proven in the same way as the proof of the lower bound (2.25) of Theorem 2.5 in the case $p \ge q$. ∎

3 Sparse-grid sampling recovery in the setting (i)

In this section, for the setting (i) in the high dimensional case ($d \ge 2$), we establish upper and lower bounds of $\varrho_n\big(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)\big)$ and construct linear sampling algorithms based on step-hyperbolic-cross grids of sparse sampled points which realize the upper bounds.

For $\boldsymbol{x} \in \mathbb{R}^d$ and $e \subset \{1,...,d\}$, let $\boldsymbol{x}^e \in \mathbb{R}^{|e|}$ be defined by $(x^e)_i := x_i$, and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$ by $(\bar{x}^e)_i := x_i$, $i \in \{1,...,d\} \setminus e$. With an abuse we write $(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) = \boldsymbol{x}$. For the proof of the following lemma, see [6, Lemma 3.2].

Lemma 3.1 *Let $1 \le p \le \infty$, $e \subset \{1,...,d\}$ and $\boldsymbol{r} \in \mathbb{N}_0^d$. Assume that f is a function on $\mathbb{R}^d$ such that for every $\boldsymbol{k} \le \boldsymbol{r}$, $D^{\boldsymbol{k}} f \in L_{p,w}(\mathbb{R}^d)$. Put for $\boldsymbol{k} \le \boldsymbol{r}$ and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$,*

$$g(\boldsymbol{x}^e) := D^{\bar{\boldsymbol{k}}^e} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e).$$

Then $D^{\boldsymbol{s}} g \in L_{p,w}(\mathbb{R}^{|e|})$ for every $\boldsymbol{s} \le \boldsymbol{k}^e$ and almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$.

Based on the operators Δ^I_k, $k \in \mathbb{N}_0$, defined in (2.18), we construct sampling operators on $\mathbb{R}^d$ by using the well-known Smolyak algorithm. For convenience, with an abuse we make use of the notation:

$$S_{2^k} := I_{m_k} \in \mathcal{S}_{2^k}, \quad k \in \mathbb{N}_0,$$

where we recall that the sampling operator I_m is given as in (2.9), and m_k is the largest number such that $m_k + 1 \le 2^k$. Then we have

$$\Delta^I_k = S_{2^k} - S_{2^{k-1}}, \quad k \in \mathbb{N}_0.$$

We also define for $k \in \mathbb{N}_0$, the one-dimensional operators

$$E_k f := f - S_{2^k}(f), \quad k \in \mathbb{N}_0.$$

For $\boldsymbol{k} \in \mathbb{N}^d$, the d-dimensional operators $S_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}$ and $E_{\boldsymbol{k}}$ are defined as the tensor product of one-dimensional operators:

$$S_{2^{\boldsymbol{k}}} := \bigotimes_{i=1}^d S_{2^{k_i}}, \quad \Delta_{\boldsymbol{k}} := \bigotimes_{i=1}^d \Delta_{k_i}, \quad E_{\boldsymbol{k}} := \bigotimes_{i=1}^d E_{k_i}, \tag{3.1}$$

where $2^{\boldsymbol{k}} := (2^{k_1}, \cdots, 2^{k_d})$ and the univariate operators $S_{2^{k_j}}$, Δ_{k_j} and E_{k_j} are successively applied to the univariate functions $\bigotimes_{i<j} S_{2^{k_i}}(f)$, $\bigotimes_{i<j} \Delta_{k_i}(f)$ and $\bigotimes_{i<j} E_{k_i}$, respectively, by considering them as functions of variable x_j with the other variables held fixed. The operators $S_{2^{\boldsymbol{k}}}$, $\Delta_{\boldsymbol{k}}$ and $E_{\boldsymbol{k}}$ are well-defined for continuous functions on $\mathbb{R}^d$, in particular for ones from $W^r_{p,w}(\mathbb{R}^d)$.

Notice that $S_{2^{\boldsymbol{k}}}$ is a sampling operator on $\mathbb{R}^d$ given by

$$S_{2^{\boldsymbol{k}}} f = \sum_{\boldsymbol{s}=\boldsymbol{1}}^{m_{\boldsymbol{k}}} f(\boldsymbol{x}_{m_{\boldsymbol{k}},\boldsymbol{s}}) \varphi_{m_{\boldsymbol{k}},\boldsymbol{s}}, \quad \{\boldsymbol{x}_{m_{\boldsymbol{k}},\boldsymbol{s}}\}_{\boldsymbol{1} \le \boldsymbol{s} \le m_{\boldsymbol{k}}} \subset \mathbb{R}^d, \tag{3.2}$$

where

$$m_{\boldsymbol{k}} := (m_{k_1}, ..., m_{k_d}), \quad \boldsymbol{x}_{m_{\boldsymbol{k}},\boldsymbol{s}} := \left(x_{m_{k_1},s_1}, ..., x_{m_{k_d},s_d}\right), \quad \varphi_{m_{\boldsymbol{k}},\boldsymbol{s}} := \bigotimes_{i=1}^{d} \varphi_{m_{k_i},s_i},$$

and the summation $\sum_{\boldsymbol{s}=\boldsymbol{1}}^{m_{\boldsymbol{k}}}$ means that the sum is taken over all $\boldsymbol{s}$ such that $\boldsymbol{1} \le \boldsymbol{s} \le m_{\boldsymbol{k}}$. Hence we derive that

$$\Delta_{\boldsymbol{k}} f = \sum_{e \subset \{1,...,d\}} (-1)^{d-|e|} S_{2^{\boldsymbol{k}(e)}} f = \sum_{e \subset \{1,...,d\}} (-1)^{d-|e|} \sum_{\boldsymbol{s}=\boldsymbol{1}}^{m_{\boldsymbol{k}(e)}} f(\boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}) \varphi_{\boldsymbol{k}(e),\boldsymbol{s}}, \tag{3.3}$$

where $\boldsymbol{k}(e) \in \mathbb{N}_0^d$ is defined by $k(e)_i = k_i$, $i \in e$, and $k(e)_i = \max(k_i - 1, 0)$, $i \notin e$. We also have

$$E_{\boldsymbol{k}}(f)(\boldsymbol{x}) = \sum_{e \subset \{1,...,d\}} (-1)^{|e|} S_{2^{\boldsymbol{k}^e}}(f(\cdot, \bar{\boldsymbol{x}}^e))(\boldsymbol{x}^e) \tag{3.4}$$

Lemma 3.2 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. Then we have that*

$$\left\| E_{\boldsymbol{k}}(f) \right\|_{L_{q,w}(\mathbb{R}^d)} \le C 2^{-r_{\lambda,p,q}|\boldsymbol{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d, \quad f \in W^r_{p,w}(\mathbb{R}^d).$$

Proof. The case $d = 1$ of the lemma follows from Lemma 2.1. For simplicity we prove the lemma for the case $d = 2$. The general case can be proven in the same way by induction on d. We make use of the temporary notation:

$$\|f\|_{W^r_{p,w}(\mathbb{R}),2}(x_1) := \|f(x_1, \cdot)\|_{W^r_{p,w}(\mathbb{R})}.$$

From Lemma 3.1 it follows that $f(\cdot, x_2) \in W^r_{p,w}(\mathbb{R})$ for every $x_2 \in \mathbb{R}$. Hence, by using Lemma 2.1 we obtain that

$$\begin{aligned}
\left\| E_{(k_1,k_2)}(f) \right\|_{L_{q,w}(\mathbb{R}^2)} &= \left\| \left\| E_{k_2}(E_{k_1}(f)) \right\|_{L_{q,w}(\mathbb{R})} \right\|_{L_{q,w}(\mathbb{R})} \le C 2^{-r_{\lambda,p,q}k_2} \left\| \left\| E_{k_1}(f) \right\|_{W^r_{p,w}(\mathbb{R}),2}(\cdot) \right\|_{L_{q,w}(\mathbb{R})} \\
&\le C 2^{-r_{\lambda,p,q}k_2} \| 2^{-r_{\lambda,p,q}k_1} \|f\|_{W^r_{p,w}(\mathbb{R}),2}(\cdot) \|_{W^r_{p,w}(\mathbb{R})} \\
&= C 2^{-r_{\lambda,p,q}|\boldsymbol{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^2)}.
\end{aligned}$$

□

We say that $\boldsymbol{k} \to \infty$, $\boldsymbol{k} \in \mathbb{N}_0^d$, if and only if $k_i \to \infty$ for every $i = 1, ..., d$.

Lemma 3.3 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. Then we have that for every $f \in W^r_{p,w}(\mathbb{R}^d)$,*

$$f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} \Delta_{\boldsymbol{k}}(f) \tag{3.5}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R}^d)$ of the series, and

$$\left\| \Delta_{\boldsymbol{k}}(f) \right\|_{L_{q,w}(\mathbb{R}^d)} \le C 2^{-r_{\lambda,p,q}|\boldsymbol{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d. \tag{3.6}$$

Proof. The operator $\Delta_{\boldsymbol{k}}$ can be represented in the form

$$\Delta_{\boldsymbol{k}}(f) = \sum_{e \subset \{1,\dots,d\}} (-1)^{|e|} E_{\boldsymbol{k}(e)}(f).$$

Therefore, by using Lemma 3.2 we derive that for every $f \in W^r_{p,w}(\mathbb{R}^d)$ and $\boldsymbol{k} \in \mathbb{N}^d_0$,

$$\begin{aligned}\left\|\Delta_{\boldsymbol{k}} f\right\|_{L_{q,w}(\mathbb{R}^d)} &\le \sum_{e \subset \{1,\dots,d\}} \left\|E_{\boldsymbol{k}(e)} f\right\|_{L_{q,w}(\mathbb{R}^d)} \\ &\le \sum_{e \subset \{1,\dots,d\}} C 2^{-r_{\lambda,p,q}|\boldsymbol{k}(e)|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \le C 2^{-r_{\lambda,p,q}|\boldsymbol{k}|_1} \|f\|_{W^r_{p,w}(\mathbb{R}^d)}\end{aligned}$$

which proves (3.6) and hence the absolute convergence of the series in (3.11) follows. Notice that

$$f - S_{2^{\boldsymbol{k}}} f = \sum_{e \subset \{1,\dots,d\},\ e \neq \varnothing} (-1)^{|e|} E_{\boldsymbol{k}^e} f,$$

where recall $\boldsymbol{k}^e \in \mathbb{N}^d_0$ is defined by $k^e_i = k_i$, $i \in e$, and $k^e_i = 0$, $i \notin e$. By using Lemma 3.2 we derive for $\boldsymbol{k} \in \mathbb{N}^d_0$ and $f \in W^r_{p,w}(\mathbb{R}^d)$,

$$\begin{aligned}\left\|f - S_{2^{\boldsymbol{k}}} f\right\|_{L_{q,w}(\mathbb{R}^d)} &\le \sum_{e \subset \{1,\dots,d\},\ e \neq \varnothing} \left\|E_{\boldsymbol{k}^e} f\right\|_{L_{q,w}(\mathbb{R}^d)} \\ &\le C \max_{e \subset \{1,\dots,d\},\ e \neq \varnothing} \ \max_{1 \le i \le d} 2^{-r_{\lambda,p,q}|k^e_i|} \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \\ &\le C \max_{1 \le i \le d} 2^{-r_{\lambda,p,q}|k_i|} \|f\|_{W^r_{p,w}(\mathbb{R}^d)},\end{aligned}$$

which is going to 0 when $\boldsymbol{k} \to \infty$. This together with the obvious equality

$$S_{2^{\boldsymbol{k}}} = \sum_{\boldsymbol{s} \le \boldsymbol{k}} \Delta_{\boldsymbol{s}}$$

proves (3.11). ∎

We now define an algorithm for sampling on sparse grids initiated by Smolyak (for detail see [11, Sections 4.2 and 5.3]). For $m \in \mathbb{N}_0$, we define the operator

$$P_m := \sum_{|\boldsymbol{k}|_1 \le m} \Delta_{\boldsymbol{k}}. \tag{3.7}$$

From (3.3) we can see that P_m is a linear sampling algorithm on $\mathbb{R}^d$ of the form (1.7) :

$$P_m(f) = \sum_{|\boldsymbol{k}|_1 \le m} \ \sum_{e \subset \{1,\dots,d\}} (-1)^{d-|e|} \sum_{\boldsymbol{s}=\boldsymbol{1}}^{2^{\boldsymbol{k}(e)}} f(\boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}) \phi_{\boldsymbol{k}(e),\boldsymbol{s}} = \sum_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(m)} f(\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}) \phi_{\boldsymbol{k},e,\boldsymbol{s}}, \tag{3.8}$$

where

$$\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}} := \boldsymbol{x}_{\boldsymbol{k}(e),\boldsymbol{s}}, \quad \phi_{\boldsymbol{k},e,\boldsymbol{s}} := (-1)^{d-|e|} \phi_{\boldsymbol{k}(e),\boldsymbol{s}}$$

and

$$G(m) := \{(\boldsymbol{k}, e, \boldsymbol{s}) : \ |\boldsymbol{k}|_1 \le m, \, e \subset \{1, ..., d\}\,, \, \boldsymbol{1} \le \boldsymbol{s} \le \boldsymbol{k}(e)\}$$

is a finite set. The set of sampled points in this operator

$$H(m) := \{\boldsymbol{x}_{\boldsymbol{k},e,\boldsymbol{s}}\}_{(\boldsymbol{k},e,\boldsymbol{s}) \in G(m)}$$

is a step hyperbolic cross in the function domain $\mathbb{R}^d$. The number of sampled in the operator P_m is

$$|H(m)| = |G(m)| = \sum_{|\boldsymbol{k}|_1 \le m} \ \sum_{e \subset \{1,\dots,d\}} 2^{|\boldsymbol{k}(e)|_1}$$

which can be estimated as

$$|H(m)| \asymp \sum_{|\boldsymbol{k}|_1 \le m} 2^{|\boldsymbol{k}|_1} \ \asymp \ 2^m m^{d-1}, \quad m \ge 1. \tag{3.9}$$

The sampling operator P_m plays a basic role in the proof of the upper bound (1.12).

Lemma 3.4 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$. Then we have for every $m > 1$ and every $f \in W^r_{p,w}(\mathbb{R}^d)$,*

$$\big\|f - P_m(f)\big\|_{L_{q,w}(\mathbb{R}^d)} \ll \|f\|_{W^r_{p,w}(\mathbb{R}^d)} \begin{cases} 2^{-r_\lambda} m^{d-1} & \text{if } \ p = q, \\ 2^{-r_{\lambda,p,q} m} m^{(d-1)/q} & \text{if } \ p \ne q < \infty, \\ 2^{-r_{\lambda,p,q} m} m^{(d-1)} & \text{if } \ p \ne q = \infty. \end{cases} \tag{3.10}$$

Proof. From Lemma 3.3 we derive that for $m \ge 1$ and $f \in W^r_{p,w}(\mathbb{R}^d)$,

$$f - P_m(f) = \sum_{|\boldsymbol{k}|_1 > m} \Delta_{\boldsymbol{k}} f, \quad \Delta_{\boldsymbol{k}} f \in \mathcal{P}_{2^{\boldsymbol{k}}}, \tag{3.11}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R}^d)$ of the series, and there holds (3.6). If $p \ne q$, applying Lemma A.2 in Appendix and (3.6), we obtain (3.10):

$$\begin{aligned} \big\|f - P_m(f)\big\|^q_{L_{q,w}(\mathbb{R}^d)} &\ll \sum_{|\boldsymbol{k}|_1 > m} \big\|2^{\delta_{\lambda,p,q}|\boldsymbol{k}|_1} \Delta_{\boldsymbol{k}} f\big\|^q_{L_{p,w}(\mathbb{R}^d)} \ll \sum_{|\boldsymbol{k}|_1 > m} 2^{-qr_{\lambda,p,q}|\boldsymbol{k}|_1} \|f\|^q_{W^r_{p,w}(\mathbb{R}^d)} \\ &= \|f\|^q_{W^r_{p,w}(\mathbb{R}^d)} \sum_{|\boldsymbol{k}|_1 > m} 2^{-qr_{\lambda,p,q}|\boldsymbol{k}|_1} \ll 2^{-qr_{\lambda,p,q}m} m^{d-1} \|f\|^q_{W^r_{p,w}(\mathbb{R}^d)}. \end{aligned}$$

If $p = q$ or $p \ne q = \infty$, the upper bound (3.10) can be derived similarly by using (3.11), (3.6) and the inequality

$$\big\|f - P_m(f)\big\|_{L_{q,w}(\mathbb{R}^d)} \le \sum_{|\boldsymbol{k}|_1 > m} \big\|\Delta_{\boldsymbol{k}} f\big\|_{L_{q,w}(\mathbb{R}^d)}.$$

∎

Theorem 3.5 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r_{\lambda,p,q} > 0$, and denote*

$$\varrho_n := \varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d))).$$

For every $n \in \mathbb{N}$, let m_n be the largest number such that $|G(m_n)| \le n$. Then P_{m_n} defines a sampling operator belonging to $\mathcal{S}_n$, and we have that

$$\varrho_n \le \sup_{f \in \boldsymbol{W}^r_{p,w}(\mathbb{R}^d)} \left\| f - P_{m_n} f \right\|_{L_{q,w}(\mathbb{R}^d)} \ll \begin{cases} n^{-r_\lambda} (\log n)^{(r_\lambda+1)(d-1)} & \text{if } \ p = q, \\ n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q}+1/q)(d-1)} & \text{if } \ p \ne q < \infty, \\ n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q}+1)(d-1)} & \text{if } \ p \ne q = \infty; \end{cases} \tag{3.12}$$

and

$$\varrho_n \gg \begin{cases} n^{-r_\lambda} (\log n)^{r_\lambda(d-1)} & \text{if } \ p = q, \\ n^{-r_{\lambda,p,q}} (\log n)^{r_{\lambda,p,q}(d-1)} & \text{if } \ p < q, \\ n^{-r_{\lambda,p,q}} & \text{if } \ p > q. \end{cases} \tag{3.13}$$

Proof. Let us prove the case $p \ne q < \infty$ of the upper bounds (3.12). The cases $p = q$ and $p \ne q = \infty$ can be proven in a similar manner. From (3.9) it follows

$$2^{m_n} m_n^{d-1} \asymp |G(m_n)| \asymp n.$$

Hence we deduce the asymptotic equivalences

$$2^{-m_n} \asymp n^{-1} (\log n)^{d-1}, \quad m_n \asymp \log n,$$

which together with Lemma 3.4 yield that

$$\begin{aligned} \varrho_n &\le \sup_{f \in \boldsymbol{W}^r_{p,w}(\mathbb{R}^d)} \left\| f - P_{m_n} f \right\|_{L_{q,w}(\mathbb{R}^d)} \\ &\le C 2^{-r_\lambda m_n} m_n^{(d-1)/q} \asymp n^{-r_{\lambda,p,q}} (\log n)^{(r_{\lambda,p,q}+1/q)(d-1)}. \end{aligned}$$

The upper bound in (3.12) for the case $p \ne q < \infty$ is proven.

We now prove the lower bounds in (3.13). The lower bound for the case $p > q$ can be derived from the lower bound in Theorem 2.5 for $d = 1$. We prove it for the cases $p = q$ and $p < q$ merged as the case $p \le q$, by using the inequality (2.23). For $M \ge 1$, we define the set

$$\Gamma_d(M) := \left\{ \boldsymbol{s} \in \mathbb{N}^d : \prod_{i=1}^d s_i \le 2M, \ s_i \ge M^{1/d}, \ i = 1, ..., d \right\}.$$

Then we have by [6, (3.15)]

$$|\Gamma_d(M)| \asymp M (\log M)^{d-1}, \quad M > 1. \tag{3.14}$$

For a given $n \in \mathbb{N}$, let $\{\boldsymbol{m}_1, ..., \boldsymbol{m}_n\} \subset \mathbb{R}^d$ be arbitrary n points. Denote by M_n the smallest number such that $|\Gamma_d(M_n)| \ge n+1$. We define the d-parallelepiped $K_{\boldsymbol{s}}$ for $\boldsymbol{s} \in \mathbb{N}_0^d$ of size

$$\delta := M_n^{\frac{1/\lambda - 1}{d}}$$

by

$$K_{\boldsymbol{s}} := \prod_{i=1}^{d} K_{s_i}, \quad K_{s_i} := (\delta s_i, \delta s_{i-1}).$$

Since $|\Gamma_d(M_n)| > n$, there exists a multi-index $\boldsymbol{s} \in \Gamma_d(M_n)$ such that $K_{\boldsymbol{s}}$ does not contain any point from $\{\boldsymbol{m}_1, ..., \boldsymbol{m}_n\}$.

As in the proof of Theorem 2.5, we take a nonnegative function $\varphi \in C_0^\infty([0,1])$, $\varphi \neq 0$, and put

$$b_s := \|\varphi^{(s)}(y)\|_{L_p([0,1])}, \;\; s = 0, 1, ..., r. \tag{3.15}$$

For $i = 1, ..., d$, we define the univariate functions g_i in variable x_i by

$$g_i(x_i) := \begin{cases} \varphi(\delta^{-1}(x_i - \delta s_{i-1})), & x_i \in K_{s_i}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.16}$$

Then the multivariate functions g and h on $\mathbb{R}^d$ are defined by

$$g(\boldsymbol{x}) := \prod_{i=1}^{d} g_i(x_i),$$

and

$$h(\boldsymbol{x}) := (gw^{-1})(\boldsymbol{x}) = \prod_{i=1}^{d} g_i(x_i) w^{-1}(x_i) =: \prod_{i=1}^{d} h_i(x_i). \tag{3.17}$$

Let us estimate the norm $\|h\|_{W^r_{p,w}(\mathbb{R}^d)}$. For every $\boldsymbol{k} \in \mathbb{N}_0^d$ with $0 \le |\boldsymbol{k}|_\infty \le r$, we prove the inequality

$$\int_{\mathbb{R}^d} \left|(D^{\boldsymbol{k}} h) w\right|^p(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \le C M_n^{(1-1/\lambda)(r-1/p)}. \tag{3.18}$$

We have

$$D^{\boldsymbol{k}} h = \prod_{i=1}^{d} h_i^{(k_i)}. \tag{3.19}$$

Similarly to (2.27)–(2.30) we derive that for every $i = 1, ..., d$,

$$h_i^{(k_i)}(x_i) w(x_i) = \sum_{\nu_i=0}^{k_i} \binom{k_i}{\nu_i} g_i^{(k_i-\nu_i)}(x_i) (\operatorname{sign}(x_i))^{\nu_i} \sum_{\eta_i=1}^{\nu_i} c_{\nu_i,\eta_i}(\lambda, a) |x_i|^{\lambda_{\nu_i,\eta_i}},$$

where

$$\lambda_{\nu_i,\nu_i} = \nu_i(\lambda - 1) > \lambda_{\nu_i,\nu_i-1} > \cdots > \lambda_{\nu_i,1} = \lambda - \nu_i,$$

and $c_{\nu_i,\eta_i}(\lambda, a)$ are polynomials in the variables λ and a of degree at most ν_i with respect to each variable. This together with (3.15), (3.16) and the inequalities $s_i \ge M_n^{\frac{1}{d}}$ and

$\lambda_{\nu_i,\nu_i} = \nu_i(\lambda - 1) \geq 0$ yields that

$$\begin{aligned}
\int_{\mathbb{R}} \left|h_i^{(k_i)}(x_i)w(x_i)\right|^p \mathrm{d}x_i &\leq C \max_{0\leq\nu_i\leq k_i} \max_{1\leq\eta_i\leq\nu_i} \int_{K_{s_i}} |x_i|^{p\lambda_{\nu_i,\eta_i}} \left|g^{(k_i-\nu_i)}(x_i)\right|^p \mathrm{d}x_i \\
&\leq C \max_{0\leq\nu_i\leq k_i} (\delta s_i)^{p\lambda_{\nu_i,\nu_i}} \int_{K_{s_i}} \left|g^{(k_i-\nu_i)}(x_i)\right|^p \mathrm{d}x_i \\
&\leq C \max_{0\leq\nu_i\leq k_i} (\delta s_i)^{p\nu_i(\lambda-1)} \delta^{-p(k_i-\nu_i-1/p)} b^p_{k_i-\nu_i} \\
&= C\delta^{-p(k_i-1/p)} \max_{0\leq\nu_i\leq k_i} \left(\delta^\lambda s_i^{\lambda-1}\right)^{p\nu_i}.
\end{aligned} \tag{3.20}$$

Since $s_i \geq M_n^{\frac{1}{d}}$ and $\delta := M_n^{\frac{1/\lambda-1}{d}}$, we have that $\delta^\lambda s_i^{\lambda-1} \geq 1$, and consequently,

$$\max_{0\leq\nu_i\leq k_i} \left(\delta^\lambda s_i^{\lambda-1}\right)^{p\nu_i} = \left(\delta^\lambda s_i^{\lambda-1}\right)^{pk_i}.$$

This equality, the estimates (3.20) and the inequalities $0 \leq k_i \leq r$ and $\delta s_i \geq 1$ yield that

$$\begin{aligned}
\int_{\mathbb{R}} \left|h_i^{(k_i)}(x_i)w(x_i)\right|^p \mathrm{d}x_i &\leq C\delta^{-p(k_i-1/p)} \left(\delta^\lambda s_i^{\lambda-1}\right)^{pk_i} = C\delta \left(\delta s_i\right)^{pk_i(\lambda-1)} \\
&\leq C\delta \left(\delta s_i\right)^{pr(\lambda-1)} = C\delta^{pr(\lambda-1)+1} s_i^{pr(\lambda-1)}.
\end{aligned}$$

Hence, by (3.19) we deduce

$$\begin{aligned}
\int_{\mathbb{R}^d} |(D^{\boldsymbol{k}}h)w|^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \prod_{i=1}^d \int_{\mathbb{R}} \left|h^{(k_i)}(x_i)w(x_i)\right|^p \mathrm{d}x_i \\
&\leq C\prod_{i=1}^d \delta^{pr(\lambda-1)+1} s_i^{pr(\lambda-1)} \leq C\delta^{d(p(r(\lambda-1)+1)} \left(\prod_{i=1}^d s_i\right)^{pr(\lambda-1)}.
\end{aligned}$$

Since $\prod_{i=1}^d s_i \leq 2M_n$, $\delta := M_n^{\frac{1/\lambda-1}{d}}$ and $\lambda > 1$, we can continue the estimation as

$$\int_{\mathbb{R}^d} |(D^{\boldsymbol{k}}h)w|^p(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq CM_n^{p(r(\lambda-1)+1/p)(1/\lambda-1)} M_n^{pr(\lambda-1)} = CM_n^{p(1-1/\lambda)(r-1/p)},$$

which completes the proof of the inequality (3.18). This inequality means that $h \in W^r_{p,w}(\mathbb{R}^d)$ and

$$\|h\|_{W^r_{p,w}(\mathbb{R}^d)} \leq CM_n^{(1-1/\lambda)(r-1/p)}.$$

If we define

$$\bar{h} := C^{-1}M_n^{-(1-1/\lambda)(r-1/p)}h,$$

then $\bar{h}$ is nonnegative, $\bar{h} \in \boldsymbol{W}^r_{p,w}(\mathbb{R})$, $\operatorname{supp}\bar{h} \subset K_{\boldsymbol{s}}$ and by (3.15)–(3.17), we have that for

$q < \infty$,

$$\int_{\mathbb{R}^d} |\bar{h}w|^q(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = C^{-q} M_n^{-q(1-1/\lambda)(r-1/p)} \int_{\mathbb{R}^d} |hw|^q(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$
$$= C^{-q} M_n^{-q(1-1/\lambda)(r-1/p)} \prod_{i=1}^d \int_{K_{s_i}} |g_i(x_i)|^q \mathrm{d}x_i$$
$$= C^{-q} M_n^{-q(1-1/\lambda)(r-1/p)} \prod_{i=1}^d \delta \int_0^1 |\varphi(x)|^q \mathrm{d}x$$
$$= C'^q M_n^{-q(1-1/\lambda)(r-1/p)} M_n^{1/\lambda-1} = C'^q M_n^{-q r_{\lambda,p,q}}.$$

From the definition of M_n and (3.14) it follows that

$$M_n(\log M_n)^{d-1} \asymp |\Gamma(M_n)| \asymp n,$$

which implies that $M_n^{-1} \asymp n^{-1}(\log n)^{d-1}$. This allows to receive the estimate

$$\|\bar{h}\|_{L_{q,w}(\mathbb{R}^d)} = C' M_n^{-r_{\lambda,p,q}} \gg n^{-r_{\lambda,p,q}} (\log n)^{r_{\lambda,p,q}(d-1)}. \tag{3.21}$$

Since the interval $K_{\boldsymbol{s}}$ does not contain any point from the set $\{\boldsymbol{m}_1, ..., \boldsymbol{m}_n\}$ which has been arbitrarily chosen, we have

$$\bar{h}(\boldsymbol{m}_k) = 0, \quad k = 1, ..., n.$$

Hence, by the inequality (2.23) and (3.21) we have that

$$\varrho_n \geq \|\bar{h}\|_{L_{q,w}(\mathbb{R}^d)} \gg n^{-r_{\lambda,p,q}} (\log n)^{r_{\lambda,p,q}(d-1)}.$$

The lower bound in (3.12) for the case $p \leq q < \infty$ is proven. It can be proven similarly for the case $p < q = \infty$ with a certain modification. □

We have proven upper and lower bounds of the sampling widths $\varrho_n(\boldsymbol{W}^r_{p,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d))$ for $1 < p < \infty$ and $1 \leq q \leq \infty$. In a similar way with a certain modification, we can upper and lower bounds of these sampling widths for $p = \infty$ and $1 \leq q \leq \infty$ in the case when the generating weights w and v are of the form (2.37) and (2.38).

More precisely, by using the same technique and by replacing (2.10) in Lemma 2.1 with (2.41) we prove

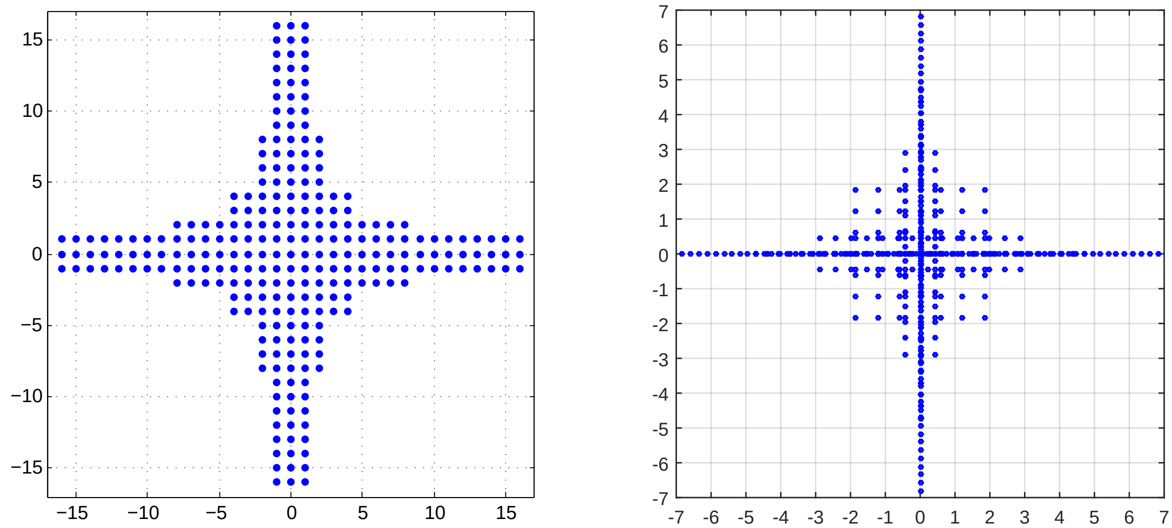
Lemma 3.6 *Let the generating weights w and v be given by (2.37) and (2.38). Let $1 \leq q \leq \infty$ and $r_{\lambda,\infty,q} > 0$. Then we have that for every $f \in W^r_{\infty,w}(\mathbb{R}^d)$,*

$$f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} \Delta_{\boldsymbol{k}}(f) \tag{3.22}$$

with absolute convergence in the space $L_{q,w}(\mathbb{R})$ of the series. Moreover,

$$\left\|\Delta_{\boldsymbol{k}} f\right\|_{L_{q,w}(\mathbb{R}^d)} \leq C 2^{-r_{\lambda,\infty,q}|\boldsymbol{k}|_1} |\boldsymbol{k}|_1^d \, \|f\|_{W^r_{\infty,w}(\mathbb{R}^d)}, \quad \boldsymbol{k} \in \mathbb{N}_0^d. \tag{3.23}$$

In the next step, similarly to Lemma 3.4, this lemma implies



A classical HC in frequency domain   A Hermite HC in function domain

Figure 1: Different hyperbolic crosses (HC)($d = 2$)

Lemma 3.7 *Under the assumptions of Lemma 3.6 we have that*

$$\left\|f - P_m(f)\right\|_{L_{q,w}(\mathbb{R}^d)} \ll \|f\|_{W^r_{\infty,w}(\mathbb{R}^d)} 2^{-r_{\lambda,\infty,q} m} m^{2d-1}, \quad m > 1, \ f \in W^r_{\infty,w}(\mathbb{R}^d). \tag{3.24}$$

Analogously to Theorem 3.5, from Lemmata 3.6 and 3.7 we deduce the following results.

Theorem 3.8 *Under the assumptions of Lemma 3.6, denote*

$$\varrho_n := \varrho_n(\boldsymbol{W}^r_{\infty,w}(\mathbb{R}^d), L_{q,w}(\mathbb{R}^d)).$$

For every $n \in \mathbb{N}$, let m_n be the largest number such that $|G(m_n)| \le n$. Then P_{m_n} defines a sampling operator belonging to $\mathcal{S}_n$, and we have that

$$n^{-r_{\lambda,\infty,q}} \ll \varrho_n \le \sup_{f \in \boldsymbol{W}^r_{\infty,w}(\mathbb{R}^d)} \left\|f - P_{m_n}(f)\right\|_{L_{q,w}(\mathbb{R}^d)} \ll n^{-r_{\lambda,\infty,q}} (\log n)^{r_{\lambda,\infty,q}(d-1)+2d-1}. \tag{3.25}$$

The grids of sample points $H(m)$ in the algorithm P_m, which are formed from the non-equidistant zeros of the orthonornal polynomials $p_m(w)$, as noticed above, is a step hyperbolic cross on the function domain $\mathbb{R}^d$. The grids $H(m)$ completely differ from classical Smolyak grids on the function domain $[-1,1]^d$ which are used in sampling recovery for functions having a mixed smoothness (see, e.g., [11, Section 5.3] for detail). In Figure 1, a step hyperbolic cross grid in the two-dimensional function domain $\mathbb{R}^2$ in the right picture is designed for the Hermite weight $w(\boldsymbol{x}) = \exp(-x_1^2 - x_2^2)$, and a classical step hyperbolic cross in the two-dimensional frequency domain is in the left picture. The classical Smolyak grid on the domain $[-1,1]^2$ is in the left picture of Figure 2. The grids $H(m)$ are very sparsely distributed inside the d-cube

$$K(m) := \left\{\boldsymbol{x} \in \mathbb{R}^d : |x_i| \le C 2^{m/\lambda}, \ i = 1, ..., d\right\},$$

for some constant $C > 0$. Its diameter which is the length of its symmetry axes is $2C2^{m/\lambda}$, i.e., the size of $K(m)$.

4 Sparse-grid sampling recovery in the setting (ii)

In this section, for the setting (ii) in the high dimensional case ($d \ge 1$), we establish upper and lower bounds of $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big)$ for $1 \le q \le p \le \infty$, and construct linear sampling algorithms which realize the upper bounds. In the one dimensional case ($d = 1$), we obtain the right convergence rate of $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R};\mu), L_q(\mathbb{R};\mu)\big)$ for $1 \le q \le p \le \infty$. We also prove in a non-constructive way the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $d \ge 2$ and $1 < q \le 2 < p \le \infty$.

Denote by $\tilde{C}(\mathbb{I}^d)$, $\tilde{L}_q(\mathbb{I}^d)$ and $\tilde{W}^r_p(\mathbb{I}^d)$ the subspaces of $C(\mathbb{I}^d)$, $L_q(\mathbb{I}^d)$ and $W^r_p(\mathbb{I}^d)$, respectively, of all functions f on $\mathbb{I}^d := \big[-\frac{1}{2},\frac{1}{2}\big]^d$, which can be extended to the whole $\mathbb{R}^d$ as 1-periodic functions in each variable (denoted again by f). Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$. Let $S_n \in \mathcal{S}_n$ be a sampling algorithm on $\mathbb{I}^d$. Assume it holds that

$$\|f - S_n(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \le C n^{-\alpha}(\log n)^{\beta}\|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{4.1}$$

Then based on S_n, we will construct a sampling algorithm on $\mathbb{R}^d$ belonging to $\mathcal{S}_n$, which approximates $f \in W^r_p(\mathbb{R}^d;\mu)$ with the same bound as in (4.1) (constant C may be different) for the approximation error measured in the norm of $L_q(\mathbb{R}^d;\mu)$. Such a sampling algorithm will be constructed by assembling sampling algorithms which are designed for the related Sobolev spaces on the integer-shifted d-cubes which cover $\mathbb{R}^d$. Let us process this construction.

Fix a number $\theta > 1$ and put $\mathbb{I}^d_\theta := \big[-\frac{\theta}{2},\frac{\theta}{2}\big]^d$. Denote by $\tilde{C}(\mathbb{I}^d_\theta)$, $\tilde{L}_q(\mathbb{I}^d_\theta)$ and $\tilde{W}^r_p(\mathbb{I}^d_\theta)$ the subspaces of $C(\mathbb{I}^d_\theta)$, $L_q(\mathbb{I}^d_\theta)$ and $W^r_p(\mathbb{I}^d_\theta)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as θ-periodic functions in each variable (denoted again by f). A sampling algorithm S_n induces the sampling algorithm $S_{\theta,n} \in \mathcal{S}_n$ on $\mathbb{I}^d_\theta$ defined for a function $f \in \tilde{C}(\mathbb{I}^d_\theta)$ by

$$S_{\theta,n}(f) := S_n(f(\cdot/\theta)).$$

From (4.1) it follows that

$$\|f - S_{\theta,n}(f)\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \le C n^{-\alpha}(\log n)^{\beta}\|f\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d_\theta).$$

We define for $n \in \mathbb{N}$,

$$m_n := \left(\delta^{-1}\lambda\alpha\log n\right)^{1/\lambda}, \tag{4.2}$$

and for $\boldsymbol{k} \in \mathbb{Z}^d$,

$$n_{\boldsymbol{k}} = \begin{cases} \lfloor \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|^\lambda} + 1 \rfloor & \text{if } |\boldsymbol{k}| < m_n, \\ 0 & \text{if } |\boldsymbol{k}| \ge m_n, \end{cases} \tag{4.3}$$

where an appropriate fixed value of parameter $\delta > 0$ will be chosen below and

$$\varrho^{-1} := \frac{2(2\pi)^{(d-1)/2}}{d!!} \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha} s^\lambda} < \infty.$$

We write $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \boldsymbol{k} + \mathbb{I}^d_\theta$ for $\boldsymbol{k} \in \mathbb{Z}^d$, and denote by $f_{\theta,\boldsymbol{k}}$ the restriction of f on $\mathbb{I}^d_{\theta,\boldsymbol{k}}$ for a function f on $\mathbb{R}^d$.

It is well-known that one can constructively define a unit partition $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ such that

(i) $\varphi_{\boldsymbol{k}} \in C_0^\infty(\mathbb{R}^d)$ and $0 \le \varphi_{\boldsymbol{k}}(\boldsymbol{x}) \le 1, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \boldsymbol{k} \in \mathbb{Z}^d$;

(ii) $\operatorname{supp} \varphi_{\boldsymbol{k}}$ are contained in the interior of $\mathbb{I}^d_{\theta,\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$;

(iii) $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_{\boldsymbol{k}}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^d$;

(iv) $\|\varphi_{\boldsymbol{k}}\|_{W^r_p(\mathbb{I}^d_{\theta,\boldsymbol{k}})} \le C_{r,d,\theta}, \quad \boldsymbol{k} \in \mathbb{Z}^d$

(see, e.g., [23, Chapter VI, 1.3]).

By using the items (ii) and (iv) we have that if $f \in W^r_p(\mathbb{R}^d;\mu)$, then

$$\tilde{f}_{\theta,\boldsymbol{k}}(\cdot) := f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \tilde{W}^r_p(\mathbb{I}^d_\theta), \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

If $q < p$, from the definition of the univariate weight w in (1.2) it follows that there are numbers C and $0 < \delta < 1/q - 1/p$ such that

$$w^{1/q}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|)w^{-1/p}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) \le Ce^{-\delta|\boldsymbol{k}|}, \quad \boldsymbol{k} \in \mathbb{Z}^d, \tag{4.4}$$

where recall, w is the generating univariate weight defined as in (1.2). Taking the sequence $(n_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ given as in (4.3) with the value of δ satisfying (4.4), we define the linear sampling algorithm $S^\mu_{\theta,n}$ on $\mathbb{R}^d$ generated from S_n by

$$\left(S^\mu_{\theta,n} f\right)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|<m_n} \left(S_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\boldsymbol{x} - \boldsymbol{k}). \tag{4.5}$$

Let us check that $S^\mu_{\theta,n} \in \mathcal{S}_n$, i.e., $m \le n$ where m denotes the number of sample points in $S^\mu_{\theta,n}$. Indeed, by using the well-known formula $\operatorname{Vol}_d(B_R) = \frac{2(2\pi)^{(d-1)/2}}{d!!} R^d$ for the volume $\operatorname{Vol}_d(B_R)$ of the d-dimensional ball B_R of radius R, we get

$$\begin{aligned} m &\le \sum_{|\boldsymbol{k}|<m_n} n_{\boldsymbol{k}} \le \sum_{|\boldsymbol{k}|=1}^{\lfloor m_n \rfloor} \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|^\lambda} \le n\varrho \sum_{s=0}^{\lfloor m_n \rfloor} \sum_{\boldsymbol{k}\in B_s} e^{-\frac{a\delta}{\alpha} s^\lambda} \le n\varrho \sum_{s=0}^{\lfloor m_n \rfloor} \operatorname{Vol}_d(B_s) e^{-\frac{a\delta}{\alpha} s^\lambda} \\ &\le \varrho n \frac{2(2\pi)^{(d-1)/2}}{d!!} \sum_{s=0}^{\lfloor m_n \rfloor} s^d e^{-\frac{a\delta}{\alpha} s^\lambda} \le n\varrho \frac{2(2\pi)^{(d-1)/2}}{d!!} \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha} s^\lambda} \le n. \end{aligned} \tag{4.6}$$

Theorem 4.1 *Let $1 \leq q < p \leq \infty$ and $\alpha > 0$, $\beta \geq 0$, $\theta > 1$. Assume that for any $n \in \mathbb{N}$, there is a linear sampling algorithm $S_n \in \mathcal{S}_n$ on $\mathbb{I}^d$ such that the convergence rate (4.1) holds. Then for any $n \in \mathbb{N}$, based on this sampling algorithm one can construct the sampling algorithm $S^\mu_{\theta,n} \in \mathcal{S}_n$ on $\mathbb{R}^d$ as in (4.5) so that*

$$\|f - S^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \leq C n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu). \tag{4.7}$$

Proof. We preliminarily decompose a function in $W^r_p(\mathbb{R}^d;\mu)$ into a sum of functions on $\mathbb{R}^d$ having support contained in integer translations of the d-cube $\mathbb{I}^d_\theta$. Then a desired sampling algorithm for $W^r_p(\mathbb{R}^d;\mu)$ will be the sum of integer-translated dilations of S_n. Notice that

$$\mathbb{R}^d = \bigcup_{\boldsymbol{k} \in \mathbb{Z}^d} \mathbb{I}^d_{\theta,\boldsymbol{k}},$$

where $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \mathbb{I}^d_\theta + \boldsymbol{k}$. From the items (ii) and (iii) in the definition of unite partion it is implied that

$$f = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} f_{\theta,\boldsymbol{k}} \varphi_{\boldsymbol{k}}, \tag{4.8}$$

where $f_{\theta,\boldsymbol{k}}$ denotes the restriction of f to $\mathbb{I}^d_{\theta,\boldsymbol{k}}$. Hence we have

$$\begin{aligned} \|f - S^\mu_{\theta,n}(f)\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \leq & \sum_{|\boldsymbol{k}| < m_n} \left\| f_{\theta,\boldsymbol{k}} \varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \\ & + \sum_{|\boldsymbol{k}| \geq m_n} \|f_{\theta,\boldsymbol{k}} \varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)}. \end{aligned} \tag{4.9}$$

Recalling the convention $1/\infty := 0$, from the definitions we derive the bound

$$\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \ll w^{-1/p}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|)\|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \tag{4.10}$$

By (4.3), (4.1) and (4.10) we derive the estimates

$$\begin{aligned} & \left\| f_{\theta,\boldsymbol{k}} \varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \\ & \ll w^{1/q}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) \left\| f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right) \right\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \\ & \ll w^{1/q}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) w^{-1/p}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) n_{\boldsymbol{k}}^{-\alpha}(\log n_{\boldsymbol{k}})^\beta \|f(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \\ & \ll w^{1/q}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) w^{-1/p}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \end{aligned}$$

By using the inequality (4.4) we get

$$\left\| f_{\theta,\boldsymbol{k}} \varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \ll e^{-\delta|\boldsymbol{k}|^\lambda} n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)},$$

which in a similar manner as (5.6) implies

$$\sum_{|\boldsymbol{k}|<m_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \ll \sum_{|\boldsymbol{k}|<m_n} e^{-\delta|\boldsymbol{k}|^\lambda} n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}$$
$$\le n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}|=1}^{\infty} e^{-\delta|\boldsymbol{k}|^\lambda}$$
$$\ll (n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}.$$

For a fixed $\varepsilon \in (0, 1-1/\lambda)$, again in a similar manner as (5.6) we have by (4.4) and the inequality $\lambda(1-\varepsilon) > 1$ that

$$\sum_{|\boldsymbol{k}|\ge m_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \ll \sum_{|\boldsymbol{k}|\ge m_n} w^{1/q}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|) w^{-1/p}(|\boldsymbol{k} + (\theta \operatorname{sign} \boldsymbol{k})|)\|f\|_{W^r_p(\mathbb{R}^d;\mu)}$$
$$\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}|\ge m_n} e^{-\delta|\boldsymbol{k}|^\lambda} \ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} e^{-\delta(1-\varepsilon)m_n^\lambda} \sum_{|\boldsymbol{k}|\ge m_n} e^{-\delta\varepsilon|\boldsymbol{k}|^\lambda}$$
$$\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} e^{-\alpha\lambda(1-\varepsilon)\log n} \ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} n^{-\alpha}.$$

From the last two estimates and (4.9) we obtain (4.7). ∎

We present Smolyak sampling algorithms for 1-periodic functions on $\mathbb{R}^d$ based B-spline quasi-interpolation which satisfy (4.1). For a given number $\ell \in \mathbb{N}$, denote by M_ℓ the cardinal B-spline of order ℓ with support $[0,\ell]$ and knots at the points $0, 1, ..., \ell$. We fixed an even number $\ell \in \mathbb{N}$ and take the cardinal B-spline $M := M_\ell$ of order ℓ. Let $\Lambda = \{\lambda(j)\}_{|j|\le\mu}$ be a given finite even sequence, i.e., $\lambda(-j) = \lambda(j)$ for some $\mu \ge \frac{\ell}{2} - 1$. We define the linear operator Q for functions f on $\mathbb{R}$ by

$$Q(f)(x) := \sum_{s\in\mathbb{Z}} \Lambda(f,s) M(x-s), \tag{4.11}$$

where

$$\Lambda(f,s) := \sum_{|j|\le\mu} \lambda(j) f(s - j + \ell/2). \tag{4.12}$$

The operator Q is local and bounded in $C(\mathbb{R})$ (see [4, p. 100–109]). An operator Q of the form (4.11)–(4.12) is called a quasi-interpolation operator in $C(\mathbb{R})$ if it reproduces $\mathcal{P}_{\ell-1}$, i.e., $Q(f) = f$ for every $f \in \mathcal{P}_{\ell-1}$, where $\mathcal{P}_{\ell-1}$ denotes the set of d-variate polynomials of degree at most $\ell - 1$ in each variable.

Since $M(\ell\, 2^k x) = 0$ for every $k \in \mathbb{N}_0$ and $x \notin (0,1)$, we can extend the restriction to the interval $[0,1]$ of the B-spline $M(\ell\, 2^k \cdot)$ to an 1-periodic function on the whole $\mathbb{R}$. Denote this periodic extension by N_k and define

$$N_{k,s}(x) := N_k(x - h^{(k)}s), \; k \in \mathbb{Z}_+, \; s \in I(k),$$

where

$$I(k) := \{0, 1, ..., \ell 2^k - 1\}.$$

Then we have for 1-periodic functions f on $\mathbb{R}$,

$$Q_k(f)(x) \ = \sum_{\boldsymbol{s}\in I(\boldsymbol{k})} a_{k,s}(f) N_{k,s}(x), \quad \forall x \in \mathbb{R}. \tag{4.13}$$

For convenience we define the univariate operator Q_{-1} by putting $Q_{-1}(f) := 0$.

We define the univariate B-spline $N_{k,s}$ by

$$N_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}) := \bigotimes_{i=1}^{d} N_{k_i,s_i}(x_i), \ \boldsymbol{k} \in \mathbb{Z}^d_+, \ \boldsymbol{s} \in I(\boldsymbol{k}),$$

where

$$I(\boldsymbol{k}) := \prod_{i=1}^{d} I(k_i).$$

Let the operators $q_{\boldsymbol{k}}$ be defined by

$$q_{\boldsymbol{k}} := \ \prod_{i=1}^{d} \Big(Q_{k_i} - Q_{k_i - 1} \Big), \ \boldsymbol{k} \in \mathbb{Z}^d_+.$$

where the univariate operator $Q_{k_i} - Q_{k_i-1}$ is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed.

From the refinement equation for the B-spline M (see, e.g., [4, (4.3.4)]), in the univariate case, we can represent the component functions $q_{\boldsymbol{k}}(f)$ as

$$q_{\boldsymbol{k}}(f) = \sum_{\boldsymbol{s}\in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}, \tag{4.14}$$

where where the coefficient functionals $c_{\boldsymbol{k},\boldsymbol{s}}(f)$ are explicitly constructed as linear combinations of at most m_0 of function values of f for some $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}$ and f.

For $m \in \mathbb{N}$, the well known periodic Smolyak grid of points $G^d(m) \subset \mathbb{T}^d$ is defined as

$$G^d(m) := \ \{\boldsymbol{x} = 2^{-\boldsymbol{k}}\boldsymbol{s} : \boldsymbol{k} \in \mathbb{N}^d, \ |\boldsymbol{k}|_1 = m, \ \boldsymbol{s} \in I(\boldsymbol{k})\}.$$

Here and in what follows, we use the notations: $\boldsymbol{x}\boldsymbol{y} := (x_1y_1, ..., x_dy_d)$ and $2^{\boldsymbol{x}} := (2^{x_1}, ..., 2^{x_d})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.

For $m \in \mathbb{N}_0$, we define the operator R_m by

$$R_m(f) := \sum_{|\boldsymbol{k}|_1 \le m} q_{\boldsymbol{k}}(f) \ = \sum_{|\boldsymbol{k}|_1 \le m} \ \sum_{\boldsymbol{s}\in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) \, N_{\boldsymbol{k},\boldsymbol{s}}.$$

From (4.14) one can see that for a functions f on $\mathbb{I}^d$, R_m defines a sampling algorithm on $\mathbb{I}^d$ of the form (1.7):

$$R_m(f) \ = \sum_{2^{-\boldsymbol{k}}\boldsymbol{s}\in G^d(m)} f(2^{-\boldsymbol{k}}\boldsymbol{s})\varphi_{\boldsymbol{k},\boldsymbol{s}},$$

where $n := |G^d(m)|$, and $\varphi_{\boldsymbol{k},\boldsymbol{s}}$ are explicitly constructed as linear combinations of at most at most m_0 B-splines $N_{\boldsymbol{k},\boldsymbol{j}}$ for some $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}, m$ and f. The operator R_m is also called Smolyak (sparse-gird) sampling algorithm initiated and used by him [22] for quadrature and interpolation for functions with mixed smoothness. It plays an important role in sampling recovery of multivariate functions and its applications (see [3], [11] for comments and bibliography).

Lemma 4.2 *Let $1 \le q \le p \le \infty$ and $1/p < r < \ell$. For $n \in \mathbb{N}$, let m_n be the largest integer number such that $|G^d(m_n)| \le n$. Then we have $R_{m_n} \in \mathcal{S}_n$ and*

$$\sup_{f \in \tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d)} \big\| f - R_{m_n}(f) \big\|_{\tilde{L}_q(\mathbb{I}^d)} \ll \begin{cases} n^{-r} (\log n)^{(r+1/2)(d-1)}, & 1 < q \le p < \infty; \\ n^{-r} (\log n)^{(r+1)(d-1)}, & \text{either } q = 1 \text{ or } p = \infty. \end{cases} \tag{4.15}$$

Proof. The case $1 < q \le p < \infty$ in (4.15) was proven in [5, Corollary 4.1]. We consider the case when either $q = 1$ or $p = \infty$ in (4.15). Let $\tilde{H}^r_p(\mathbb{I}^d)$ be the Hölder-Nikol'skii space of 1-periodic functions on $\mathbb{R}^d$ of mixed smoothness r bounded in the space $\tilde{L}_p(\mathbb{I}^d)$ (see, e.g., [11] for the definition). Then the bound (4.15) follows from the embedding $\tilde{W}^r_p(\mathbb{I}^d)$ into $\tilde{H}^r_p(\mathbb{I}^d)$ and the bound

$$\sup_{f \in \tilde{\boldsymbol{H}}^r_p(\mathbb{I}^d)} \big\| f - R_{m_n}(f) \big\|_{\tilde{L}_q(\mathbb{I}^d)} \ll n^{-r} (\log n)^{(r+1)(d-1)},$$

which can be proven in the same way as the non-periodic version [9, Theorem 3.4]. ☐

If $S_n = R_{m_n}$ where the operator R_{m_n} is defined as in Lemma 4.2, we write $S^\mu_{\theta,n} := R^\mu_{\theta,n}$.

We will need some results on hyperbolic cross trigonometric approximation of periodic functions. For $\boldsymbol{s} \in \mathbb{N}_0$ and $f \in \tilde{L}_p(\mathbb{I}^d)$ denote by $\hat{f}(\boldsymbol{s})$ the usual $\boldsymbol{s}$th Fourier coefficient of f in distributional sense. We define the function

$$\delta_{\boldsymbol{k}}(f)(\boldsymbol{x}) := \sum_{\boldsymbol{s} \in \Pi(\boldsymbol{k})} \hat{f}(\boldsymbol{s}) e^{\pi i (\boldsymbol{s}, \boldsymbol{x})},$$

where for $\boldsymbol{k} \in \mathbb{N}_0$,

$$\Pi(\boldsymbol{k}) = \big\{ \boldsymbol{s} \in \mathbb{Z}^d : \lfloor 2^{k_j - 1} \rfloor \le |s_j| < 2^{k_j},\ j = 1, ..., d \big\}.$$

For $m \in \mathbb{N}_0$, let

$$\Delta(m) := \bigcup_{|\boldsymbol{k}|_1 \le m} \Pi(\boldsymbol{k})$$

be the (step) hyperbolic cross set of multi-indices in $\mathbb{Z}^d$. For $f \in \tilde{L}_2(\mathbb{I}^d)$, we introduce the hyperbolic cross Fourier operator $F_{\Delta(m)}$ by

$$F_{\Delta(m)}(f)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|_1 \le m} \delta_{\boldsymbol{k}}(f)(\boldsymbol{x}) = \sum_{\boldsymbol{s} \in \Delta(m)} c_{\boldsymbol{s}} e^{\pi i (\boldsymbol{s}, \boldsymbol{x})}.$$

Notice that if $1 < p < \infty$, by the well-known Littlewood-Paley theorem $F_{\Delta(m)}$ is a bounded linear operator from $\tilde{L}_p(\mathbb{I}^d)$ to the space $\mathcal{T}_{\Delta(m)}$ of hyperbolic cross polynomials φ of the form

$$\varphi(\boldsymbol{x}) = \sum_{\boldsymbol{s} \in \Delta(m)} c_{\boldsymbol{s}} e^{\pi i (\boldsymbol{s}, \boldsymbol{x})}.$$

Also,

$$\dim \mathcal{T}_{\Delta(m)} = |\Delta(m)| \asymp 2^m m^{d-1}.$$

Lemma 4.3 *Let $1 \le q < p \le \infty$ For every $n \in \mathbb{N}$, let m_n be the largest number such that $|\Delta(m_n)| \le n$. Then $F_n := F_{\Delta(m_n)}$ defines a linear operator of rank $\le n$ from $\tilde{L}_p(\mathbb{I}^d)$ to the space $\mathcal{T}_{\Delta(m_n)}$ of dimension $\le n$ such that*

$$d_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \asymp \sup_{f \in \tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d)} \|f - F_n(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \asymp n^{-r} (\log n)^{(d-1)r}. \tag{4.16}$$

For detail on the proof of this lemma see, e.g., in [11, Theorems 4.2.5, 4.3.1, 4.3.6 & 4.3.7] and related comments on the asymptotic optimality of the hyperbolic cross approximation.

Theorem 4.4 *Let $1 \le q \le p \le \infty$ and $1/p < r < \ell$ and denote*

$$\varrho_n := \varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)).$$

Then we have the following.

(i) *If $1 < p = q < \infty$ and $r_\lambda > 0$, for every $n \in \mathbb{N}$, let m_n be the largest number such that $|G(m_n)| \le n$. Then P_{m_n} given as in (3.7), defines a sampling algorithm on $\mathbb{R}^d$ belonging to $\mathcal{S}_n$, and we have that*

$$n^{-r_\lambda} (\log n)^{r_\lambda(d-1)} \ll \varrho_n \le \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \big\|f - P_{m_n}(f)\big\|_{L_p(\mathbb{R}^d;\mu)} \ll n^{-r_\lambda} (\log n)^{(r_\lambda+1)(d-1)}. \tag{4.17}$$

(ii) *If $1 \le q < p \le \infty$ and $1/p < r < \ell$, for any $n \in \mathbb{N}$, based on the sampling algorithm R_{m_n} in Lemma 4.2, one can construct the sampling algorithm $R^\mu_{\theta,n} \in \mathcal{S}_n$ on $\mathbb{R}^d$ as in (4.5) so that*

$$\varrho_n \le \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \big\|f - R^\mu_{\theta,n}(f)\big\|_{L_q(\mathbb{R}^d;\mu)} \ll \begin{cases} n^{-r} (\log n)^{(r+1/2)(d-1)}, & 1 < q < p < \infty; \\ n^{-r} (\log n)^{(r+1)(d-1)}, & \text{either } q = 1 \text{ or } p = \infty. \end{cases} \tag{4.18}$$

Moreover,

$$\varrho_n \gg n^{-r} (\log n)^{r(d-1)}. \tag{4.19}$$

Proof. The claim (i) immediately follows from Theorem 2.5 since as noticed $W^r_p(\mathbb{R}^d;\mu) = W^r_{p,w^{1/p}}(\mathbb{R}^d)$ and $L_q(\mathbb{R}^d;\mu) = L_{q,w^{1/p}}(\mathbb{R}^d)$. Let us prove the upper bounds (4.18) in the claim (ii). For a fixed $\theta > 1$, we define $S^\mu_n := S^\mu_{\theta,n}$ as the sampling algorithm described in Theorem 4.1. The upper bounds (4.18) follow from Lemma 4.2 and Theorem 4.1 with $\alpha = r$ and $\beta = (d-1)(r+1/2)$ for $1 < q < p < \infty$, and $\beta = (d-1)(r+1)$ for the other cases.

We next prove the lower bound (4.19) in the claim (i). If f is a 1-periodic function on $\mathbb{R}^d$ and $f \in \tilde{W}^r_p(\mathbb{I}^d)$, then for $1 < p < \infty$,

$$\begin{aligned}
\|f\|^p_{W^r_p(\mathbb{R}^d;\mu)} &= \sum_{|\boldsymbol{r}|_\infty \le r} \int_{\mathbb{R}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\
&= \sum_{|\boldsymbol{r}|_\infty \le r} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x}+\boldsymbol{k})|^p w(\boldsymbol{x}+\boldsymbol{k}) \mathrm{d}\boldsymbol{x} \\
&\ll \sum_{|\boldsymbol{r}|_\infty \le r} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} w(|\boldsymbol{k} - (\operatorname{sign} \boldsymbol{k})|) \\
&\ll \|f\|^p_{\tilde{W}^r_p(\mathbb{I}^d)}.
\end{aligned}$$

The bound $\|f\|_{W^r_\infty(\mathbb{R}^d;\mu)} \ll \|f\|_{\tilde{W}^r_\infty(\mathbb{I}^d)}$ can be proven similarly with a slight modification. On the other hand,

$$\|f\|_{\tilde{L}_q(\mathbb{I}^d)} = \|f(\cdot + \mathbf{1})\|_{\tilde{L}_q(\mathbb{I}^d)} \ll \|f\|_{L_q(\mathbb{R}^d;\mu)},$$

where $\mathbf{1} := (1, 1, ..., 1) \in \mathbb{R}^d$. Hence we get by Lemma 4.3 the lower bound (4.19):

$$\begin{aligned}
\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) &\ge d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \\
&\gg d_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \gg n^{-r} (\log n)^{r(d-1)}.
\end{aligned}$$

□

In Figure 2, a classical Smolyak grid on the domain $[-1,1]^2$ is in the left picture, and an assembled grid in the two-dimensional function domain $\mathbb{R}^2$ in the right picture is designed for the setting (ii) based on classical Smolyak grids. It is completely different from the step hyperbolic cross gird in the two-dimensional function domain $\mathbb{R}^2$ in the right picture of Figure 1 designed for the setting (i).

In the one-dimensional case we have the following sharpened results.

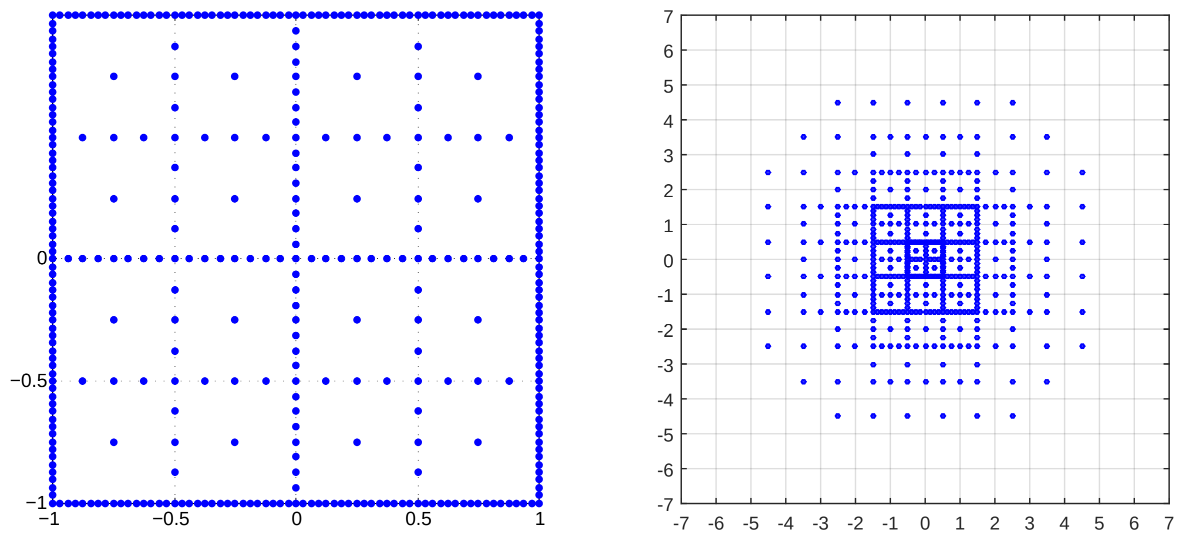
Theorem 4.5 *For $\varrho_n := \varrho_n(\boldsymbol{W}^r_p(\mathbb{R};\mu), L_q(\mathbb{R};\mu))$, we have the following.*

(i) *If $1 \le q < p \le \infty$ and $1/p < r < \ell$, for any $n \in \mathbb{N}$, based on the sampling algorithm R_{m_n} in Lemma 4.2, one can construct the sampling algorithm $R^\mu_{\theta,n} \in \mathcal{S}_n$ on $\mathbb{R}$ as in (4.5) so that*

$$\varrho_n \asymp \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R};\mu)} \left\| f - R^\mu_{\theta,n}(f) \right\|_{L_q(\mathbb{R}^d;\mu)} \asymp n^{-r}. \tag{4.20}$$

(ii) *If $1 < p = q < \infty$ and $r_\lambda > 0$, for any $n \in \mathbb{N}$, letting $k(n)$ be the largest integer such that $2^{k(n)} \le n$. Then the sampling algorithms on $\mathbb{R}$ $I_{m_{k(n)}} \in \mathcal{S}_n$, $n \in \mathbb{N}$, defined as in (2.9), are asymptotically optimal for the sampling n-widths ϱ_n and*

$$\varrho_n \asymp \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R};\mu)} \left\| f - I_{m_{k(n)}}(f) \right\|_{L_p(\mathbb{R};\mu)} \asymp n^{-r_\lambda}. \tag{4.21}$$



A classical Smolyak grid   An assembling of Smolyak grids

Figure 2: Different sparse grids on function domains ($d = 2$)

Proof. The claim (i) is in Theorem 4.4 for $d = 1$. The claim (ii) follows from Theorem 2.5. □

5 Right convergence rate of sampling widths

In this section, we prove the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $d \geq 2$ and $1 < q \leq 2 < p < \infty$. We also prove the RKHS property of the Hilbert space $W^r_2(\mathbb{R}^d;\mu_w)$ and the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\mu), L_p(\mathbb{R}^d;\mu))$ $(d \geq 2)$ for $1 < q \leq p = 2$ in the particular case when the generating univariate Freud-type weight w is given by

$$w(x) := \exp\left(-ax^4 + b\right), \ a > 0, \ b \in \mathbb{R}, \tag{5.1}$$

Assumption A. Let $\boldsymbol{W}$ be a class of complex-valued functions on the measurable set $\Omega \subset \mathbb{R}^d$. We say that $\boldsymbol{W}$ satisfies Assumption A, if there is a metric on $\boldsymbol{W}$ such that $\boldsymbol{W}$ is continuously embedded into the separable space with measure $L_2(\Omega;\mu)$, and for each $\boldsymbol{x} \in \Omega$, the evaluation functional $f \mapsto f(\boldsymbol{x})$ is continuous on $\boldsymbol{W}$.

The following lemma is a consequence of [14, Corollary 4].

Lemma 5.1 *Assume that* $\boldsymbol{W}$ *satisfies Assumption* A *and that*

$$d_n(\boldsymbol{W}, L_2(\Omega;\mu)) \ \ll \ n^{-\alpha} \log^{-\beta} n \tag{5.2}$$

for some $\alpha > 1/2$ *and* $\beta \in \mathbb{R}$. *Then*

$$\varrho_n(\boldsymbol{W}, L_2(\Omega;\mu)) \ \ll \ n^{-\alpha} \log^{-\beta} n. \tag{5.3}$$

Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$. Let A_n be a linear operator in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\le n$. Assume it holds that

$$\|f - A_n(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \le C n^{-\alpha} (\log n)^\beta \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{5.4}$$

Then based on A_n, in the same way as the construction of the sampling operator $S^\mu_{\theta,n}$ defined as in (4.5), we construct a linear operator $A^\mu_{\theta,n}$ in $L_q(\mathbb{R}^d;\mu)$ of rank $\le n$ which approximates $f \in W^r_p(\mathbb{R}^d,\gamma)$ with the same bound as in (5.4). In the construction of $A^\mu_{\theta,n}$, the sampling operators $S_{\theta,n}$ are substituted by the linear operator $A_{\theta,n}$ which are defined similarly. For a given $n \in \mathbb{N}$, taking the sequence $(n_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ given in (4.3) with the value of δ satisfying (4.4) an m_n as in (4.2), we define the linear operator $A^\mu_{\theta,n}$ generated from the linear operator A_n by

$$A^\mu_{\theta,n}(f)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|<m_n} \left(A_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\boldsymbol{x}-\boldsymbol{k}). \tag{5.5}$$

If $A_n = F_n$ where the operator F_n is defined as in Lemma 4.3, we write $A^\mu_{\theta,n} := F^\mu_{\theta,n}$.

The rank of the operator $A^\mu_{\theta,n}$ is not greater than n. Indeed, by the definition of $A^w_{\theta,n}$ similarly to (5.6) we have

$$\operatorname{rank} A^\mu_{\theta,n} \le \sum_{|\boldsymbol{k}|<m_n} \operatorname{rank} A_{\theta,n_{\boldsymbol{k}}} \le \sum_{|\boldsymbol{k}|<m_n} n_{\boldsymbol{k}} \le n. \tag{5.6}$$

In a way similar to the proof of Theorem 4.1 we derive the following result.

Theorem 5.2 *Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$, $1 < \theta < 2$. Assume that for any $n \in \mathbb{N}$, there is a linear operator A_n in $\tilde{L}_q(\mathbb{I}^d)$ of rank $\le n$ such that the bound (5.4) holds. Then for any $n \in \mathbb{N}$, based on this linear operator one can construct the linear operator $A^\mu_{\theta,n}$ in $L_q(\mathbb{R}^d;\mu)$ of rank $\le n$ as in (5.5) so that*

$$\|f - A^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \le C n^{-\alpha} (\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu).$$

In a way similar to the proof of Theorem 4.1 from Lemma 4.3 and Theorem 5.2 we derive the following result.

Theorem 5.3 *Let $1 \le q < p \le \infty$. Then for any $n \in \mathbb{N}$, based on the linear operator F_n of rank $\le n$ in $\tilde{L}_q(\mathbb{I}^d)$ defined as in Lemma 4.3, one can construct the linear operator $F^\mu_{\theta,n}$ in $L_q(\mathbb{R}^d;\mu)$ of rank $\le n$ as in (5.5) so that there holds the right convergence rate*

$$d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp \sup_{f\in\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \|f - F^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \asymp n^{-r}(\log n)^{(d-1)r}. \tag{5.7}$$

Theorem 5.4 *Let $r \in \mathbb{N}$ and $1 \le q \le 2 < p \le \infty$. Then there holds the right convergence rate*

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp n^{-r}(\log n)^{(d-1)r}. \tag{5.8}$$

Proof. The lower bound in (5.8) is implied from the inequality (2.24) and (5.7). By the norm inequality $\|\cdot\|_{L_q(\mathbb{R}^d;\mu)} \ll \|\cdot\|_{L_2(\mathbb{R}^d;\mu)}$ for $1 \le q \le 2$, it is sufficient to prove the upper bound in (5.8) for $q = 2$. By (5.7) we have that

$$d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \ll n^{-r}(\log n)^{(d-1)r}. \tag{5.9}$$

Notice that the separable normed space $W^r_p(\mathbb{R}^d;\mu)$ is continuously embedded into $L_2(\mathbb{R}^d;\mu)$, and the evaluation functional $f \mapsto f(\boldsymbol{x})$ is continuous on the space $W^r_p(\mathbb{R}^d;\mu)$ for each $\boldsymbol{x} \in \mathbb{R}^d$. This means that the set $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ satisfies Assumption A. By Lemma 5.1 and (5.9) we prove the upper bound:

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \ll d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \ll n^{-r}(\log n)^{(d-1)r}.$$

□

Let $(\phi_m)_{m\in\mathbb{N}_0}$ be the sequence of orthonormal polynomials with respect to the univariate Freud-type weight

$$v(x) := w^2(x) = \exp\left(-2a|x|^\lambda + 2b\right). \tag{5.10}$$

For every multi-degree $\boldsymbol{k} \in \mathbb{N}^d_0$, the d-variate polynomial $\phi_{\boldsymbol{k}}$, we define

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^d \phi_{k_j}(x_j), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

For the generating weight w given as in (5.10), the polynomials $\{\phi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^d_0}$ constitute an orthonormal basis of the Hilbert space $L_2(\mathbb{R}^d;\mu)$, and every $f \in L_2(\mathbb{R}^d;\mu)$ can be represented by the polynomial series

$$f = \sum_{\boldsymbol{k}\in\mathbb{N}^d_0} \hat{f}(\boldsymbol{k})\phi_{\boldsymbol{k}} \quad \text{with} \quad \hat{f}(\boldsymbol{k}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \phi_{\boldsymbol{k}}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

converging in the norm of $L_2(\mathbb{R}^d;\mu)$. Moreover, there holds Parseval's identity

$$\|f\|^2_{L_2(\mathbb{R}^d;\mu)} = \sum_{\boldsymbol{k}\in\mathbb{N}^d_0} |\hat{f}(\boldsymbol{k})|^2.$$

For $r > 0$ and $\boldsymbol{k} \in \mathbb{N}^d_0$, we define

$$\rho_{\lambda,r,\boldsymbol{k}} := \prod_{j=1}^d (k_j + 1)^{r_\lambda}.$$

Denote by $\mathcal{H}^{r_\lambda}(\mathbb{R}^d)$ the space of all functions $f \in L_2(\mathbb{R}^d;\mu)$ represented by the series (5.34) for which the norm

$$\|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{N}^d_0} |\rho_{\lambda,r,\boldsymbol{k}}\hat{f}(\boldsymbol{k})|^2\right)^{1/2}$$

is finite. Notice that for $r > \frac{\lambda}{2(\lambda-1)}$, we have $r_\lambda > 1/2$ and therefore, $\mathcal{H}^{r_\lambda}(\mathbb{R}^d)$ is a separable RKHS with the reproducing kernel

$$K(\boldsymbol{x},\boldsymbol{y}) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho_{\lambda,r,\boldsymbol{k}}^{-2}\phi_{\boldsymbol{k}}(\boldsymbol{x})\phi_{\boldsymbol{k}}(\boldsymbol{y}). \tag{5.11}$$

Theorem 5.5 *We have for any $\lambda > 1$ and $r > \frac{\lambda}{2(\lambda-1)}$,*

$$\varrho_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{5.12}$$

Proof. The proof of theorem is similar to the proof of [10, Theorem 3.5]. For completeness, we shortly perform it. We need the following results on Kolmogorov widths $d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu))$ (see, e.g., for the definition of Kolmogorov widths) which can be proven in the same manner as the proof of [10, (3.18)]. We have for $r > 0$,

$$d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{5.13}$$

The lower bound of (5.12) follows from (5.13) and the inequality

$$\varrho_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \geq d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)).$$

We check the upper bound of (5.12). By (5.13) we get

$$d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{5.14}$$

From the orthonormality of the system $\{\phi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ it is easy to see that $K(\boldsymbol{x},\boldsymbol{y})$ satisfies the finite trace assumption

$$\int_{\mathbb{R}^d} K(\boldsymbol{x},\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ < \ \infty. \tag{5.15}$$

Hence by (5.14) and [14, Corollary 2] we obtain

$$\varrho_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ll d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}.$$

□

Lemma 5.6 *Let λ be an even integer for the generating univariate weight w as in (1.2). Then we have the inequality*

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \ll \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}, \quad f \in W_2^r(\mathbb{R};\mu). \tag{5.16}$$

Proof. We will use the following representation of the derivative of the polynomials ϕ_m for $m \in \mathbb{N}$, which was proven in [1, Lemma 3]:

$$\phi_m' = \sum_{k=m-\lambda+1}^{m-1} a_{m,k}\phi_k, \tag{5.17}$$

where

$$a_{m,k} := \lambda \int_{\mathbb{R}} \phi_m(x)\phi_k(x)x^{\lambda-1}w(x)\mathrm{d}x \tag{5.18}$$

satisfying the inequalities

$$|a_{m,k}| \le Cm^{1-1/\lambda} \tag{5.19}$$

for some positive constant C independent of m, k.

We first prove the lemma for $r = 1$. Given $f \in W_2^1(\mathbb{R};\mu)$, we denote $g := f'$. From Parseval's identity and the equality (5.17) we have

$$g = \sum_{m\in\mathbb{N}_0} \hat{f}(m)\phi_m' = \sum_{m\in\mathbb{N}_0} \hat{f}(m) \sum_{k=m-\lambda+1}^{m-1} a_{m,k}\phi_k = \sum_{k\in\mathbb{N}} \phi_k \sum_{m=k+1}^{k+\lambda-1} a_{m,k}\hat{f}(m), \tag{5.20}$$

and consequently, for every $k \in \mathbb{N}$,

$$\hat{g}(k) = \sum_{m=k+1}^{k+\lambda-1} a_{m,k}\hat{f}(m). \tag{5.21}$$

Hence, by (5.19)

$$|\hat{g}(k)|^2 \le (\lambda-1) \sum_{m=k+1}^{k+\lambda-1} |a_{m,k}\hat{f}(m)|^2 \le C(\lambda-1) \sum_{m=k+1}^{k+\lambda-1} |m^{1-1/\lambda}\hat{f}(m)|^2. \tag{5.22}$$

This and Parseval's identity yield

$$\begin{aligned} \|g\|_{L_2(\mathbb{R};\mu)}^2 &= \sum_{k\in\mathbb{N}_0} |\hat{g}(k)|^2 \le C(\lambda-1) \sum_{k\in\mathbb{N}_0} \sum_{m=k+1}^{k+\lambda-1} |m^{1-1/\lambda}\hat{f}(m)|^2 \\ &\le C_\lambda \sum_{k\in\mathbb{N}_0} |k^{1-1/\lambda}|\hat{f}(k)|^2 \\ &\ll \sum_{k\in\mathbb{N}_0} |\rho_{\lambda,1,k}\hat{f}_k|^2 = \|f\|_{\mathcal{H}^1(\mathbb{R})}^2, \end{aligned} \tag{5.23}$$

which implies

$$\|f\|_{W_2^1(\mathbb{R};\mu)} = \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{L_2(\mathbb{R};\mu)} \ll \|f\|_{\mathcal{H}^1(\mathbb{R})}. \tag{5.24}$$

This proves the theorem in the case $r = 1$. In the general case it can be obtained by induction on r. Assuming that (5.16) is true for $r-1$, we prove it for r. Again, given $f \in W_2^r(\mathbb{R};\mu)$, we denote $g := f' \in W_2^{r-1}(\mathbb{R};\mu)$. From the induction assumption, in a way similar to (5.23) we derive

$$\begin{aligned} \|g\|_{W_2^{r-1}(\mathbb{R};\mu)}^2 &\asymp \|g\|_{\mathcal{H}^{r-1}(\mathbb{R})} = \sum_{k\in\mathbb{N}_0} |\rho_{\lambda,r-1,k}\hat{g}_k|^2 \\ &\ll \sum_{k\in\mathbb{N}_0} |k^{1-1/\lambda}\rho_{\lambda,r-1,k}\hat{f}(k)|^2 \le \|f\|_{\mathcal{H}^{r}\lambda(\mathbb{R})}^2. \end{aligned} \tag{5.25}$$

Hence,

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \asymp \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{W_2^{r-1}(\mathbb{R};\mu)} \le \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,. \tag{5.26}$$

□

We show the equivalence between the norm of the space $W_2^r(\mathbb{R};\mu)$ and the norm of $H_w^{r_\lambda}(\mathbb{R}^d)$ in the case $\lambda = 4$ in the weight (1.2) by proving the inequality inverse to (5.16). In order to do this, we need some properties of the polynomials ϕ_m. Denote by $\gamma_m > 0$ the leading coefficient of the polynomial ϕ_m, i.e., $\phi_m(x) := \gamma_m x^m + \varphi$ for some $\varphi \in \mathcal{P}_{m-1}$. We put $\alpha_m := \gamma_{m-1}/\gamma_m$ for $m \in \mathbb{N}$. Then we have the following equalities for $\lambda = 4$.

(i)

$$\phi'_m = \frac{m}{\alpha_m}\,\phi_{m-1} + 4a\,\alpha_m\alpha_{m-1}\alpha_{m-2}\,\phi_{m-3}. \tag{5.27}$$

(ii)

$$4a\,\alpha_m^2(\alpha_{m+1}^2 + \alpha_m^2 + \alpha_{m-1}^2) = m. \tag{5.28}$$

(iii)

$$\lim_{m\to\infty}\left(\frac{12}{m}\right)^{1/4}\alpha_m = 1. \tag{5.29}$$

Here the parameter a is the same as in (1.2). The claims (i) and (ii) were proven in [2], the claim (iii) in [15].

Theorem 5.7 *Let $\lambda = 4$ for the generating univariate weight w as in (**??**). Then we have the norm equivalence*

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,, \quad f \in W_2^r(\mathbb{R};\mu). \tag{5.30}$$

Proof. By the inequality (5.16) of Lemma 5.6, to prove the theorem it is sufficient to show the inverse inequality

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \gg \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,, \quad f \in W_2^r(\mathbb{R};\mu). \tag{5.31}$$

We first prove this inequality for $r = 1$. Given $f \in W_2^1(\mathbb{R};\mu)$, we denote $g := f'$. Put

$$b_k := \frac{k}{\alpha_k}, \quad c_k := 4a\,\alpha_k\alpha_{k-1}\alpha_{k-2},$$

where recall, a is the parameter in the definition (1.2) of the generating univariate weight w. For any fixed $m_0 \in \mathbb{Z}$, from the equality $\lim_{m\to\infty}\frac{m+m_0}{m} = 1$, (5.28) and (5.29) it follows that

$$\lim_{m\to\infty}\left(\frac{12}{m}\right)^{3/4} b_{m+m_0} = 12a, \quad \lim_{m\to\infty}\left(\frac{12}{m}\right)^{3/4} c_{m+m_0} = 4a, \tag{5.32}$$

and

$$b_{m+m_0} \asymp c_{m+m_0} \asymp m^{3/4}, \quad k \in \mathbb{N}_0. \tag{5.33}$$

By using the equality (5.27) and

$$g = \sum_{m \in \mathbb{N}_0^d} \hat{f}(m)\phi'_m, \tag{5.34}$$

we have for every $k \in \mathbb{N}_0$,

$$\hat{g}(k) = b_k \hat{f}_{k+1} + c_k \hat{f}_{k+3}, \quad k \in \mathbb{N}_0. \tag{5.35}$$

By (5.28) there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$,

$$\left(\frac{12}{k}\right)^{3/4} b_{k-1} \geq 9a, \quad \left(\frac{12}{k}\right)^{3/4} c_{k-3} \leq 6a, \tag{5.36}$$

Hence by Parseval's identity and (5.33) we obtain

$$\begin{aligned}
\|g\|_{L_2(\mathbb{R};\mu)} &\geq \left(\sum_{k\geq 1} |b_{k-1}\hat{f}_k|^2\right)^{1/2} - \left(\sum_{k\geq 3} |c_{k-3}\hat{f}_k|^2\right)^{1/2} \\
&\geq \left(\sum_{k> k_0} |b_{k-1}\hat{f}_k|^2\right)^{1/2} - \left(\sum_{k> k_0} |c_{k-3}\hat{f}_k|^2\right)^{1/2} - \left(\sum_{k\leq k_0} |c_{k-3}\hat{f}_k|^2\right)^{1/2} \\
&\geq 3a(12)^{-3/4}\left(\sum_{k> k_0} |(k+1)^{3/4}\hat{f}_k|^2\right)^{1/2} - \max_{0\leq k\leq k_0} |c_{k-3}| \left(\sum_{k\leq k_0} |\hat{f}_k|^2\right)^{1/2} \\
&\geq 3a(12)^{-3/4}\left(\sum_{k\in\mathbb{N}_0} |(k+1)^{3/4}\hat{f}_k|^2\right)^{1/2} \\
&\quad - 3a(12)^{-3/4}(k_0+1)^{3/4}\left(\sum_{k\leq k_0} |\hat{f}_k|^2\right)^{1/2} - \max_{0\leq k\leq k_0} |c_{k-3}| \left(\sum_{k\leq k_0} |\hat{f}_k|^2\right)^{1/2} \\
&\geq C_1 \|f\|_{\mathcal{H}^1(\mathbb{R})} - C_2 \|f\|_{L_2(\mathbb{R};\mu)},
\end{aligned} \tag{5.37}$$

where

$$C_1 := 3a(12)^{-3/4} > 0, \quad C_2 := 3a(12)^{-3/4}(k_0+1)^{3/4} + \max_{0\leq k\leq k_0} |c_{k-3}| > 0.$$

This yields that

$$\begin{aligned}
\|f\|_{W_2^1(\mathbb{R};\mu)} &\asymp C_2 \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{L_2(\mathbb{R};\mu)} \\
&\geq C_2 \|f\|_{L_2(\mathbb{R};\mu)} + C_1 \|f\|_{\mathcal{H}^1(\mathbb{R})} - C_2 \|f\|_{L_2(\mathbb{R};\mu)} = C_1 \|f\|_{\mathcal{H}^1_w(\mathbb{R})}.
\end{aligned} \tag{5.38}$$

This and (5.24) prove the theorem in the case $r = 1$. In the general case it can be established by induction on r. Assuming that (5.31) is true for $r - 1$, we prove it for

r. Again, given $f \in W_2^r(\mathbb{R};\mu)$, we denote $g := f' \in W_2^{r-1}(\mathbb{R};\mu)$. From the induction assumption and (5.35), in a way similar to (5.37) we derive

$$\begin{aligned}
\|g\|_{W_2^{r-1}(\mathbb{R};\mu)} \asymp \|g\|_{\mathcal{H}^{r-1}(\mathbb{R})} &= \left(\sum_{k\in\mathbb{N}_0} \rho_{r-1,k}|\hat{g}_k|^2\right)^{1/2} \\
&\geq \left(\sum_{k\geq 1} \rho_{\lambda,r-1,k}|b_{k-1}\hat{f}_k|^2\right)^{1/2} - \left(\sum_{k\geq 3} \rho_{\lambda,r-1,k}|c_{k-3}\hat{f}_k|^2\right)^{1/2} \\
&\geq 3a(12)^{-3/4}\left(\sum_{k\in\mathbb{N}_0} \rho_{\lambda,r-1,k}|(k+1)^{3/4}\hat{f}_k|^2\right)^{1/2} \\
&- 3a(12)^{-3/4}(k_0+1)^{3/4}\rho_{\lambda,r-1,k_0}\left(\sum_{k\leq k_0} |\hat{f}_k|^2\right)^{1/2} \\
&- \rho_{\lambda,r-1,k_0}\max_{0\leq k\leq k_0}|c_{k-3}|\left(\sum_{k\leq k_0} |\hat{f}_k|^2\right)^{1/2} \\
&\geq C_1\|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})} - C_2\rho_{\lambda,r-1,k_0}\|f\|_{L_2(\mathbb{R};\mu)},
\end{aligned} \tag{5.39}$$

where k_0, C_1 and C_2 are the same constants as in (5.37). Hence, similarly to (5.38) we obtain that

$$\begin{aligned}
\|f\|_{W_2^r(\mathbb{R};\mu)} &\asymp C_2\rho_{\lambda,r-1,k_0}\|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{W_2^{r-1}(\mathbb{R};\mu)} \\
&\geq C_2\rho_{\lambda,r-1,k_0}\|f\|_{L_2(\mathbb{R};\mu)} + C_1\|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})} - C_2\rho_{\lambda,r-1,k_0}\|f\|_{L_2(\mathbb{R};\mu)} \\
&= C_1\|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}.
\end{aligned} \tag{5.40}$$

∎

Theorem 5.8 *Let λ be an even integer for the generating univariate weight w as in (1.2). Then we have the inequality*

$$\|f\|_{W_2^r(\mathbb{R}^d;\mu)} \ll \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W_2^r(\mathbb{R}^d;\mu). \tag{5.41}$$

Moreover, we have the norm equivalence for $\lambda = 4$,

$$\|f\|_{W_2^r(\mathbb{R}^d;\mu)} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W_2^r(\mathbb{R}^d;\mu). \tag{5.42}$$

Proof. In the case $d = 1$, this theorem combines Lemma 5.6 and Theorem 5.7. Both the relations (5.41) and (5.42) can be proven in the same way. For simplicity we prove (5.42) for the case $d = 2$. The general case can be proven by induction on d. We make use of the temporary notation:

$$\|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}),2}(x_1) := \|f(x_1,\cdot)\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}.$$

Let $f \in W^r_{2,w}(\mathbb{R}^2)$. From Lemma 3.1 it follows that $f(\cdot, x_2) \in W^r_2(\mathbb{R}; \mu)$, and consequently, by Theorem 5.7 for $d = 1$ $f(\cdot, x_2) \in \mathcal{H}^{r_\lambda}(\mathbb{R})$ for almost everywhere $x_2 \in \mathbb{R}$. Hence,

$$
\begin{aligned}
\|f\|_{W^r_{2,w}(\mathbb{R}^2)} &= \left\| \|f\|_{W^r_{2,w}} \right\|_{W^r_2(\mathbb{R};\mu)} \asymp \left\| \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}),2}(\cdot) \right\|_{W^r_2(\mathbb{R};\mu)} \\
&\asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}),2}(\cdot)\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^2)}.
\end{aligned}
$$

□

Notice that the norm equivalence (5.42) for $\lambda = 2$ has been proven in [10, Lemma 3.4] (see also [13, pages 687–689]) .

Due to the norm equivalence (5.42) in Theorem 5.8, we identify $W^r_2(\mathbb{R}^d; \mu)$ with $\mathcal{H}^{r_\lambda}(\mathbb{R}^d)$ for the case when $\lambda = 4$ and $r \in \mathbb{N}$. From Theorem 5.5 and the norm inequality $\|\cdot\|_{L_p(\mathbb{R}^d;\mu} \le C \|\cdot\|_{L_2(\mathbb{R}^d;\mu}$ for $1 \le q < 2$ we derive the following result on right convergence rate of sampling n-widths.

Theorem 5.9 *We have for $1 \le q < 2$, $r \in \mathbb{N}$ and $\lambda = 4$,*

$$
\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d; \mu), L_2(\mathbb{R}^d; \mu)) \asymp n^{-\frac{3r}{4}} (\log n)^{\frac{3(d-1)r}{4}}. \tag{5.43}
$$

We finish this section with some conjectures.

Conjecture 5.10 *We have for any $r \in \mathbb{N}$ and even integer $\lambda > 4$,*

$$
\|f\|_{W^r_2(\mathbb{R}^d;\mu)} \gg \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W^r_2(\mathbb{R}^d; \mu). \tag{5.44}
$$

If Conjecture 5.10 holds true, then from this conjecture and Theorems 5.8 and 5.5 we can deduce the following results.

Conjecture 5.11 *We have for any $r \in \mathbb{N}$ and even integer $\lambda > 4$,*

$$
\|f\|_{W^r_2(\mathbb{R}^d;\mu)} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W^r_2(\mathbb{R}^d; \mu). \tag{5.45}
$$

Conjecture 5.12 *We have for any $d \ge 2$, , $r \in \mathbb{N}$ and even integer $\lambda > 4$,*

$$
\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d; \mu), L_2(\mathbb{R}^d; \mu)) \asymp n^{-r_\lambda} (\log n)^{r_\lambda(d-1)}.
$$

A Appendix

For $\boldsymbol{p} = (p_1, ..., p_d) \in [1, \infty)^d$, we defined the mixed integral norm $\|\cdot\|_{L_{\boldsymbol{p},w}(\mathbb{R}^d)}$ for functions on $\mathbb{R}^d$ as follows

$$
\|f\|_{L_{\boldsymbol{p},w}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(\boldsymbol{x}) w(\boldsymbol{x})|^{p_1} \mathrm{d}x_1 \right)^{p_2/p_1} \mathrm{d}x_2 \cdots \right)^{p_d/p_{d-1}} \mathrm{d}x_d \right)^{1/p_d},
$$

and put $\mathbf{1/p} := (1/p_1, ..., 1/p_d)$. For $\boldsymbol{m} \in \mathbb{N}_0^d$, denote by $\mathcal{P}_{\boldsymbol{m}}$ the space of polynomials on $\mathbb{R}^d$ of degree at most m_j in the variable x_j, $j = 1, ..., d$. If $\boldsymbol{p}, \boldsymbol{q} \in [1, \infty)^d$, from Lemma 2.3 one can deduce the Nikol'skii's inequalities

$$\|\varphi\|_{L_{\boldsymbol{q},w}(\mathbb{R}^d)} \ \le \ C2^{(1-1/\lambda)|(\mathbf{1/p}-\mathbf{1/q})\boldsymbol{k}|_1}\|f\|_{L_{\boldsymbol{p},w}(\mathbb{R}^d)} \quad \forall \varphi \in \mathcal{P}_{2^{\boldsymbol{k}}} \quad (\boldsymbol{p} < \boldsymbol{q}), \tag{A.1}$$

and

$$\|\varphi\|_{L_{w,\boldsymbol{q}}(\mathbb{R}^d)} \ \le \ C2^{(1/\lambda)|(\mathbf{1/q}-\mathbf{1/p})\boldsymbol{k}|_1}\|f\|_{L_{\boldsymbol{p},w}(\mathbb{R}^d)} \quad \forall \varphi \in \mathcal{P}_{2^{\boldsymbol{k}}} \quad (\boldsymbol{p} > \boldsymbol{q}), \tag{A.2}$$

with constants C depending on $\lambda, \boldsymbol{p}, \boldsymbol{q}, d$ only, where we denote $\boldsymbol{xy} := (x_1y_1, ..., x_dy_d)$.

Lemma A.1 *Let $1 \le p, q < \infty$, $p \ne q$ and $\tau := |1/2 - p/(p+q)|$. Then there holds the inequality*

$$\int_{\mathbb{R}^d} |\varphi_{\boldsymbol{k}}(\boldsymbol{x})\varphi_{\boldsymbol{s}}(\boldsymbol{x})|^{q/2} w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ \le \ CS_{\boldsymbol{k}}S_{\boldsymbol{s}}2^{-\tau|\boldsymbol{k}-\boldsymbol{s}|_1} \quad \forall \varphi_{\boldsymbol{k}} \in \mathcal{P}_{2^{\boldsymbol{k}}}, \ \varphi_{\boldsymbol{s}} \in \mathcal{P}_{2^{\boldsymbol{s}}}, \ \boldsymbol{k}, \boldsymbol{s} \in \mathbb{N}_0^d,$$

with some constant C depending at most on λ, p, q, d, where

$$S_{\boldsymbol{k}} := \ \left(2^{\delta_{\lambda,p,q}|\boldsymbol{k}|_1}\|\varphi_{\boldsymbol{k}}\|_{L_{p,w}(\mathbb{R}^d)}\right)^{q/2}.$$

Proof. We prove the lemma for the case $p > q$. It can be proven in a similar way with some modification for the $p < q$. Let $\nu := (p+q)/p$. Then $\tau = 1/\nu - 1/2$ and $1 < \nu < 2$. Put $\nu' := \nu/(\nu - 1)$, we have $1/\nu + 1/\nu' = 1$ and $1 < \nu' < 2$. Let $\boldsymbol{u}, \boldsymbol{v} \in (1, \infty)^d$ be defined by $\boldsymbol{u} := q\boldsymbol{v}/2$ and $v_i = \nu$ if $k_i < s_i$, and $v_i = \nu'$ if $k_i \ge s_i$ for $i = 1, ..., d$. Let $\boldsymbol{u}'$ and $\boldsymbol{v}'$ be given by $\mathbf{1/u} + \mathbf{1/u'} = \mathbf{1}$ and $\mathbf{1/v} + \mathbf{1/v'} = \mathbf{1}$, respectively. Notice that $\boldsymbol{v} \in (1, \infty)^d$. Applying the Hölder inequality for the mixed norm $\|\cdot\|_{L_{w,\boldsymbol{v}}(\mathbb{R}^d)}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |\varphi_{\boldsymbol{k}}(\boldsymbol{x})\varphi_{\boldsymbol{s}}(\boldsymbol{x})|^{q/2} w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ &\le \ \||\varphi_{\boldsymbol{k}}|^{q/2}\|_{L_{\boldsymbol{v},w}(\mathbb{R}^d)}\||\varphi_{\boldsymbol{s}}|^{q/2}\|_{L_{\boldsymbol{v}',w}(\mathbb{R}^d)} \\ &= \ \|\varphi_{\boldsymbol{k}}\|^{q/2}_{L_{\boldsymbol{u},w}(\mathbb{R}^d)}\|\varphi_{\boldsymbol{s}}\|^{q/2}_{L_{\boldsymbol{u}',w}(\mathbb{R}^d)}. \end{aligned} \tag{A.3}$$

Since $\boldsymbol{u} < p\mathbf{1}$ and $\boldsymbol{u}' < p\mathbf{1}$, by inequality (A.2) we have

$$\begin{aligned} \|\varphi_{\boldsymbol{k}}\|_{L_{\boldsymbol{u},w}(\mathbb{R}^d)} \ &\le \ 2^{(1/\lambda)|(\mathbf{1/u}-\mathbf{1}/p)\boldsymbol{k}|_1}\|\varphi_{\boldsymbol{k}}\|_{L_{p,w}(\mathbb{R}^d)}, \\ \|\varphi_{\boldsymbol{s}}\|_{L_{\boldsymbol{u}',w}(\mathbb{R}^d)} \ &\le \ 2^{(1/\lambda)|(\mathbf{1/u'}-\mathbf{1}/p)\boldsymbol{s}|_1}\|\varphi_{\boldsymbol{s}}\|_{L_{p,w}(\mathbb{R}^d)}. \end{aligned} \tag{A.4}$$

From (A.3) and (A.4) we prove the lemma. □

Lemma A.2 *Let $1 \le p, q < \infty$, $p \ne q$ and $f \in L_{q,w}(\mathbb{R}^d)$ be represented by the series*

$$f \ = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} \varphi_{\boldsymbol{k}}, \ \ \varphi_{\boldsymbol{k}} \in \mathcal{P}_{2^{\boldsymbol{k}}}$$

converging in $L_{q,w}(\mathbb{R}^d)$. Then there holds the inequality

$$\|f\|_{L_{q,w}(\mathbb{R}^d)} \ \le C\left(\sum_{k \in \mathbb{Z}_+^d} \|2^{\delta_{\lambda,p,q}|\boldsymbol{k}|_1}\varphi_{\boldsymbol{k}}\|^q_{L_{p,w}(\mathbb{R}^d)}\right)^{1/q}, \tag{A.5}$$

with some constant C depending at most on λ, p, q, d, whenever the right side is finite.

Proof. It is sufficient to show the inequality (A.5) for f of the form

$$f = \sum_{\boldsymbol{k}\le \boldsymbol{m}} \varphi_{\boldsymbol{k}}, \ \varphi_{\boldsymbol{k}} \in \mathcal{P}_{2^{\boldsymbol{k}}},$$

for any $\boldsymbol{m} \in \mathbb{N}_0^d$. We will prove this for the case $1 \le q < p < \infty$. The case $1 \le p < q < \infty$ can be proven analogously.

Put $n := [q] + 1$. Then $0 < q/n \le 1$. By the Jensen inequality we have

$$\left|\sum_{\boldsymbol{k}\le \boldsymbol{m}} \varphi_{\boldsymbol{k}}(\boldsymbol{x})\right|^q = \left(\left|\sum_{\boldsymbol{k}\le \boldsymbol{m}} \varphi_{\boldsymbol{k}}(\boldsymbol{x})\right|^{q/n}\right)^n$$
$$\le \left(\sum_{\boldsymbol{k}\le \boldsymbol{m}} |\varphi_{\boldsymbol{k}}(\boldsymbol{x})|^{q/n}\right)^n = \sum_{\boldsymbol{k}^1\le m} \cdots \sum_{\boldsymbol{k}^n\le m} \prod_{j=1}^n |\varphi_{\boldsymbol{k}^j}(\boldsymbol{x})|^{q/n}.$$

and therefore,

$$\|f\|_{L_{q,w}(\mathbb{R}^d)}^q \le \sum_{\boldsymbol{k}^1\le \boldsymbol{m}} \cdots \sum_{\boldsymbol{k}^n\le \boldsymbol{m}} \int_{\mathbb{R}^d} \prod_{j=1}^n |\varphi_{\boldsymbol{k}^j}(\boldsymbol{x})|^{q/n} w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{A.6}$$

From the identity

$$\prod_{j=1}^n a_j = \left(\prod_{i\ne j} a_i a_j\right)^{1/2(n-1)} \tag{A.7}$$

for non-negative numbers $a_1, ..., a_n$, we obtain

$$\mathcal{I} := \int_{\mathbb{R}^d} \prod_{j=1}^n |\varphi_{\boldsymbol{k}^j}(\boldsymbol{x})|^{q/n} w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \prod_{i\ne j} |\varphi_{\boldsymbol{k}^i}(\boldsymbol{x})\varphi_{\boldsymbol{k}^j}(\boldsymbol{x})|^{q/2n(n-1)} w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Hence, applying the Hölder inequality to $n(n-1)$ functions in the right side of the last equality, Lemma A.1 and (A.7) give

$$\mathcal{I} \le \prod_{i\ne j}\left(\int_{\mathbb{R}^d} |\varphi_{\boldsymbol{k}^i}(\boldsymbol{x})\varphi_{\boldsymbol{k}^j}(\boldsymbol{x})|^{q/2} w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\right)^{1/n(n-1)} \le \prod_{i\ne j}\left(S_{\boldsymbol{k}^i} S_{\boldsymbol{k}^j} 2^{-\tau|\boldsymbol{k}^i - \boldsymbol{k}^j|_1}\right)^{1/n(n-1)}$$
$$= \prod_{i\ne j}(S_{\boldsymbol{k}^i} S_{\boldsymbol{k}^j})^{1/n(n-1)} \left\{\left(\prod_{i\ne j}\prod_{i'=1}^n 2^{-\tau|\boldsymbol{k}^i - \boldsymbol{k}^{i'}|_1} \prod_{j'=1}^n 2^{-\tau|\boldsymbol{k}^j - \boldsymbol{k}^{j'}|_1}\right)^{1/2(n-1)}\right\}^{1/n(n-1)}$$
$$= \left\{\prod_{i\ne j}(S_{\boldsymbol{k}^i} S_{\boldsymbol{k}^j})^{2/n} \left(\prod_{i'=1}^n 2^{-\tau|\boldsymbol{k}^i - \boldsymbol{k}^{i'}|_1} \prod_{j'=1}^n 2^{-\tau|\boldsymbol{k}^j - \boldsymbol{k}^{j'}|_1}\right)^{1/n(n-1)}\right\}^{1/2(n-1)}$$
$$= \prod_{j=1}^n S_{\boldsymbol{k}^j}^{2/n} \left(\prod_{i=1}^n 2^{-\tau|\boldsymbol{k}^j - \boldsymbol{k}^i|_1}\right)^{1/n(n-1)} = \left(\prod_{j=1}^n S_{\boldsymbol{k}^j}^2 \prod_{i=1}^n 2^{-\eta\tau|\boldsymbol{k}^j - \boldsymbol{k}^i|_1}\right)^{1/n},$$

where $\eta := \tau/(n-1) > 0$. (Here and below we use the notation in Lemma A.1.) Therefore, from (A.6) and the Hölder inequality we obtain

$$\begin{aligned} \|f\|_{L_{q,w}(\mathbb{R}^d)}^q &\le \sum_{\boldsymbol{k}^1 \le \boldsymbol{m}} \cdots \sum_{\boldsymbol{k}^n \le \boldsymbol{m}} \left(\prod_{j=1}^n S_{\boldsymbol{k}^j}^2 \prod_{i=1}^n 2^{-\eta\tau|\boldsymbol{k}^j - \boldsymbol{k}^i|_1} \right)^{1/n} \\ &\le \prod_{j=1}^n \left(\sum_{\boldsymbol{k}^1 \le \boldsymbol{m}} \cdots \sum_{\boldsymbol{k}^n \le \boldsymbol{m}} S_{\boldsymbol{k}^j}^2 \prod_{j=1}^n 2^{-\eta\tau|\boldsymbol{k}^j - \boldsymbol{k}^i|_1} \right)^{1/n} =: \prod_{j=1}^n B_j. \end{aligned} \tag{A.8}$$

We have

$$\begin{aligned} B_j &= \sum_{\boldsymbol{k}^j \le \boldsymbol{m}} S_{\boldsymbol{k}^j}^2 \sum_{\boldsymbol{k}^1 \le \boldsymbol{m}} \cdots \sum_{\boldsymbol{k}^{j-1} \le \boldsymbol{m}} \sum_{\boldsymbol{k}^{j+1} \le \boldsymbol{m}} \cdots \sum_{\boldsymbol{k}^n \le \boldsymbol{m}} \prod_{i=1}^n 2^{-\eta\tau|\boldsymbol{k}^j - \boldsymbol{k}^i|_1} \\ &= \sum_{\boldsymbol{k}^j \le \boldsymbol{m}} S_{\boldsymbol{k}^j}^2 \left(\sum_{\boldsymbol{s} \le \boldsymbol{m}} 2^{-\eta\tau|\boldsymbol{k}^j - \boldsymbol{s}|_1} \right)^{n-1} \le C \sum_{\boldsymbol{k}^j \le \boldsymbol{m}} S_{\boldsymbol{k}^j}^2. \end{aligned}$$

Using the last bound for B_j, continuing the estimates (A.8) we finish the proof of the lemma:

$$\begin{aligned} \|f\|_{L_{q,w}(\mathbb{R}^d)}^q &\le \prod_{j=1}^n B_j^{1/n} \le C \sum_{\boldsymbol{k} \le \boldsymbol{m}} S_{\boldsymbol{k}}^2 \\ &= C \sum_{\boldsymbol{k} \le \boldsymbol{m}} \|2^{\delta_{\lambda,p,q}|\boldsymbol{k}|_1} \varphi_{\boldsymbol{k}}\|_{L_{p,w}(\mathbb{R}^d)}^q. \end{aligned}$$

□

In the proofs of Lemma A.2, we have used a modification of a technique employed in [24] and [8] in the proofs of a trigonometric version ($1 \le p < q < \infty$) and of a B-spline version ($0 < p < q < \infty$) of the results of this lemma.

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