ON THE PERTURBATIONS OF NOETHERIAN LOCAL DOMAINS

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Abstract. We study how the properties of being reduced, an integral domain, and normal, behave under small perturbations of the defining equations of a noetherian local ring. It is not hard to show that the property of being a local integral domain (reduced, normal ring) is not stable under small perturbations in general. We prove that perturbation stability holds in the following situations: (1) perturbation of being an integral domain for factorial excellent Henselian local rings; (2) perturbation of normality for excellent local complete intersections containing a field of characteristic zero; and (3) perturbation of reducedness for excellent local complete intersections containing a field of characteristic zero, and for factorial Nagata local rings.

1. Introduction

A noetherian local ring $(R, \mathfrak{m})$ is said to *have isolated singularity* if $R_\mathfrak{p}$ is regular for any non-maximal prime ideal $\mathfrak{p}$. Let $\Bbbk$ be a field and $S = \Bbbk[\![y_1, \ldots, y_s]\!]$ be a power series ring where $s \geq 1$. In 1956, Samuel [30] showed that for any series $f \in S$ such that $S/(f)$ has isolated singularity, there exists an integer $N \geq 1$ such that for all $\delta \in (y_1, \ldots, y_s)^N$, there is an automorphism of S sending f to $f + \delta$. Thus if $S/(f)$ is a hypersurface ring with isolated singularity, then replacing f by an element in a sufficiently small adic neighborhood of f, does not affect the properties of the ring. Subsequently, many authors have considered the behaviour of various ring- and module-theoretic properties and numerical invariants under small perturbations; see, e.g. [5, 6, 7, 9, 11, 17, 20, 27, 31]. We have information about the behaviour under perturbation of Hilbert–Samuel functions [20, 28, 31], homology of complexes [9], regular sequences [17], positive characteristic singularities [6], among others.

Given noetherian local ring $(R, \mathfrak{m})$, we say that a ring property $\mathcal{P}$ is *stable under perturbations* for R (or *perturbation of* $\mathcal{P}$ *holds* for R) if for any regular element a such that $R/(a)$ satisfies $\mathcal{P}$, we also have that $R/(b)$ satisfies $\mathcal{P}$ for all $N \gg 0$ and all $b \in a + \mathfrak{m}^N$, i.e., for all b lying in a sufficiently small neighborhood of a. We say that the property $\mathcal{P}$ is *stable under perturbations* for a class $\mathcal{C}$ of noetherian local rings (resp., for all noetherian local rings), if $\mathcal{P}$ is stable under perturbations for every noetherian local ring $(R, \mathfrak{m})$ in $\mathcal{C}$ (resp. for every such ring). As proved in [20, Theorem 3.5], the transformations sending $R/(a)$ to $R/(a + \delta)$ where δ is a sufficiently small error, preserves the *Hilbert–Samuel function* of $R/(a)$, namely the Hilbert function of the associated graded ring $\operatorname{gr}_\mathfrak{m}(R/(a))$. Not all ring-theoretic properties behave alike under perturbations and many basic questions remain open. For instance, in a recent work on F-singularities in positive characteristics [6], De Stefani and Smirnov showed the stability of F-rational singularities, as well as some subclass of F-injective singularities under perturbations. On the other hand, they exhibited in [6,

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Example 5.3] that F-purity and (strong) F-regularity are not stable under perturbations. It is not clear whether F-injectivity always perturbs.

In this note, we look at very basic properties of rings: integrality (being a domain), reducedness and normality, and ask how they behave under fine perturbations. As far as we know, this problem was first proposed by L. Duarte in his Ph.D. thesis [7, Question 5.2.2]. It is not hard to show that these questions generally have a negative answer. In fact, there exists a regular local ring $(R, \mathfrak{m})$, a non-zerodivisor a such that $R/(a)$ is a domain, however arbitrarily small perturbations of it need not be domains (see Example 2.7). Using a classical construction of Nagata, we also produce a normal local domain $R/(a)$ whose arbitrarily small perturbations may fail to be normal (see Example 3.5 and also Remark 3.6 on the instability of reducedness). As a consequence, we get negative answers to the last part of [7, Questions 5.2.1 and 5.2.2]. In all these examples, the ring R is regular but not $\mathfrak{m}$-adically complete, so we are naturally led to the following

Question 1.1. *Let $(R, \mathfrak{m})$ be a complete noetherian local ring, and $a \in \mathfrak{m}$ a regular element. Suppose $R/(a)$ is an integral domain (res. reduced, normal ring). Is it true that $R/(a+\delta)$ is an integral domain (res. reduced, normal ring) for all $N \gg 0$ and all $\delta \in \mathfrak{m}^N$?*

We do not know the full answer to this question. On the other hand, using strong Artin approximation, a recent generalization of Samuel's work [12], and various Bertini-type theorems, we prove several results supporting a positive answer for it.

Main Theorem. *The following statements hold true.*

1. *Perturbation of integrality holds for excellent factorial Henselian local rings, in particular, for complete regular local rings (see Theorem 3.1 and Corollary 3.2 for details).*
2. *Perturbation of normality holds for excellent local complete intersections that contain a field of characteristic zero (Theorem 3.8).*
3. *Perturbation of reducedness holds for excellent local complete intersections that contain a field of characteristic zero and for factorial Nagata local rings(Theorems 3.17 and 3.19).*

Note that Theorem 3.1 answers affirmatively a question in Duarte's thesis [8, Question 5.2.3]. Note also that in our main results, the base ring R is usually "nice" but not necessarily complete. At the moment, we do not know whether perturbation of integrality holds for $\mathfrak{m}$-adically complete local complete intersections, even those that contain a field.

Our results on the perturbation stability of normality and reducedness (Theorem 3.8 and Theorem 3.17) are consequences of a more general statement.

Theorem 1.2 (= Theorem 3.9)**.** *Let $(R, \mathfrak{m})$ be an excellent local complete intersection that contains a field of characteristic zero. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ satisfies Serre's regularity condition (R_ν) for some non-negative integer ν. Then there exists $N \geq 1$ such that for every $\delta \in \mathfrak{m}^N$, $R/(a+\delta)$ also satisfies (R_ν).*

Since we will encounter types of rings such as (quasi-)excellent, Nagata, and Henselian ones quite often in this note, we recall their definitions in Section 2. In Section 3, we display examples on the instability of (normal) integral domains, thereby answering in the negative [6, Question 6.1]. We also provide the proofs of our main results in this section, hoping to inspire further study on the tantalizing Question 1.1.

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2. Background

Let $\mathcal{P}$ be a given property of noetherian local rings. We say that the property $\mathcal{P}$ is *stable under perturbations* (or *perturbation of $\mathcal{P}$ holds*) for a noetherian local ring $(R, \mathfrak{m})$ if for any non-zerodivisor $a \in \mathfrak{m}$ such that $R/(a)$ satisfies $\mathcal{P}$, then $R/(a+\delta)$ satisfies $\mathcal{P}$ for all $N \gg 0$ and all $\delta \in \mathfrak{m}^N$. Similarly, we define when $\mathcal{P}$ is stable under perturbations for a class of (resp. for all) noetherian local rings. Related to the notion of stability under perturbations, we record the well-known fact that regular sequences perturb; see [9, Corollary 1] and [17].

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a noetherian local ring, and $a \in \mathfrak{m}$ an R-regular element. Then there exists an $N \geq 1$ such that for all $\delta \in \mathfrak{m}^N$, $a + \delta$ is R-regular.*

We say that *deformation of $\mathcal{P}$ holds* for a local ring $(R, \mathfrak{m})$ if R satisfies $\mathcal{P}$, as soon as there exists an R-regular element $a \in \mathfrak{m}$ such that $R/(a)$ satisfies $\mathcal{P}$. Define in a similar manner when *deformation of $\mathcal{P}$ holds for all local rings.* Perturbation and deformation are closely related as shown in [6, Theorem 2.4]. In some sense, perturbation of $\mathcal{P}$ can be regarded as an $\mathfrak{m}$-adic analogue of deformation.

Theorem 2.2 (De Stefani and Smirnov [6, Theorem 2.4])**.** *Let $\mathcal{P}$ be a property of local rings such that:*

(i) *if a local ring $(R, \mathfrak{m})$ satisfies $\mathcal{P}$, and T is a variable, then $R[T]_{(\mathfrak{m},T)}$ satisfies $\mathcal{P}$; and,*
(ii) *if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is faithfully flat, and S satisfies $\mathcal{P}$, then so does R.*

If perturbation of $\mathcal{P}$ holds for all noetherian local rings, so does deformation of $\mathcal{P}$.

Recall that for an non-negative integer ν, a noetherian ring A satisfies *Serre's regularity condition* (R_ν) if $A_\mathfrak{p}$ is a regular local ring for any $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\operatorname{ht} \mathfrak{p} \leq \nu$. The ring A satisfies *Serre's condition* (S_ν) if $\operatorname{depth} A_\mathfrak{p} \geq \min\{\nu, \dim A_\mathfrak{p}\}$ for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$. Recall the following fundamental facts.

Lemma 2.3 (See [23, §23] and [10, Theorem 11.5])**.** *Let A be a noetherian ring. The following statements hold.*

(i) *A is reduced if and only if it satisfies both (R_0) and (S_1).*
(ii) (Serre's normality criterion) *A is normal if and only if it satisfies both (R_1) and (S_2).*

We will employ the next deformation result quite often. For a proof of part (ii), see [15, Theorem, pp. 439-440]. Part (iii) is contained in [3, Lemma 0]. See also the much more general and sweeping result of Li [19, Theorem 1.4].

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local ring and let $a \in \mathfrak{m}$ be a non-zerodivisor.*

(i) *If $R/(a)$ is an integral domain, then so is R.*

(ii) *If $R/(a)$ is normal, then so is R.*
(iii) *More generally, if $R/(a)$ satisfies Serre's conditions (R_ν) and $(S_{\nu+1})$, then R also satisfies both (R_ν) and $(S_{\nu+1})$.*
(iv) *If $R/(a)$ is reduced, then so is R.*

Remark 2.5. (1) Given a noetherian local ring $(R, \mathfrak{m})$, we say that a property $\mathcal{Q}$ of principal ideals of R is *stable under perturbations* if any principal R-ideal (a) generated by an R-regular element, with the property $\mathcal{Q}$, there exists $N \geq 1$ such that for all $\delta \in \mathfrak{m}^N$, the ideal $(a + \delta)$ also satisfies $\mathcal{Q}$.

In this language, the part concerning integral domains in Question 1.1 can be rephrased as follows: *Let $(R, \mathfrak{m})$ be a complete local ring. Then is the property of being a principal prime ideal stable under perturbations?*

The following example shows that the property of being an irreducible (or primary) principal ideal is not stable under perturbations.

Let $R = \Bbbk[\![x, y]\!]$ be the ring of formal power series over a field $\Bbbk$ with $\operatorname{char} \Bbbk \neq 2$. We have (x^2) is an irreducible ideal. However for all $\delta = -y^{2n}, n \geq 1$, as $(x^2 - y^{2n}) = (x + y^n)(x - y^n)$, $(x^2 - y^{2n})$ is not even primary.

(2) Similarly, one can define the ideal-theoretic version of deformation. Given a noetherian local ring $(R, \mathfrak{m})$, we say that a property $\mathcal{Q}$ of ideals of R is *stable under deformations* if for any R-ideal I and any element $x \in R$ that is (R/I)-regular, such that $I + (x)$ has the property $\mathcal{Q}$, then I also has property $\mathcal{Q}$.

Thus Proposition 2.4 implies that the property of being a radical (prime) ideal is stable under deformations. On the other hand, the following example shows that the property of being an irreducible (or primary) ideal is not stable under deformations.

Let $R = \Bbbk[\![x, y]\!]$ and $I = (xy)$. Then $a = x + y$ is a non-zerodivisor on R/I. Moreover $I + (x + y) = (x + y, x^2)$ is an irreducible ideal of R. However, I is not every primary.

The authors of [6] asked if the converse of Theorem 2.2 also holds true.

Question 2.6 ([6, Question 6.1])**.** *Let $\mathcal{P}$ be a property of local rings that satisfies the assumptions of Theorem 2.2. Are deformation and perturbation of $\mathcal{P}$ equivalent?*

The following example provides a negative answer to this question. The example also shows that the integrality property is generally unstable under perturbations.

Example 2.7. Let $\Bbbk$ be a field of characteristic 0. Note that the element $1 + x$ is not a square in $\Bbbk[x]_{(x)}$ since otherwise $1 + x = \dfrac{P^2}{Q^2}$ for some $P, Q \in \Bbbk[x]$, so $2 \deg P = 1 + 2 \deg Q$, which is impossible. However in $\Bbbk[\![x]\!]$ there exists an element $u = 1 + a_1 x + a_2 x^2 + \cdots$ such that $u^2 = 1 + x$, thanks to the Taylor expansion of $(1 + x)^{1/2}$. Let $R = \Bbbk[x, y]_{(x,y)}$. We have $a = x^2 + x^3 - y^2$ is an irreducible polynomial in $\Bbbk[x, y]$, so $R/(a)$ is a domain. Taking the (x, y)-adic completion, we have

$$x^2 + x^3 - y^2 = (ux)^2 - y^2 = (ux + y)(ux - y),$$

which is reducible as a power series. For each $N \geq 2$, set

$$b_{1,N} = x + a_1 x^2 + \cdots + a_{N-1} x^N + y \in \Bbbk[x, y],$$

$$b_{2,N} = x + a_1 x^2 + \cdots + a_{N-1} x^N - y \in \Bbbk[x, y].$$

It is easy to check that $a = b_{1,N}b_{2,N} - \delta_N$ with $\delta_N \in (x, y)^{N+1}R$. Hence for every $N \geq 1$ we have an element $\delta_N \in (x, y)^{N+1}R$ such that $a + \delta_N$ is reducible.

Corollary 2.8. *There exists a property $\mathcal{P}$ of local rings that satisfies the assumptions of Theorem 2.2 such that deformation and perturbation of $\mathcal{P}$ are not equivalent.*

Proof. Let $\mathcal{P}$ be the property of "integrality". It is easy to see that $\mathcal{P}$ satisfies the assumptions of Theorem 2.2. It follows from Proposition 2.4 that deformation of $\mathcal{P}$ holds for local rings, but perturbation of $\mathcal{P}$ does not hold in general for local rings according to Example 2.7. $\square$

We will see from Example 3.5 that the property of "normality" also satisfies the assumptions of Theorem 2.2, but deformation holds for normality while perturbation does not.

Next, let us recall the notions of quasi-excellent, excellent and Nagata rings, following the Stacks Project [32, 33].

Definition 2.9. Let $\mathbb{k}$ be a field, and A be a $\mathbb{k}$-algebra.

(1) We say that A is *geometrically normal* over $\mathbb{k}$, if for every field extension $\mathbb{k}'$ of $\mathbb{k}$, $\mathbb{k}' \otimes_{\mathbb{k}} A$ is a normal ring. Equivalently, A is geometrically normal over $\mathbb{k}$ if $\mathbb{k}' \otimes_{\mathbb{k}} A$ is a normal ring for every finitely generated field extension $\mathbb{k}'$ of $\mathbb{k}$.
(2) We say that A is *geometrically regular* over $\mathbb{k}$ if for every finitely generated field extension $\mathbb{k}'$ of $\mathbb{k}$, $\mathbb{k}' \otimes_{\mathbb{k}} A$ is a regular ring.

If $(R, \mathfrak{m})$ is a local ring, let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R, and the *formal fiber* of R over $\mathfrak{p} \in \operatorname{Spec}(R)$ is defined as $\widehat{R} \otimes_R \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ denotes the residue field of $R_{\mathfrak{p}}$.

Definition 2.10 (G-rings)**.** A ring R is called a *G-ring* if R is noetherian and for every prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$, the formal fiber of $R_{\mathfrak{p}}$ over $\mathfrak{q}R_{\mathfrak{p}}$, namely $\widehat{R_{\mathfrak{p}}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{q})$, is geometrically regular over $\kappa(\mathfrak{q})$.

For a ring S, the *regular locus* of $\operatorname{Spec}(S)$ is

$$\operatorname{Reg}(\operatorname{Spec}(S)) = \{\mathfrak{q} \in \operatorname{Spec}(S) \mid S_{\mathfrak{q}} \text{ is a regular local ring}\}.$$

The *singular locus* $\operatorname{Sing}(\operatorname{Spec}(S))$ of $\operatorname{Spec}(S)$ is the complement of $\operatorname{Reg}(\operatorname{Spec}(S))$ inside $\operatorname{Spec}(S)$.

Definition 2.11. Let R be a noetherian ring, and $X = \operatorname{Spec}(R)$. We say that R is a *J-2* ring if for any finite type R-algebra S, the regular locus $\operatorname{Reg}(\operatorname{Spec}(S))$ is an open subset of $\operatorname{Spec}(S)$.

Now we come to the following crucial notions.

Definition 2.12 (Quasi-excellent and excellent rings)**.** Let R be a ring. We say that R is *quasi-excellent* if R is noetherian, a G-ring, and J-2 ring. We say that R an *excellent* ring, if R is quasi-excellent, and universally catenary.

Remark 2.13. By [33, Lemma 15.52.2, Tag 07QU and Lemma 15.52.3, Tag 07QW], the following types of rings are excellent:

(1) fields, $\mathbb{Z}$,
(2) complete noetherian local rings,
(3) Dedekind domains with fraction field of characteristics zero,

(4) localizations of a finite type algebra over an excellent ring.

The following lemma follows from the definition of quasi-excellent rings.

Lemma 2.14. *Let $(R, \mathfrak{m})$ be a quasi-excellent noetherian local ring. Let $\nu \geq 0$ be an integer. Then R satisfies (R_ν) if and only if $\widehat{R}$ satisfies (R_ν).*

Proof. Since R is quasi-excellent, it is a G-ring, so the formal fibers $\widehat{R} \otimes_R \kappa(\mathfrak{p})$ are regular for all $\mathfrak{p} \in \operatorname{Spec}(R)$. In particular, all the formal fibers satisfy (R_ν). Since $R \to \widehat{R}$ is a flat morphism, by the base change result for the condition (R_ν) [4, Proposition 2.2.21], R satisfies (R_ν) if and only if $\widehat{R}$ does. $\square$

Recall that an *N-2 ring* (or *Japanese ring*) is an integral domain R such for every finite field extension L of the fraction field $Q(R)$ of R, the integral closure of R in L is R-finite. A ring R is said to be *Nagata* if R is noetherian and for every prime ideal $\mathfrak{p}$ the ring $R/\mathfrak{p}$ is N-2 ([32, Definition 10.162.1] and [22, Definition 31.A]).

Remark 2.15. By [32, Section 162], examples of Nagata rings include:

(1) fields, $\mathbb{Z}$,
(2) complete noetherian local rings,
(3) Dedekind domains with fraction field of characteristic zero,
(4) localizations of a finite type algebra over a Nagata ring.

While perturbation of integral domains fails in general, fortunately, we can prove that perturbation of integrality holds for excellent factorial Henselian local rings (e.g. formal power series rings over fields). The key technique is a version of Artin approximation theorem ([1, 13, 25, 26]).

Definition 2.16 (Henselian rings)**.** A noetherian local ring $(R, \mathfrak{m}, \mathbb{k})$ is called *Henselian* if for every monic polynomial $f \in R[T]$ and every root $a_0 \in \mathbb{k}$ of $\overline{f} := f + \mathfrak{m}R[T] \in \mathbb{k}[T]$ such that a_0 is not a root of the derivative $\overline{f}'$ (namely $\overline{f}'(a_0) \neq 0$), there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = a + \mathfrak{m}$.

Remark 2.17. Note that by [32, Lemma 10.153.9, Tag 04GM and Lemma 10.153.10, Tag 06RS], the class of Henselian local rings contains complete local rings and artinian local rings and the rings of algebraic (convergent) power series over (valued) fields.

The version of Artin approximation that we will use in this note is

Theorem 2.18 (Strong Artin approximation, [13, Corollary 3.17])**.** *Let $(R, \mathfrak{m})$ be an excellent Henselian local ring. Assume that $r, s \geq 1$ are integers, $f_1, \ldots, f_r \in R[y] := R[y_1, \ldots, y_s]$ are polynomials in the indeterminates $y_1, \ldots, y_s$ with coefficients in R. Denote $f(y) = (f_1(y), \ldots, f_r(y)) \in R[y]^{\oplus r}$ (finite Cartesian product). Then there exists a function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ with the property: for every $c \in \mathbb{N}$ and every $\bar{y} \in R^s$ such that $f(\bar{y}) \in (\mathfrak{m}^{\beta(c)})^{\oplus r}$, there exists $\widetilde{y} \in R^s$ such that $f(\widetilde{y}) = \underbrace{(0, 0, \ldots, 0)}_{r \text{ times}}$ and $\widetilde{y} - \bar{y} \in (\mathfrak{m}^c)^{\oplus r}$.*

We also recall the notions of analytically unramified/irreducible local rings.

Definition 2.19. Let $(R, \mathfrak{m})$ be a noetherian local ring. We say that R is *analytically unramified* if $\widehat{R}$ is a reduced ring. We say that *analytically irreducible* if $\widehat{R}$ is an integral domain.

Remark 2.20. Following from [33, Lemma 15.52.5, Tag 07QV], any quasi-excellent ring is Nagata. By [32, Lemma 10.162.13, Tag 0331], any Nagata local domain is analytically unramified. Hence any quasi-excellent local domain is analytically unramified.

Moreover, combining the last statement with [32, Lemma 10.162.10, Tag 032Y], we see that any quasi-excellent reduced local ring is analytically unramified.

An important fact is that quasi-excellent *normal* local domains are analytically irreducible. While we will not use this result explicitly in the sequel, we record it to showcase the relevance of (quasi-)excellency to our study of normality in this paper.

Theorem 2.21 (See [29, Corollaire, page 99], [2, Theorem 2.3])**.** *Let $(R, \mathfrak{m})$ be noetherian local domain such that its formal fibers are geometrically normal (e.g., R is quasi-excellent). Then the number of minimal primes of the $\mathfrak{m}$-adic completion $\widehat{R}$ equals exactly the number of maximal ideals of the integral closure $\overline{R}$.*

If moreover $(R, \mathfrak{m})$ is a quasi-excellent noetherian local domain, then R is analytically irreducible if and only if $\overline{R}$ is a (noetherian) local ring. In particular, any quasi-excellent normal local domain is analytically irreducible.

3. Perturbations of (normal) domains and reduced rings

Our first main result provides a partial positive answer on the perturbation of integrality when R is an excellent factorial Henselian local ring.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be an excellent factorial Henselian local ring. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ is an integral domain. Then there exists $N \geq 1$ such that $R/(a + \delta)$ is a domain for all $\delta \in \mathfrak{m}^N$.*

Proof. Since R is a factorial domain, it suffices to show that if $a \in \mathfrak{m}, a \neq 0$ is irreducible then so is every sufficiently small perturbation $a + \delta$ of it.

Since R is excellent Henselian, applying Theorem 2.18 for $f(y_1, y_2) = y_1 y_2 - a$ in $R[y_1, y_2]$, there exists a function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that: For all $c \in \mathbb{N}$, for all $\bar{y} = (\bar{y}_1, \bar{y}_2) \in R^2$ such that $f(\bar{y}) \in \mathfrak{m}^N$, where $N = \beta(c)$, there exists $\widetilde{y} \in R^2$ such that $f(\widetilde{y}) = 0$ and $\widetilde{y} - \bar{y} \in (\mathfrak{m}^c)^{\oplus 2}$.

We will show that $a + \delta$ is irreducible for all $\delta \in \mathfrak{m}^N$, where $N := \beta(1)$. Assume by contradiction that, $a + \delta$ is not irreducible for some $\delta \in \mathfrak{m}^N$. Then

$$a + \delta = \bar{y}_1 \bar{y}_2 \quad \text{for some } \bar{y}_1, \bar{y}_2 \in \mathfrak{m}.$$

This implies that

$$f(\bar{y}_1, \bar{y}_2) = \bar{y}_1 \bar{y}_2 - a = \delta \in \mathfrak{m}^N.$$

Then there exist $y = (y_1, y_2) \in R^2$ such that $f(y_1, y_2) = 0$ and $y - \bar{y} \in \mathfrak{m}^{\oplus 2}$. This means that a is reducible, which is a contradiction. This completes the proof. □

An immediate corollary is the following positive answer to a question of Duarte [8, Question 5.2.3].

Corollary 3.2. *Let R be a complete regular local ring, e.g. a power series ring $\Bbbk[\![y_1, \ldots, y_s]\!]$ over a field $\Bbbk$. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ is an integral domain. Then we can find an $N \geq 1$ such that $R/(a + \delta)$ is a domain for every error term $\delta \in \mathfrak{m}^N$.*

Proof. This follows from Theorem 3.1 since any regular local ring is factorial, and any complete noetherian local ring is Henselian. □

Let us recall the notion of abstract local complete intersections [4, Definition 2.3.1].

Definition 3.3. Let $(R, \mathfrak{m})$ be a noetherian local ring. We say R is a *complete intersection* if there exists a regular local ring $(S, \mathfrak{m}_S)$ and an S-regular sequence $x_1, \dots, x_c$ in $\mathfrak{m}_S$ such that the $\mathfrak{m}$-adic completion of R satisfies $\widehat{R} \cong S/(x_1, \dots, x_c)$.

In view of Corollary 3.2, it seems to be natural to ask whether integrality perturbs for $\mathfrak{m}$-adically complete, local complete intersections.

Question 3.4. Let $(R, \mathfrak{m})$ be an $\mathfrak{m}$-adically complete, local complete intersection. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ is an integral domain. Is it true that there exists $N \geq 1$ such that for all $\delta \in \mathfrak{m}^N$, $R/(a+\delta)$ is also an integral domain?

We do not know the answer to this question. On the other hand, we will prove in Theorem 3.8 that normality perturbs for excellent local complete intersections containing a field of characteristic zero.

A natural question is whether perturbation of normality always holds for local rings. We provide a negative answer to this question in the following example, based on a classical construction of Nagata. We are largely influenced by the exposition of Heinzer, Rotthaus and S. Wiegand [14, Example 4.15].

Example 3.5 (Nagata's example). Let $\mathbb{k}$ be a field of characteristic zero. We start with the power series ring $\mathbb{k}[\![x, y]\!]$. Let $\tau \in x\mathbb{k}[\![x]\!]$ be the following element that is transcendental over $\mathbb{k}(x, y)$:

$$\tau = e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

Let $f := (y+\tau)^2$, and consider the rings

$$A = \mathbb{k}(x, y, f) \cap \mathbb{k}[\![x, y]\!] \quad \text{and} \quad R = A[z]_{(x,y,z)}.$$

Denote $a = z^2 - f \in R$. We claim that the following statements hold true.

(1) R is a 3-dimensional regular local ring with the unique maximal ideal $\mathfrak{m} = (x, y, z)R$. Moreover, R is not excellent.
(2) a is R-regular and $R/(a)$ is a normal domain.
(3) For each $n \geq 1$, there exists $\delta_n \in \mathfrak{m}^n$ such that $R/(a+\delta_n)$ is not a domain, hence not normal.

Proof. We first note that A is identical to the following ring in [14, Proposition 6.19]:

$$A' = \mathbb{k}(x, y, f) \cap \mathbb{k}[y]_{(y)}[\![x]\!].$$

Indeed, clearly A' is contained in A. Conversely, any element $p \in A$ can be written as a power series in x:

$$p = a_0(y) + a_1(y)x + a_2(y)x^2 + \cdots, \quad \text{where } a_i(y) \in \mathbb{k}[\![y]\!] \text{ for each } i.$$

To prove $A \subseteq A'$, it remains to establish the next

Observation: We have $a_i(y) \in \mathbb{k}[y]_{(y)}$ for each $i \geq 0$.

Proof of the observation: Indeed, it is harmless to assume that $a_0(y) \neq 0$, as otherwise we may replace p by $p/x \in A$. Note that each element of $k[x, y, f] \subseteq \mathbb{k}[y][\![x]\!]$ has the form $g(x) = g_0(y) + g_1(y)x + g_2(y)x^2 + \cdots$, where $g_i(y) \in \mathbb{k}[y]$.

Now p belongs to $\Bbbk(x, y, f)$ means that $p = \dfrac{h(x)}{g(x)}$, where $g(x), h(x) \in \Bbbk[y][\![x]\!]$, and $g(x) \neq 0$. Write $h(x) = h_0(y) + h_1(y)x + h_2(y)x^2 + \cdots$, where $h_i(y) \in \Bbbk[y]$. Now since $pg(x) = h(x) \in \Bbbk[y][\![x]\!]$, clearing common powers of x, we may assume that $g_0(y) \neq 0$. Comparing the two sides of the last equation and arguing by induction, this yields $a_i(y) \in \Bbbk[y]\left[\dfrac{1}{g_0(y)}\right]$ for every $i \geq 0$. In particular, for all such i,

$$a_i(y) \in \Bbbk[y]\left[\frac{1}{g_0(y)}\right] \cap \Bbbk[\![y]\!] \subseteq \Bbbk[y]_{(y)},$$

as desired. Return now to the claims (1)–(3).

For (1): By [14, Proposition 6.19], $A = A'$ is a 2-dimensional regular local ring with the maximal ideal $(x, y)A$. Hence R is a 3-dimensional regular local ring with the maximal ideal $(x, y, z)R$. Per [14, Remark 4.16], $R/(z) \cong A$ is not an excellent ring, so by Remark 2.13, neither is R.

For (2): By [24, Example 7, pp. 209–211],

$$D = \frac{A[z]}{(z^2 - f)A[z]}$$

is a 2-dimensional normal local domain. In particular, the unique maximal ideal of D is $(x, y, z)D$, hence

$$R/(a) = \frac{A[z]_{(x,y,z)}}{(z^2 - f)A[z]_{(x,y,z)}} \cong D_{(x,y,z)D} \cong D$$

is a 2-dimensional normal domain. In particular, a is R-regular.

For (3): For each $n \geq 1$, write $f = (y_n + x^{n+1}u_n)^2$, where $y_n \in \Bbbk[x, y] \subseteq A, u_n \in \Bbbk[\![x, y]\!]$ are given by

$$y_n = y + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \quad u_n = \sum_{i=0}^{\infty} \frac{x^i}{(i + n + 1)!}.$$

We note that $f = y_n^2 + x^{n+1}u_n(2y_n + x^{n+1}u_n) = y_n^2 + x^{n+1}v_n$, where $v_n := u_n(2y_n + x^{n+1}u_n) \in \Bbbk[\![x, y]\!]$. Since

$$v_n = \frac{f - y_n^2}{x^{n+1}} \in \Bbbk(x, y, f) \cap \Bbbk[\![x, y]\!] = A,$$

the element $\delta_n = x^{n+1}v_n$ belongs to $\mathfrak{m}^{n+1}$. Thus

$$a + \delta_n = z^2 - f + x^{n+1}v_n = z^2 - y_n^2 = (z - y_n)(z + y_n)$$

is a product of two elements in $\mathfrak{m}$. In particular, $R/(a + \delta_n)$ is not a domain. Being a noetherian local ring, the last ring therefore cannot be normal. $\square$

Remark 3.6. Keep the notations as in Example 3.5. Per [14, Remark 4.16], $(A, (x, y)A)$ is a 2-dimension regular local domain with $\widehat{A} = \Bbbk[\![x, y]\!]$. Moreover, $A/(f)$ is an integral domain. We show that there exist arbitrarily small perturbations of $A/(f)$ that are not reduced.

As in Example 3.5, $v_n \in A$, so $\delta_n = x^{n+1}v_n \in (x, y)^{n+1}$. Now

$$f - \delta_n = y_n^2 \in (x, y)^2$$

so $A/(f - \delta_n)$ is not reduced. Thus reducedness is generally unstable under perturbations.

In view of Example 3.5 in which R is regular local but not excellent, the most optimistic question about perturbation of normality seems to be

Question 3.7. *Let $(R, \mathfrak{m})$ be an excellent local ring. Suppose $a \in \mathfrak{m}$ is a regular element such that $R/(a)$ is a normal ring. Does there exist an $N \geq 1$ such that for all $\delta \in \mathfrak{m}^N$, the ring $R/(a+\delta)$ is also normal?*

We answer Question 3.7 positively when R is an excellent complete intersection that contains a field of characteristic zero. More generally, we have

Theorem 3.8. *Let $(R, \mathfrak{m})$ be an excellent local complete intersection that contains a field of characteristic zero. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ is a normal domain. Then there exists $N \geq 1$ such that for every $\delta \in \mathfrak{m}^N$, $R/(a+\delta)$ is again a normal domain.*

Because of Serre's normality criterion (Lemma 2.3) and the fact that Cohen–Macaulay local rings always satisfy (S_ν) for every ν, Theorem 3.8 is an immediate consequence of the following stability result for the regularity condition (R_ν).

Theorem 3.9. *Let $(R, \mathfrak{m})$ be an excellent local complete intersection that contains a field $\mathbb{k}$ of characteristic zero. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ satisfies Serre's condition (R_ν) for some non-negative integer ν. Then there exists $N \geq 1$ such that for every $\delta \in \mathfrak{m}^N$, $R/(a+\delta)$ also satisfies (R_ν).*

There are two main ingredients in the proof. The first result, due to Greuel–Pham [12], generalizes Samuel's theorem [30] from hypersurfaces to complete intersections over a field. Before stating the result of Greuel–Pham in the generality that is suitable for our purpose, we have the following remark.

Remark 3.10 (On two notions of "isolated singularities"). Let $\mathbb{k}$ be a perfect field. Let I be a proper ideal of $S = \mathbb{k}[\![y_1, \ldots, y_s]\!]$ that is minimally generated by $f_1, \ldots, f_c$. Let h be the height of I. Let $I_h\left(\left[\dfrac{\partial f_i}{\partial x_j}\right]\right)$ be the ideal of $h \times h$ minors of the Jacobian matrix $\left[\dfrac{\partial f_i}{\partial x_j}\right]$ of I, and let $j(I) = I + I_h\left(\left[\dfrac{\partial f_i}{\partial x_j}\right]\right)$ be the *Jacobian ideal* of I. Assume further that S/I is *equidimensional*, namely for every minimal prime $\mathfrak{p} \in \operatorname{Min}(I)$, we have $\dim(S/\mathfrak{p}) = \dim(S/I)$.

(1) With the above assumptions, for any prime ideal $P \in \operatorname{Spec}(S)$ containing I, the following statements are equivalent:

(i) $(S/I)_P$ is a regular local ring;
(ii) P does not contain $j(I)$.

So the last statement can be rephrased as

$$\operatorname{Sing}(\operatorname{Spec}(S/I)) = V(j(I)/I).$$

For the equivalence of statements (i) and (ii), see [18, Corollary 3.14(2)(iii)] and [35, Section 4].

(2) In particular, consider the local ring $(R, \mathfrak{m}) := (S/I, (y_1, \ldots, y_s)/I)$. Then the following statements are equivalent:

(i) $(R, \mathfrak{m})$ is an isolated singularity, namely $\operatorname{Sing}(\operatorname{Spec}(R)) \subseteq \{\mathfrak{m}\}$;
(ii) $j(I)$ contains a power of $(y_1, \ldots, y_s)$.

Several authors use condition (ii) as the definition of *isolated singularity*; this is the case with the notion of *isolated complete intersection singularity* [12, p. 198, lines 13–16]. So for complete intersections that are quotients of formal power series rings over a perfect field, our notion of "isolated singularities" is compatible with that of [12].

(3) If $\Bbbk$ is not perfect, the two notions of isolated singularities are distinct in general. For example, for a prime p, let $\Bbbk = \mathbb{F}_p(t)$ and $I = (x_1^p - tx_2^p) \subseteq S = \Bbbk[x_1, x_2]$. Letting $R = S/I$, then $\operatorname{Sing}(\operatorname{Spec}(R)) = \{\mathfrak{m}\}$ but $j(I) = I$ is not (x_1, x_2)-primary.

Here is a version of Greuel–Pham's [12, Theorem 4.6(2)] that we will use; note that we require $\Bbbk$ to be a perfect field.

Theorem 3.11. *Let $\Bbbk$ be a perfect field. Let I be a proper ideal of $S = \Bbbk[\![y_1, \ldots, y_s]\!]$ that is minimally generated by $f_1, \ldots, f_c$. If either*

(i) $\dim S/I = 0$*, or*

(ii) $\dim(S/I) > 0$*, and S/I is an isolated complete intersection,*

then there exists $N \geq 1$ such that

$$S/I \cong S/(f_1 + \delta_1, \ldots, f_c + \delta_c)$$

for all elements $\delta_1, \ldots, \delta_c \in (y_1, \ldots, y_s)^N$.

Remark 3.12. Note that by the remark in Problem 4.7 in the same paper, the implication (iii) $\Longrightarrow$ (i) in [12, Theorem 4.6(2)], which is the statement of Theorem 3.11, is always valid even without $\Bbbk$ being infinite.

The second main ingredient in the proof of Theorem 3.9 is the following result on the relationship between a ring R and a relative hypersurface $R[t_1, \ldots, t_n]/(F)$, where $F \in R[t_1, \ldots, t_n]$. In such a situation, we denote by I_F the ideal of R generated the coefficients of F. We are mostly interested in the case $F = t_1x_1 + \cdots + t_nx_n$, where $x_1, \ldots, x_n \in R$ generate an $\mathfrak{m}$-primary ideal, in which $I_F = (x_1, \ldots, x_n)$. The study of such relative hypersurfaces has been taken up by Hochster [16]. The following statement is a Bertini-type theorem for the (R_s) property, proved by Trung [34, Page 222–223] and rediscovered in [21, Lemma 10].

Lemma 3.13. *Let $(R, \mathfrak{m})$ be a noetherian local ring satisfying (R_ν) such that* $\dim R \geq \nu + 2$. *If $\mathfrak{m} = (x_1, \ldots, x_n)$, then $\dfrac{R(t_1, \ldots, t_n)}{(t_1x_1 + \cdots + t_nx_n)}$ still satisfies (R_ν). Here $R(t_1, \ldots, t_n)$ denotes the localization of the ring $R[t_1, \ldots, t_n]$ at the ideal $\mathfrak{m}R[t_1, \ldots, t_n]$.*

We also record a criterion for a relative hypersurface section to be reduced and irreducible.

Lemma 3.14 ([34, Korollar 2.3])**.** *Let R be a noetherian domain. Let $n \geq 1$ an integer, $F \in R[t_1, \ldots, t_n]$ such that* $\deg F \geq 1$. *Denote by $Q(R)$ the field of fractions of R. Then the following statements are equivalent:*

(i) $(F) \subseteq R[t_1, \ldots, t_n]$ *is a prime ideal;*

(ii) $\operatorname{grade}(I_F, R) \geq 2$ *and F is irreducible as a polynomial in $Q(R)[t_1, \ldots, t_n]$.*

As a direct consequence of Lemma 3.14, we get

Lemma 3.15. *Let $(R, \mathfrak{m})$ be noetherian local domain, where $\mathfrak{m} \neq (0)$. Assume that the elements $x_1, \ldots, x_n \in R$ generate an $\mathfrak{m}$-primary ideal. Then $\dfrac{R[t_1, \ldots, t_n]}{(t_1x_1 + \cdots + t_nx_n)}$ is an integral domain if and only if* $\operatorname{depth} R \geq 2$.

Proof. Note that for $F = x_1t_1 + \cdots + x_nt_n$, $I_F = (x_1, \ldots, x_n)$ is $\mathfrak{m}$-primary so $\operatorname{grade}(I_F, R) = \operatorname{grade}(\mathfrak{m}, R) = \operatorname{depth} R$. Since $\mathfrak{m} \neq (0)$, $x_1, \ldots, x_n$ are not all zero, so $\deg F \geq 1$. The conclusion follows from Lemma 3.14 since as a linear form in $Q(R)[t_1, \ldots, t_n]$, F is always irreducible. □

We are ready to present the

Proof of Theorem 3.9. In the first step, we reduce to the case $\dim R \leq \nu + 2$. In fact, assume that $\dim R \geq \nu + 3$. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ and let $F := t_1x_1 + \cdots + t_nx_n$ for new variables $t_1, \ldots, t_n$. Since $R/(a)$ is Cohen–Macaulay of dimension $\dim R - 1 \geq \nu + 2$, (F) is a prime ideal of $(R/(a))(t_1, \ldots, t_n)$ per Lemma 3.15. In particular, a, F is a regular sequence on $R(t_1, \ldots, t_n)$. Since $R/(a)$ satisfies (R_ν), by the local Bertini theorem Lemma 3.13, the ring

$$(R(t_1, \ldots, t_n)/(F))\,/(a) \cong (R/(a)(t_1, \ldots, t_n))\,/(F)$$

satisfies (R_ν) as well. Note that $R(t_1, \ldots, t_n)/(F)$ is again a complete intersection containing $\Bbbk$, that is excellent thanks to Remark 2.13. If we have proved that the ring $\dfrac{R(t_1, \ldots, t_n)/(F)}{(a+\delta)}$ satisfies (R_ν), for all small δ, then being Cohen–Macaulay, the last ring satisfies both (R_ν) and $(S_{\nu+1})$. Then Proposition 2.4(c) implies that the conditions (R_ν) and $(S_{\nu+1})$ are also fulfilled by $(R/(a+\delta))\,(t_1, \ldots, t_n)$. Descending along the flat morphism $R/(a+\delta) \to (R/(a+\delta))\,(t_1, \ldots, t_n)$, we deduce that $R/(a+\delta)$ fulfills (R_ν). Therefore we may replace R by $R(t_1, \ldots, t_n)/(F)$ and reduce the dimension of R.

Assume that $\dim R \leq \nu + 2$ and $R/(a)$ satisfies (R_ν). Note that being a quotient of an excellent ring, $R/(a)$ is excellent by Remark 2.13. It follows from Lemma 2.14 that $\widehat{R}/(a)$ also satisfies (R_ν). Since $\widehat{R}$ contains a field, its coefficient ring is its residue field $K = \widehat{R}/\mathfrak{m}\widehat{R} \cong R/\mathfrak{m}$. By the Cohen structure theorem for local rings containing a field and the fact that $\widehat{R}$ is a complete intersection,

$$\widehat{R} \cong S/(f_1, \ldots, f_c),$$

where $S = K[\![y_1, \ldots, y_s]\!]$. We may assume that $f_1, \ldots, f_c$ form a regular sequence in S thanks to [4, Theorem 2.3.3]. By abuse of notations, we regard $a \in R \subseteq \widehat{R}$ as an element of S. Then the ring

$$\widehat{R}/(a) \cong K[\![y_1, \ldots, y_s]\!]/(a, f_1, \ldots, f_c)$$

satisfies condition (R_ν) and has dimension $\dim R - 1 \leq \nu + 1$. This implies that it has only isolated singularity. Containing $\Bbbk$ as a subfield, $K \cong R/\mathfrak{m}$ has characteristic zero, and hence is perfect, so applying Theorem 3.11 to the ideal $(a, f_1, \ldots, f_c)$, we get an $N \geq 1$ such that

$$K[\![y_1, \ldots, y_s]\!]/(a, f_1, \ldots, f_c) \cong K[\![y_1, \ldots, y_s]\!]/(a + \delta_0, f_1 + \delta_1, \ldots, f_c + \delta_c)$$

for all $\delta_0, \delta_1, \ldots, \delta_c \in (y_1, \ldots, y_s)^N$. We now take any $\delta \in \mathfrak{m}^N R$, which again can be seen as an element of S. Then

$$\widehat{R}/(a+\delta) \cong K[\![y_1, \ldots, y_s]\!]/(a+\delta, f_1, \ldots, f_c) \cong K[\![y_1, \ldots, y_s]\!]/(a, f_1, \ldots, f_c) \cong \widehat{R}/(a),$$

which yields the fulfillment of (R_ν) of $\widehat{R}/(a+\delta)$. Since $\mathfrak{m}$-adic completion is a flat extension, the ring $R/(a+\delta)$ itself fulfills (R_ν) for every $\delta \in \mathfrak{m}^N$. The proof is concluded. □

Remark 3.16. The hypothesis that R is excellent in Theorem 3.8 cannot be dropped, as we have seen in Example 3.5, where R is a regular local ring that is not excellent.

As a consequence of Lemma 2.3 and Theorem 3.9, we get that reducedness perturbs for excellent local complete intersections that contain a field of characteristic zero.

Theorem 3.17. *Let $(R, \mathfrak{m})$ be an excellent local complete intersection containing a field of characteristic zero. Let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ is reduced. Then there exists $N \geq 1$ such that for every $\delta \in \mathfrak{m}^N$, $R/(a+\delta)$ is also reduced.*

Proof. The ring $R/(a)$ is Cohen–Macaulay, so it always satisfies (S_ν) for every ν. Thus $R/(a)$ is reduced if and only if it satisfies (R_0). Everything left is to apply Theorem 3.9. □

Our next theorem claims that reducedness is stable under perturbations when R is factorial Nagata. An element x of a local ring $(R, \mathfrak{m})$ is *square-free* if it cannot be written as $x = y^2 z$ for elements $y, z \in R$ where $y \in \mathfrak{m}$.

Remark 3.18. Note that if $(R, \mathfrak{m})$ is a factorial local ring, then for $x \in \mathfrak{m}$, the quotient $R/(x)$ is reduced if and only if x is square-free.

Here comes our last main result, which is concerned with stability of reduced rings.

Theorem 3.19. *Reducedness is stable under perturbations for factorial Nagata local rings. More precisely, let $(R, \mathfrak{m})$ be a factorial Nagata local ring and let $a \in \mathfrak{m}$ be a regular element such that $R/(a)$ is reduced. Then there exists $N \geq 1$ such that $R/(a+\delta)$ is reduced for all $\delta \in \mathfrak{m}^N$.*

Proof. By Remark 3.18, we only need to show that if a is square-free, then there exists $N \geq 1$ such that $a + \delta$ is square-free for all $\delta \in \mathfrak{m}^N$.

We first assume that R is a complete local ring. Let us denote by

$$f(y) = f(y_1, y_2) = a - y_1 y_2^2$$

a polynomial in $R[y_1, y_2]$. By Theorem 2.18, there exists a function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ such that:

For all $c \in \mathbb{N}$, for all $\bar{y} \in R^2$ such that $f(\bar{y}) \in \mathfrak{m}^N, N = \beta(c)$, there exists $\widetilde{y} \in R^2$ such that $f(\widetilde{y}) = 0$ and $\widetilde{y} - \bar{y} \in (\mathfrak{m}^c)^{\oplus 2}$.

We will show that for $N := \beta(1)$, and all $\delta \in \mathfrak{m}^N$, the element $a + \delta$ is square-free. Indeed, assume by contradiction that $a + \delta$ is not square-free for some $\delta \in \mathfrak{m}^N$. Then there exist $\bar{y}_1$ and $\bar{y}_2$ in R such that $a + \delta = \bar{y}_1 \bar{y}_2^2$. This implies that $f(\bar{y}_1, \bar{y}_2) \in \mathfrak{m}^N$. Then there exist $\widetilde{y} \in R^2$ such that $f(\widetilde{y}) = 0$ and $\widetilde{y} - \bar{y} \in \mathfrak{m}^{\oplus 2}$. It follows that $a = \widetilde{y}_1 \widetilde{y}_2^2$ is not square-free, which is a contradiction. This gives the desired conclusion as soon as R is a complete local ring.

Now, assume that R is a Nagata local ring and $a \in R$ is square-free. We consider a as an element in $\widehat{R}$ and define the number N as above. We shall show that for all $\delta \in \mathfrak{m}^N$, $a + \delta$ is square-free. Assume by contradiction that $a + \delta$ is not square-free, then a is not square-free in $\widehat{R}$ by the above argument. This together with Remark 3.18 implies that the ring $\widehat{R}/(a)$ is not reduced. But so $R/(a)$ is not reduced (see, ([22, Theorem 70] or [33, Lemma 15.43.6, Tag 07NZ]), a contradiction. This concludes the proof. □

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