

Constructing Orthogonal Minimally Aliased Response Surface Designs Using Circulant Weighing Matrices

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Abstract

In chemical and engineering sciences, most factors are quantitative, benefiting from investigation at three distinct levels when studying quadratic or non-linear relationships. Consequently, definitive screening designs (DSDs), a class of 3-level foldover designs introduced by Jones and Nachtsheim (2011), have gained prominence over traditional 2-level designs such as fractional factorial and Plackett–Burman designs.

This paper extends the framework of conference matrix-based DSDs by introducing a new class of circulant weighing matrix-based screening designs, offering greater design flexibility. Like DSDs and the OMARS designs recently proposed by Núñez Ares and Goos (2020), these new designs preserve orthogonality among main effects and between main effects and second-order effects (i.e., quadratic effects and two-factor interactions), while ensuring the absence of full aliasing among second-order effects. We refer to these designs as OMARS designs to highlight their ability to support both factor screening and response surface exploration in a single step.

We present a comprehensive catalog of 149 new OMARS designs with desirable projection capability, accommodating up to 50 factors.

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1 Introduction

Screening designs are employed in the early stages of experimentation to identify the most important factors influencing a response. These designs are cost-effective, requiring substantially fewer runs than full factorial designs. Standard screening designs include: (i) 2-level fractional factorial designs (FFDs), (ii) Plackett–Burman designs (PBDs), and (iii) definitive screening designs (DSDs). The first two are well-suited for qualitative 2-level factors, while DSDs, introduced by Jones and Nachtsheim (2011), are more appropriate for quantitative factors. For a detailed discussion on FFDs and PBDs, refer to Mee (2009) and Plackett and Burman (1946).

A DSD is a class of 3-level screening designs whose design matrix $\mathbf{D}$ is structured as follows:

$$\mathbf{D} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \\ -\mathbf{C} \end{pmatrix} \quad (1)$$

where $\mathbf{C}$, the half-design fraction (HDF), is a *conference* matrix of order m , and $\mathbf{0}$ is a row vector of 0's (cf. Stylianou 2010, 2011; Xiao, Lin, and Bai 2012; Nguyen and Stylianou 2013). Beyond pairwise orthogonal main effects (MEs), DSDs offer several key advantages:

- (i) The design is mean orthogonal, i.e., each factor has the same number of ± 1 's.
- (ii) When m is even, the design is saturated for estimating the intercept, m MEs, and m quadratic effects (QEs).
- (iii) The MEs are orthogonal to the second-order effects (SOEs), i.e., QEs and the 2-factor interaction effects (IEs).

(iv) The QEs can be estimated and are not fully aliased with the IEs.

(v) The IEs are not fully aliased with one another.

Note that FFDs of resolutions III and IV do not possess properties (ii)–(v). Similarly, PBDs do not possess properties (ii)–(iv).

A conference matrix $\mathbf{C}$ in (1) is a special case of a weighing matrix of order m and weight $m - 1$. This relationship motivated Georgiou, Stylianou, and Aggarwal (2013) to develop a new class of 3-level weighing matrix-based screening designs by generalizing from conference matrices to weighing matrices of the same order but with weight $w < m - 1$. A weighing matrix $\mathbf{W}$ of order m and weight $w = m - s$ ($s > 1$) is an $m \times m$ square matrix with entries in $\{0, \pm 1\}$, where all rows and columns are pairwise orthogonal. Each row and column contains exactly s zeros. Formally, this orthogonality condition is expressed as $\mathbf{W}\mathbf{W}' = (m - s)\mathbf{I}_m$, where $\mathbf{W}'$ denotes the transpose of $\mathbf{W}$, and $\mathbf{I}_m$ is the identity matrix. In this paper, we refer to our new class of designs as OMARS designs to emphasize their ability to support both factor screening and response surface exploration in a single step (cf. Núñez Ares and Goos, 2020; Goos, 2025). We use OMARS- s to denote an OMARS design constructed from a weighing matrix of weight $m - s$, where each row and column contains exactly s zeros. Thus, a conference matrix-based DSD corresponds to an OMARS-1 design.

0+--0-0
00+--0-
-00+--0
0-00+--
-0-00+-
--0-00+
+--0-00
0000000
0-++0+0
00-++0+
+00-++0
0+00-++
+0+00-+
++0+00-
--+0+00

Table 1: An OMARS-3 design constructed from a weighing matrix of order seven and weight four.

Table 1 shows an OMARS-3 design constructed from a weighing matrix of order seven and weight four (the first seven rows) where + stands for 1 and - stands for -1.

The primary limitation of conference matrix-based DSDs is their restriction to three mid-levels (0's) for each factor, allowing only one zero for each HDF. While DSDs are effective at identifying key factors efficiently, this constraint limits their ability to detect curvature effects when there are many factors. In contrast, our newly developed OMARS designs preserve the strengths of conference matrix-based DSDs while providing greater flexibility, as they are not bound by the structural limitations of conference matrices.

This paper aims to identify suitable weighing matrices for constructing new OMARS designs. Section 2 introduces circulant matrices and circulant weighing matrices derived from circulant cores. Section 3 reviews the statistical model for an OMARS design and projection capabilities as design selection criteria. Section 4 outlines an algorithm for constructing new OMARS designs. Section 5 discusses newly identified OMARS designs and provides two examples using the algorithm from Section 4. Section 6 concludes the paper.

2 Circulant matrices and circulant weighing matrices

A circulant matrix $\mathbf{C} = (c_{ij})$ is a square matrix of order m generated by a vector $\mathbf{c} = (c_0, c_1, \dots, c_{m-1})$, with the following structure:

$$\mathbf{C} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{m-1} \\ c_{m-1} & c_0 & \cdots & c_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{pmatrix}. \quad (2)$$

Circulant matrices possess four fundamental properties: they form a closed set under addition and multiplication; their products commute; their transposes remain circulant; and the product of a circulant matrix with its transpose yields a symmetric circulant matrix.

Let the matrix $\mathbf{A} = (a_{ij}) = \mathbf{C}'\mathbf{C} (= \mathbf{C}\mathbf{C}')$. The generating vector of the symmetric circulant matrix $\mathbf{A}$ is $\mathbf{a} = (a_0, a_1, \dots, a_{m-1})$, with each element a_k :

$$a_k = \sum_{j=0}^{m-1} c_j c_{(j+k) \bmod m}. \quad (3)$$

The weighing matrix used for the OMARS design in Table 1 is an example of a circulant matrix $\mathbf{C}$, generated by the vector $\mathbf{c} = (0+--0-0)$. Readers can verify that $\mathbf{A} = \mathbf{C}'\mathbf{C}$ is the circulant matrix generated by the vector $\mathbf{a} = (4, 0, 0, 0, 0, 0, 0)$. A weighing matrix of the same order and weight but not of the circulant type can be found on p. 27 of Núñez Ares and Goos (2020).

A circulant weighing matrix (CWM) is characterized by its circulant structure and can be constructed using one, two, or four circulant cores (matrices), with each core generated by one cyclic generator. The CWM used to build the OMARS design in Table 1 is based on a single circulant core. The following paragraphs outline the construction of CWMs of order m and weight $m - s$ using two and four cores. For a more comprehensive discussion, see Kotsireas, Koukouvinos, and Seberry (2012) and Koukouvinos and Seberry (2002).

If $\mathbf{C}_1$ and $\mathbf{C}_2$ are two circulant $(0, \pm 1)$ matrices of order $l = \frac{m}{2}$ such that $\mathbf{C}_1'\mathbf{C}_1 + \mathbf{C}_2'\mathbf{C}_2 = (m - s)\mathbf{I}_l$, then the following 2-core $m \times m$ CWM can be constructed:

$$\begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_2' & -\mathbf{C}_1' \end{pmatrix}. \quad (4)$$

Similarly, if $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, and $\mathbf{C}_4$ are four circulant $(0, \pm 1)$ matrices of order $l = \frac{m}{4}$ such that $\mathbf{C}_1'\mathbf{C}_1 + \mathbf{C}_2'\mathbf{C}_2 + \mathbf{C}_3'\mathbf{C}_3 + \mathbf{C}_4'\mathbf{C}_4 = (m - s)\mathbf{I}_l$, then a 4-core $m \times m$ CWM can be constructed as follows:

$$\begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2\mathbf{R} & \mathbf{C}_3\mathbf{R} & \mathbf{C}_4\mathbf{R} \\ -\mathbf{C}_2\mathbf{R} & \mathbf{C}_1 & \mathbf{C}'_4\mathbf{R} & -\mathbf{C}'_3\mathbf{R} \\ -\mathbf{C}_3\mathbf{R} & -\mathbf{C}'_4\mathbf{R} & \mathbf{C}_1 & \mathbf{C}'_2\mathbf{R} \\ -\mathbf{C}_4\mathbf{R} & \mathbf{C}'_3\mathbf{R} & -\mathbf{C}'_2\mathbf{R} & \mathbf{C}_1 \end{pmatrix}. \quad (5)$$

Here, $\mathbf{R}$ denotes the anti-diagonal identity matrix of order l , which features 1's along the anti-diagonal (running from the top-right to the bottom-left corner) and 0's elsewhere.

Table 2 shows (a) a conference matrix of order 20 from Nguyen and Stylianou (2013), (b) a 2-core CWM of order 20 and weight 13, and (c) a 4-core CWM of order 20 and weight 17.

Figure 1 consists of three panels, (a), (b), and (c), each showing a 1D Ising chain of 20 sites over 20 time steps. The sites are represented by '+' and '-' signs, and the time steps are represented by rows. Panel (a) shows a single domain wall moving from left to right. Panel (b) shows two domain walls moving in opposite directions. Panel (c) shows a more complex evolution with multiple domain walls and a central region of disorder.

Table 2: A conference matrix of order 20 and two CWMs of order 20 with weights 13 and 17 respectively.

Let $\mathbf{C}_i$ ($i = 1, \dots, r$) be a circulant matrix associated with the generating vector $\mathbf{c}_i = (c_{i0}, c_{i1}, \dots, c_{i(l-1)})$, where r denotes the number of cores (1, 2, or 4). The generating vector corresponding to the circulant matrix $\mathbf{A}_i = \mathbf{C}_i' \mathbf{C}_i$ is $\mathbf{a}_i = (a_{i0}, a_{i1}, \dots, a_{i(l-1)})$. A CWM with parameters (m, s) constructed from the core matrices $\mathbf{C}_i$ using Eq. (4) or (5) will satisfy $\sum_{i=1}^r a_{i0} = m - s$, representing the sum of the squared ± 1 values in the vectors $\mathbf{c}_i$, and

$$\sum_{i=1}^r a_{ik} = 0, \quad (6)$$

for $k = 1, \dots, l - 1$. Here, the a_{ik} 's are computed using Eq. (3) for each i and k . Eq. (6) is employed to verify whether a matrix constructed from a single core, two cores as in Eq. (4), or four cores as in Eq. (5) qualifies as a CWM.

The 4-core OMARS-3 in Table 2(c), for example, was constructed from the following generating vectors: $\mathbf{c}_1 = (+--0+)$, $\mathbf{c}_2 = (-0-+-)$, $\mathbf{c}_3 = (+----)$, $\mathbf{c}_4 = (+--0-)$. The associated vectors $\mathbf{a}_i$ are: $\mathbf{a}_1 = (4, 1, -3, -3, 1)$, $\mathbf{a}_2 = (4, -1, 1, 1, -1)$, $\mathbf{a}_3 = (5, 1, 1, 1, 1)$, $\mathbf{a}_4 = (4, -1, 1, 1, -1)$. Thus, $\sum_{i=1}^4 a_{i0} = 17$ and $\sum_{i=1}^4 a_{ik} = 0$, for $k = 1, \dots, 4$.

3 The Statistical Model and Projection Capabilities as Design Selection Criteria

Consider a CWM-based OMARS design involving m 3-level factors and $n = 2m + 1$ runs (assuming only one center run). The design matrix $\mathbf{D}_{n \times m} = (d_{ij})$ is constructed using Eq. (1), with a weighing matrix $\mathbf{W}_{m \times m} = (w_{ij})$ of weight $w = m - s$ replacing the conference matrix $\mathbf{C}$, and includes a zero-row vector $\mathbf{0}_{1 \times m}$ to represent the center run. In this paper, s is restricted to the range $1, 2, \dots, \frac{m}{2}$ as designs with many zeros have low power for detecting MEs and IEs. The statistical model for this OMARS design is expressed as

$$\mathbf{Y}_n = \mathbf{X}\boldsymbol{\beta}_p + \boldsymbol{\epsilon}_n, \quad (7)$$

where $\mathbf{Y}_n$ is the response vector; $\mathbf{X}_{n \times p}$ is the expanded design matrix derived by augmenting $\mathbf{D}$ to include the terms required for the statistical model; p is the number of parameters; $\boldsymbol{\beta}_p$ is the vector of fixed-effect coefficients; and $\boldsymbol{\epsilon}_n$ is the error vector with mean zero and variance-covariance matrix $\sigma^2 \mathbf{I}_n$.

The design’s D -efficiency, which measures how well the design estimates the model parameters (with higher values indicating better performance), is calculated as:

$$D = |\mathbf{X}'\mathbf{X}|^{1/p} / n. \quad (8)$$

We use D_{ME} , $D_{\text{ME+QE}}$, $D_{\text{ME+IE}}$, and D_{SOE} to denote the D -efficiency when the model includes the intercept plus: MEs, MEs + QEs, MEs + IEs, or MEs + QEs + IEs, respectively. Recall that MEs, QEs, and IEs refer to main effects, quadratic effects, and two-factor interaction effects, respectively.

In experimental design, a design’s *projection capabilities* indicate its performance when analyzing subsets of k factors out of the total m factors. This is crucial for screening experiments, where typically only a few factors are significant. Loeppky, Sitter, and Tang (2007) introduced two metrics to evaluate these capabilities: projection estimation capacity (PEC) and projection information capacity (PIC).

(i) For a design $\mathbf{D}_{n \times m}$, let $\mathcal{P}_k(\mathbf{D}_{n \times m})$ be the number of estimable SOE models involving k factors out of the $\binom{m}{k}$ possible models. Then $\text{PEC}_k^{\text{SOE}}$ is computed as:

$$\text{PEC}_k^{\text{SOE}} = \frac{\mathcal{P}_k(\mathbf{D}_{n \times m})}{\binom{m}{k}}. \quad (9)$$

(ii) $\text{PIC}_k^{\text{SOE}}$ measures the average D -efficiency of these $\binom{m}{k}$ projections for the SOE models, quantifying the precision of parameter estimation from the projected subsets.

Our methodology prioritizes design selection using the $\text{PEC}_k^{\text{SOE}}$ and $\text{PIC}_k^{\text{SOE}}$ criteria. We first identify candidate designs with $\text{PEC}_k^{\text{SOE}}$ values close to 1, and then, among these, maximize $\text{PIC}_k^{\text{SOE}}$. These metrics are particularly powerful because they align with the *principle of hierarchy or effect heredity* in statistical modeling. This principle recognizes that higher-order interactions are typically meaningful only when their parent MEs are significant, implying that not all potential IEs demand equal scrutiny. By focusing on the design’s projection properties, we can systematically evaluate how effectively the design

performs across different factor subsets.

We set $k = \min(\max(\text{round}(m/5), 3), 8)$, where k is one-fifth of m (rounded to the nearest integer), constrained between 3 and 8. For example, our OMARS design with 34 factors and 69 runs ensures that any SOE model involving seven factors ($\approx 34/5$) or fewer is estimable. Experimenters can select different k values by choosing appropriate run sizes. Section 5 provides two examples: a 15-factor OMARS design involving 37–45 runs, and a 6-factor OMARS design involving 33 runs, both with $\text{PEC}_5^{\text{SOE}} = 1$.

4 The COMARS Algorithm

The following steps describe the construction of an r -core CWM of order $m = rl$ and weight $w = m - s$, where $r \in \{1, 2, 4\}$ and $s = 1, 2, \dots, \frac{m}{2}$ by the **COMARS** algorithm (CWM-based OMARS design generation algorithm). This CWM can then be used to construct an OMARS- s design for m factors with $n = 2m + 1$ runs. In the resulting design matrix, each column contains w entries equal to 1, w entries equal to -1 , and $(2s + 1)$ entries equal to 0.

1. **Initialization:** Create an initial matrix $\mathbf{c}$ of size $r \times l$. Set the first s positions to 0 and fill the remaining positions randomly with ± 1 . Randomize the entire matrix. Calculate the initial sum metric S :

$$S = \sum_{k=1}^{l-1} \left(\sum_{i=1}^r a_{ik} \right)^2, \quad (10)$$

where the a_{ik} values are calculated according to Eq. (3). When S reaches zero, the matrix constructed by Eq. (2), (4) or (5) for $r \in \{1, 2, 4\}$ qualifies as a CWM.

2. **2-Exchange Optimization:** Iteratively search for pairs of elements in $\mathbf{c}$ that can be swapped to reduce S . When an improving swap is found, execute it and update S . Continue until either no improving swaps exist or $S = 0$.

3. **Coordinate Optimization:** If $S > 0$, search for individual non-zero elements

that can be flipped (from $+1$ to -1 or vice versa). When an improving flip is found, execute it and update S . Continue until either no improving flips exist or $S = 0$.

(a)	(b)	(c)	(d)
+--++	---++	---0+	+--0+
-0---	-0-+-	-0-+-	-0-+-
0-----	0-----	+-----	+-----
+--0-	+--0-	+--0-	+--0-

Table 3: Three iterations of **COMARS** constructing a 4-core CWM with

$$(m, s) = (20, 3).$$

For illustration, consider constructing an OMARS-3 design, as shown in Table 2(c). Table 3(a) shows the initial 4×5 matrix containing 17 ± 1 entries and three 0s. The counts of 1s and -1 s need not be equal. Initially, $\mathbf{a} = (17, 6, 4, 4, 6)$ with $S = 104$. Table 3(b) shows the first iteration: swapping elements at positions (1,1) and (2,4) yields $\mathbf{a} = (17, 2, 2, 2, 2)$ with $S = 16$. In Table 3(c), the second iteration swaps elements at positions (1,4) and (3,1), yielding $\mathbf{a} = (17, 0, 2, 2, 0)$ with $S = 8$. The final iteration, shown in Table 3(d), flips the sign of the element at position (1,1), resulting in $\mathbf{a} = (17, 0, 0, 0, 0)$ with $S = 0$. This confirms that the vectors in this matrix can construct a 4-core CWM with $(m, s) = (20, 3)$.

Remarks

1. Step 2 implements the *interchange* algorithm from Nguyen (1996). Step 3 uses the *coordinate-exchange* algorithm, proposed by Kounias and Chadjiapantelis (1983) and further developed by Meyer and Nachtsheim (1995).

2. The three steps of the **COMARS** algorithm constitute a single *try*. For each try that yields a valid CWM, we sequentially apply three selection criteria. The first condition is that $\max(r_{\text{QQ}}, r_{\text{QI}}, r_{\text{II}}) < 1$, where $r_{\text{QQ}}, r_{\text{QI}}, r_{\text{II}}$ denote the maximum absolute correlations among QEs, between a QE and an IE, and among IEs, respectively, helps prevent perfect aliasing. Second, $D_{\text{ME+QE}} > 0$, where $D_{\text{ME+QE}}$ is the D-efficiency of the ME+QE model, ensures that the model is estimable. Third, the constraint $v_{\text{QE}} \leq c$,

where v_{QE} denotes the maximum variance among estimated QEs, is introduced to avoid ill-conditioned designs that yield unstable QE estimates. The cutoff $c = 1$ is chosen empirically for normalized design matrices $(-1, 0, +1)$ coding).

Only designs that satisfy all three conditions are subjected to the computation of PEC_k^{SOE} and PIC_k^{SOE} , where $k = \text{round}(m/5)$, bounded between 3 and 8 or user-defined. The design that sequentially maximizes PEC_k^{SOE} and PIC_k^{SOE} is selected.

3. For large m and k , calculating PEC_k^{SOE} and PIC_k^{SOE} for all possible $\binom{m}{k}$ SOE models is computationally infeasible. Therefore, we sample $\frac{Z^2 pq/E^2}{1+(Z^2 pq/E^2)/\binom{m}{k}}$ projections onto k factors, where $Z = 1.96$ (95% confidence level), $p = q = 0.5$ (maximum variability), $E = 0.01$ (1% margin of error), and $\binom{m}{k}$ is the population size. This sampling approach provides results with 95% confidence level and 1% margin of error, allowing us to draw statistically valid conclusions about the properties of all possible k -factor projections while keeping the computational effort manageable. For instance, when $m = 42$ and $k = 8$, where $\binom{42}{8} = 118,030,185$, we only need to examine 9,603 samples instead of the entire population.

4. For designs involving fewer factors $m' < m$, the **COMARS** algorithm randomly selects m' columns from the full design of m factors before computing the quality measures described in Remark 2 and selecting the final design. This random selection approach is most likely suboptimal (cf. Vázquez et al., 2019).

5 Results and Discussion

5.1 New OMARS Designs

The Appendix presents quality measures and generating vectors for the HDF of the CWM-based OMARS designs using one, two, and four generators (cores). Each OMARS design is characterized by (r, m, s) , where r is the number of cores, m is the number of factors, and s is the number of zeros per column and per row in the CWM. The number

of runs is $n = 2m + 1$ (including one center run), except for two OMARS designs marked with ‡: $(r, m, s) = (1, 7, 3)$ and $(1, 31, 15)$, which have two center runs. These two designs, along with $(r, m, s) = (1, 13, 4)$, achieve $r_{\text{QQ}} = 0$, satisfying the orthogonality of QEs.

Quality measures include D_{ME} and $D_{\text{ME+QE}}$ (D-values for the ME and ME + QE models), r_{QQ} , r_{QI} , r_{II} , and $\text{PEC}_k^{\text{SOE}}$ and $\text{PIC}_k^{\text{SOE}}$. Here, $k = \text{round}(m/5)$, bounded between 3 and 8: $k = 3$ for $m \leq 17$; $k = 4$ for $18 \leq m \leq 22$; $k = 5$ for $23 \leq m \leq 27$; $k = 6$ for $28 \leq m \leq 32$; $k = 7$ for $33 \leq m \leq 37$; and $k = 8$ for $m \geq 38$.

The Appendix presents 149 CWM-based OMARS designs: seven single-core, 52 two-core, and 90 four-core designs. OMARS designs with $s > 1$ typically show higher $D_{\text{ME+QE}}$ compared to those with the same m and $s = 1$, though they often exhibit higher correlation values (r_{QQ} , r_{QI} , r_{II}). However, increasing s toward $\frac{m}{2}$ does not necessarily guarantee improved $D_{\text{ME+QE}}$ and projection capability, as illustrated, for example, by OMARS-2 designs with $(m, s) = (26, 1), (26, 8), \dots, (26, 13)$, and by OMARS-4 designs with $(m, s) = (36, 1), (36, 3), \dots, (36, 18)$.

5.2 Example 1: OMARS Designs for a large number of factors

Ahuja, Ferreira, and Moreira (2004) employed a 20-run Plackett–Burman design (PBD) to screen components affecting growth. They then applied response surface methodology to optimize a growth medium for the shipworm bacterium, *Teredinobacter turnirae*, specifically targeting its aggregated morphology. Their study examined 18 medium components (factors), including NaCl, KCl, $\text{MgSO}_4 \cdot 7\text{H}_2\text{O}$, and NH_4Cl (detailed in their Table 1). The experimental design, comprising both the PBD and its folded variant, is presented in their Table 2 (see also Mee 2009, Section 11.1). Of the 18 factors studied, 15 were tested at two levels (with the high level set at four times the low level), while three factors maintained constant levels throughout the experiment.

As an alternative to the 15 columns (columns 2–16) of the conference matrix of order 20 in Table 2(a), Table 4 presents HDFs of four additional candidate OMARS designs for

15 factors, all achieving $\text{PEC}_5^{\text{SOE}} = 1$. These designs were generated by the **COMARS** algorithm (see Remark 4, Section 4). OMARS-1 and OMARS-5 were derived from 2-core CWMs of order 18 and 22, respectively, while OMARS-6 and OMARS-7 are derived from 4-core CWMs of order 20.

OMARS-1	OMARS-6	OMARS-7	OMARS-5
+++++---++-+++	-00+0-00++--+	++0+0+-+---0+0	--00-0+-0+0+---
++-+++++0+++-	-+00+00+-+---0	00++-0-00+++++	0---0-0-0+0+0-0
-----++-+++++	+0+0+0+0-0---+	-00+-+--000+-0-	-0-00-+-+---0-
0-++++++-+++++	0+0+-+0+00+-0	++-0-0+-+---+0-	-0+-0-0+0+0+-0
---++-+++++0-+-	+0+00+-+0+0+-0	-+00-00-0+-00+	-0-000-+-+0+0+
++-0++-+-+---+	+0++00---+0-+-0	0-+-0+0-+---+0	0000-+-+---++0
++-+++++0+-+	+00+0-0+-0+-0-	-+-+0-000-000-+	0-00-+-0+---++
++-+-+---++0+-	--00-00+-00+++	--0---00-0-0+0	00-+-0++-0-+-+
---+0-++-+++++	++-000-0+0-++0	-+-+000-0++-0-	-+-0-+-+---00+-0
+0-+++++---+	+++--00+00-+0	000+-+0-00---0	0-0+0-+-+00---
-0-+++++---+	0+-+---+0-+-+0	0-+-+0+-+0+00+	+00-+-+---+0+-+
+0-+++++---+	0-+-+0-0---+0	+++00-0+0+0+++	-+-+---0+0+++
+-+++++---++0+	-+0-+---+0+-0-0	0+-+0+0+00+00+0	+++++0-0+0++0-
+++++0-++-+-	0+0+-+---+00+-	0+++0+0-+-+00+	++-+-+0+-+0-0-
++++-+-+---++0	-0+-+---0-+-+0	-00+0+0+00+0+0+	--+-+---+0+0-
+++++0+-+---+	0+-+---+0-+00+	0+0-+-+0-+0+0	+---+-+0+-+00+
+++++0-+-+---+	-+++0-+-+0+0+	+0-00-+-+0+-+0	-+-+---0-+0+0+
+++++0-+-+---+	-+++0+-+0-00+	+0-+-+0+0-000-	++-+-+0+-+0+0-
+++++0-+-+---+	+---+0-+-+0+00+	+--00-+-+0+0-	-+-+---0+0+0-
+++++0-+-+---+	0+-+---+0+00+	-000+0+-+000+-	+++++00-0-+-+
+++++0-+-+---+	0+-+---+0+00+		++-+-+0+-+0+0-
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Table 4: Four candidate screening designs as represented by their HDFs.

Table 5 presents the quality measures for these designs. In this table, $\text{PIC}_5^{\text{SOE}}$ represents the average of the D-efficiencies of the $\binom{15}{5} = 3003$ SOE models on five factors. Figure 1 displays the dot plots of these D-efficiencies. The DSD and OMARS-1 exhibit higher $\text{PIC}_5^{\text{SOE}}$ and D_{ME} values and smaller r_{QI} values, but significantly larger v_{QE} values compared to the other OMARS designs.

The average maximum v_{QE} and v_{IE} across the $\binom{15}{5}$ SOE models on five factors are (1.081, 1.243, 0.711, 0.655, 0.371) and (0.068, 0.084, 0.385, 0.572, 0.142), respectively. Users prioritizing QEs should consider OMARS-5, while those focusing on IEs may prefer the DSD or OMARS-1.

Table 5: Quality measures of 15-factor candidate DSD and OMARS designs with $\text{PEC}_5^{\text{SOE}} = 1$

Design	n	D_{ME}	$D_{\text{ME+QE}}$	v_{ME}	v_{QE}	r_{QQ}	r_{QI}	r_{II}	$\text{PIC}_5^{\text{SOE}}$
DSD	41	0.931	0.238	0.026	0.472	0.281	0.200	0.444	0.397
OMARS-1	37	0.924	0.249	0.029	0.472	0.275	0.213	0.375	0.392
OMARS-6	41	0.699	0.317	0.036	0.327	0.549	0.671	0.700	0.332
OMARS-7	41	0.652	0.329	0.038	0.242	0.472	0.648	0.750	0.305
OMARS-5	45	0.769	0.355	0.029	0.155	0.278	0.425	0.500	0.390

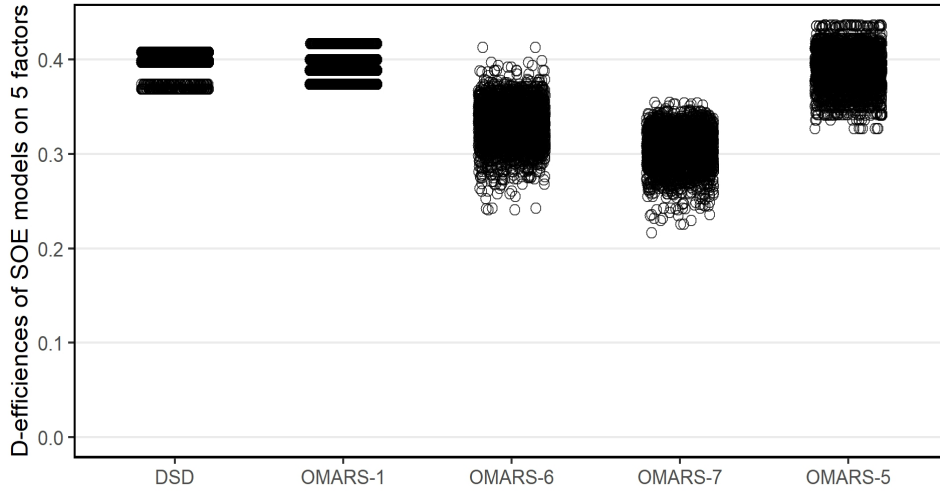


Figure 1: Dot plots showing D-efficiencies of 3003 SOE models on five factors from the five designs in Table 5.

5.3 Example 2: OMARS Designs for Six Factors

Núñez Ares and Goos (2020) introduced the term OMARS designs, along with a construction method and an extensive catalog of designs for 3–7 factors. In addition to the key properties shared with DSDs and those proposed in this paper, these designs also satisfy a specific *sparsity property*, which ensures that the number of zeros n_0^{ME} is identical across all ME columns, and the number of zeros n_0^{IE} is identical across all IE columns. Our OMARS designs with $r = 1$ (i.e., a single circulant core) or with $s = 1$ always satisfy this sparsity property, whereas those with $r > 1$ and $s > 1$ may not.

Table 6 presents the HDFs of five screening designs for six factors in 33 runs (including

one center run) with $PEC_5^{\text{SOE}} = 1$: the DSD contains six columns of a conference matrix of order 16 in Nguyen and Stylianou (2013); the EFFEX design, provided by a reviewer, was produced by the EFFEX software (<https://www.effex.app/>); bd.6.32.4.8.4 was obtained from <https://bitbucket.org/josenunezares/omars/>, OMARS-2a was constructed by the **COMARS** algorithm with the constraint $D_{\text{ME+IE}} > 0$; and OMARS-2b was also constructed by the **COMARS** algorithm, but with the constraint $D_{\text{ME+QE}} > 0$.

DSD	EFFEX	bd.6.32.4.8.4	OMARS-2a	OMARS-2b
-----	-----	----0-	-0+--	++0++
++++--	----++	----0+	-+0--	-0++-
+++++-	---++	--0++	++--	++--
++--++	---++	---++	++--0	++--0
0+---	--+-0	---++	-0---	---0++
---++	--+-0	---++	+-0++	++--
---++	-0+--	-0+++	---++	0++++
++++--	-0+++	-0+++	---+0	++--0-
+++++0	++--0	++--0	++--0-	++--
---++	++--0	++--0	++--+	---0-
++--++	++--0	++--0	0-+++	---+0+
+-0--	++--0	++--0	++--0+	0-++-
+++0--	++--0	++--0	++--0+	++--
---+0-	++--0	++--0	++--0+	++--
---++	++--0	++--0	++--0+	++--
-0+++	0-+++	0-+++	0++++	++--
++--	0-+++	0-+++	++--0-	++--

Table 6: Five candidate designs as represented by their HDFs.

Table 7 shows the quality measures of the designs in Table 6. In this table, PIC_5^{SOE} represents the average of the D-efficiencies of the $\binom{6}{5} = 6$ models on five factors. Figure 2 displays the dot plots of these D-efficiencies. Similar to the previous example, the DSD displays higher PIC_5^{SOE} , D_{ME} , and $D_{\text{ME+IE}}$ values as well as lower r_{QI} values. Among the last four designs, the EFFEX design and OMARS-2b outperform the other two with respect to PIC_5^{SOE} values. Although the PIC_5^{SOE} value of OMARS-2b surpassed that of the EFFEX design (0.390 vs. 0.384), users might opt for the latter as it can estimate both the ME + IE and ME + QE models.

The average maximum v_{QE} and v_{IE} across $\binom{6}{5}$ SOE models involving five of the six factors are (2.006, 2.212, 3.19, 3.215, 2.045) and (0.098, 0.489, 0.526, 0.405, 0.232), respectively. Users prioritizing QEs should consider the DSD, the EFFEX design, and the OMARS-2b design, while those focusing on IEs may prefer the DSD.

Table 7: Quality measures of 6-factor candidate designs involving 33 runs with $\text{PEC}_5^{\text{SOE}} = 1$

Design [†]	$(n_0^{\text{ME}}, n_0^{\text{IE}})$	D_{ME}	$D_{\text{ME+IE}}$	v_{ME}	v_{IE}	r_{QQ}	r_{QI}	r_{II}	$\text{PIC}_5^{\text{SOE}}$
DSD	(2, 4)	0.922	0.771	0.033	0.057	0.267	0.229	0.286	0.392
EFFEX	(4, 8)	0.869	0.699	0.036	0.078	0.057	0.396	0.333	0.384
bd.6.32.4.8	(4, 8)	0.869	0.581	0.036	0.238	0.057	0.396	0.333	0.365
OMARS-2a	(4, 8)	0.869	0.611	0.036	0.442	0.057	0.396	0.333	0.374
OMARS-2b	(4, 8)	0.869	–	0.036	–	0.057	0.396	0.333	0.390

[†] Except for OMARS-2b, all designs were constructed under the constraint $D_{\text{ME+IE}} > 0$. The $(D_{\text{ME+QE}}, v_{\text{QE}})$ values for the DSD and the remaining designs are (0.287, 0.443) and (0.358, 0.239), respectively.

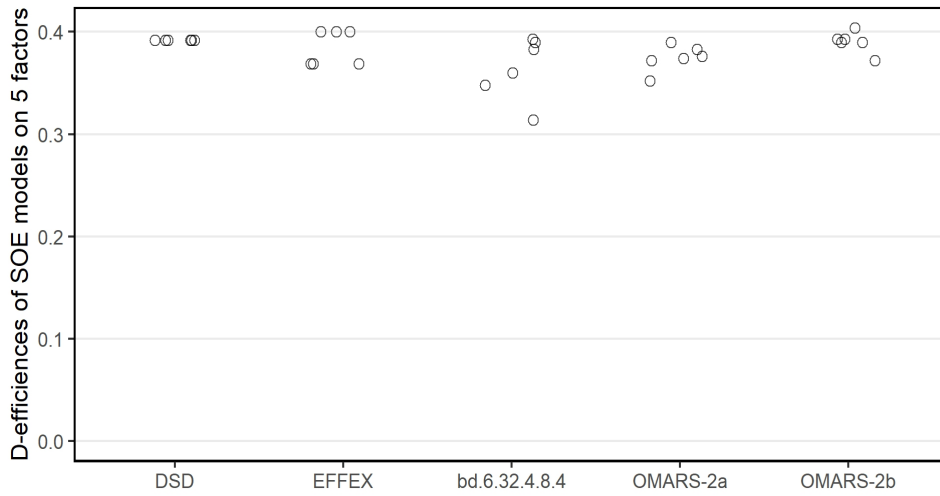


Figure 2: Dot plots showing D-efficiencies of six SOE models involving five factors for the five designs in Table 6.

6 Conclusions

This paper introduces a new class of CWM-based OMARS designs that complement the existing OMARS designs and the conference matrix-based DSDs in the literature. The **COMARS** algorithm outlined in Section 4 was used to construct 149 new OMARS designs with desirable projection capability, as detailed in the Appendix. Notably, this includes designs for some odd numbers of factors (see the first seven DSDs with $r = 1$ in the Appendix) and multiple solutions for 22 and 34 factors in this Appendix. Along

with the existing conference matrix-based DSDs and the OMARS designs in the EFFEX software, these new designs for a larger number of factors offer researchers increased flexibility in their design choices.

The **COMARS** algorithm, detailed in Section 4, showcases notable computational efficiency. Constructing the 149 OMARS designs listed in the Appendix took over a day, with the bulk of the time spent computing the $(\text{PEC}_k^{\text{SOE}}, \text{PIC}_k^{\text{SOE}})$ pairs rather than the designs themselves. On an iMac with an M1 chip, the **COMARS** algorithm generates 1,000 single-core OMARS designs with $(m, s) = (13, 4)$ in under half a second, producing 72 designs with $\text{PEC}_3^{\text{SOE}} = 1$. In contrast, constructing 10,000 2-core OMARS designs with $(m, s) = (34, 14)$ on the same machine takes less than 10 seconds, yet only 12 of these achieve $\text{PEC}_7^{\text{SOE}} = 1$. This highlights the growing difficulty of identifying OMARS designs with desirable projection capability as the dimensions increase.

Designs like DSDs, OMARS designs, and the 2-level orthogonally minimally aliased designs (Nguyen, Pham, and Vuong 2023) are all examples of minimally aliased designs, as advocated by Jones and Nachtsheim (2011), Núñez Ares and Goos (2020), and Goos (2025). They preserve orthogonality among MEs and between MEs and SOEs, while eliminating full aliasing among SOEs.

Data from OMARS designs with complex aliasing patterns can be analyzed using procedures by Hamada and Wu (1992), Vazquez, Schoen, and Goos (2021), and Hameed, Núñez Ares, and Goos (2023). Analyses typically rely on effect sparsity (assuming few factors affect the response) through a two-stage approach: screening for main and quadratic effects, then fitting SOE models with significant terms. The standard technique combines forward stepwise regression with effect heredity, though all-subsets regression and regularization methods (LASSO, Ridge, Elastic Net) are also used. These leverage OMARS designs' advantages: unbiased main effects, projection capability, minimal runs, quadratic effect estimation, and near-orthogonality, with implementation in the Design-Expert, Effex, JMP, Minitab, and R packages.

OMARS designs and DSDs with a large number of factors and runs may require blocking, a topic recently addressed by Núñez Ares and Goos (2023).

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Data Availability

The supplementary materials for this paper, available for download at <https://tinyurl.com/2sns4dp2>, include: (i) the 149 HDFs generated by the vectors in Appendix; (ii) the pseudo-code for the **COMARS** algorithm; and (iii) the **COMARS** program, which implements the **COMARS** algorithm discussed in this paper, along with the associated Java class files and (iv) the **COMARS** user notes.

Disclosure of interest

We have no conflicts of interest to disclose.

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Appendix: New OMARS designs with one, two and four circulant core†

r	m	s	D_{ME}	D_{ME+QE}	r_{QQ}	r_{QI}	r_{II}	$PIC_k^{SOE}§$	Generating vector(s)
1	7	3	0.577	0.386	0.071	0.518	0.5	0.307 (3)	0-0---+0
1	7‡	3	0.545	0.379	0.0	0.5	0.5	0.297 (3)	0-0---+0
1	13	4	0.686	0.399	0.0	0.471	0.5	0.42 (3)	++++0--+00+0
1	21	5	0.754	0.384	0.023	0.428	0.583	0.432 (4)	+---00--+0+-----+0-0
1	31	6	0.799	0.365	0.031	0.492	0.65	0.407 (6)	++-00++0+0+----+0-+++++0+-----
1	31	15	0.519	0.361	0.016	0.252	0.25	0.23*(6)	0---+0-+00-++0000--0+0+00-000-
1	31‡	15	0.511	0.359	0.0	0.25	0.25	0.23*(6)	0---+0-+00-++0000--0+0+00-000-
2	10	1	0.869	0.323	0.222	0.312	0.75	0.445 (3)	-+---;+-0-
2	14	1	0.903	0.277	0.256	0.249	0.5	0.43 (3)	----+++;+---0--
2	14	4	0.707	0.352	0.356	0.463	0.875	0.416 (3)	-----+;00++0+0
2	18	1	0.923	0.245	0.275	0.213	0.375	0.42 (4)	-----+;+-+0+---
2	18	5	0.716	0.292	0.741	0.54	0.8	0.382 (4)	+0+-----0;0-+++++00
2	22	5	0.765	0.347	0.278	0.425	0.5	0.429 (4)	-000-0+-0-;+-----+-----
2	22	6	0.722	0.278	0.298	0.588	0.6	0.412 (4)	+---+0+-----;0+000+-+0--
2	22	9	0.592	0.262	0.275	0.604	0.577	0.339 (4)	+++0+-0+-0;0-000--0-0+
2	24	7	0.704	0.31	0.424	0.506	0.8	0.368 (5)	0-0+0+-----;0+-----+00-++0
2	24	11	0.544	0.256	0.508	0.573	0.75	0.257 (5)	+++0000-0+0;0-0+--0-00+-
2	26	1	0.945	0.204	0.293	0.172	0.417	0.407 (5)	-----+-----;+---0-+-----
2	26	8	0.689	0.247	0.386	0.481	0.772	0.361 (5)	+000+0++00-;00+-+-----+--
2	26	9	0.652	0.272	0.477	0.512	0.6	0.342 (5)	0--0-+-0+0-0-;0-+-----0+-00
2	26	10	0.615	0.298	0.369	0.502	0.671	0.321 (5)	+0+0+-----0+-;0+---0000+00--
2	26	13	0.504	0.264	0.359	0.476	0.667	0.24 (5)	0-0+-----+0+000;00-00--0+00+-
2	28	10	0.642	0.321	0.206	0.56	0.8	0.294 (6)	+++++---000+---;0-+00-0+0-00-
2	28	11	0.607	0.285	0.312	0.483	0.671	0.27 (6)	0+---000+0-00+;-0-+-0+00++++
2	30	5	0.825	0.268	0.778	0.502	0.6	0.395 (6)	0+---0-+-----+;0+-----+0-+0+-
2	30	10	0.665	0.273	0.419	0.509	0.617	0.318 (6)	++0+-0+-----+0-;--0-+-0000+-00-
2	30	12	0.6	0.305	0.356	0.466	0.6	0.275 (6)	+00-0+0-+0+000;---+00++0+0+0+
2	30	13	0.568	0.319	0.336	0.461	0.6	0.255 (6)	-0+0-00000+0+---;+-+00-+0+00+++
2	30	14	0.536	0.261	0.474	0.513	0.866	0.23 (6)	-0-0+0-0+0+---;000-000-+0---00
2	32	7	0.775	0.269	0.48	0.491	0.667	0.39 (6)	-----+000++++;0+0+-----+0-+0-
2	32	14	0.564	0.291	0.253	0.446	0.75	0.257 (6)	++000000+00+-0-;0-0-0-0-+0+---
2	32	15	0.533	0.276	0.383	0.497	0.8	0.238 (6)	000-+00+---+00--;0000-+---0+0--0-0
2	34	9	0.731	0.236	0.307	0.476	0.667	0.319 (7)	-----+---+00+0;-----+000-+-00--0
2	34	14	0.589	0.302	0.428	0.498	0.6	0.232 (7)	+---000+-0+-----0;0-0+-000---0-000+
2	34	16	0.532	0.258	0.394	0.492	0.6	0.195 (7)	+---00++0--0-0-0;000000+0+000+-----
2	34	17	0.503	0.275	0.42	0.482	0.667	0.177 (7)	00+0+-0+-0+0-0000;---00++0-+00-00
2	36	7	0.799	0.236	0.832	0.437	0.667	0.359*(7)	--0+---0-0+0+---0-0;+++++-----+0-
2	36	10	0.719	0.269	0.337	0.517	0.6	0.326 (7)	0+0+00-0+-----+0-0;0+-----+0+---+--
2	36	11	0.692	0.284	0.365	0.504	0.589	0.309 (7)	-000-+++0--0+-0+---;00-0+-----+0+0
2	36	16	0.557	0.268	0.336	0.526	0.676	0.222 (7)	-+-0000+00++0+0+000;-0-0+-0+-----+0+0-0
2	38	9	0.759	0.237	0.581	0.558	0.727	0.277*(8)	--0++000+-0+-----+;+-----000+-----0
2	38	12	0.682	0.26	0.422	0.552	0.625	0.238*(8)	0-0-0-+0000+-----0;-----+00+-00--
2	38	13	0.657	0.224	0.483	0.542	0.63	0.223*(8)	++0+0+++0-+-0+---;0-+---00-+00-0--0+00
2	38	18	0.528	0.232	0.457	0.559	0.667	0.154*(8)	-+00+0+-00+---0+0+-;0+0+0+00+000-00-0++

†The run size n of each design is $2m + 1$ (one center run), except for those marked with ‡, which have two center runs.

§ The number in parentheses is k for which $PEC_k^{SOE} = 1$; if $0.99 < PEC_k^{SOE} < 1$, PIC_k^{SOE} is followed by an asterisk (*).

When $m \geq 28$, a sample from $\binom{m}{k}$ projections on to k factors is selected to compute $(PEC_k^{SOE}, PIC_k^{SOE})$ (see Section 4).

Appendix: New OMARS designs with one, two and four circulant core†

r	m	s	D_{ME}	D_{ME+QE}	r_{QQ}	r_{QI}	r_{II}	$PIC_k^{SOE}§$	Generating vector(s)
2	40	3	0.916	0.203	0.687	0.288	0.588	0.351*(8)	+++++--+0+-+-----+;+++++--+0+-----0
2	40	11	0.722	0.241	0.393	0.498	0.8	0.279 (8)	--00-0++0+-+0-00+-0;--+0+-----+0-----0
2	40	14	0.649	0.239	0.504	0.541	0.668	0.234 (8)	++-+-----0--0+-0;0+-00-0+0-00--000-0
2	40	15	0.625	0.283	0.359	0.432	0.571	0.225 (8)	0+00++0+-0-+++00000-;0+-----0+-0-+++0-0
2	40	20	0.502	0.285	0.21	0.444	0.5	0.155 (8)	-0000-0+00000-++0-0-;0-+-0-00--0+-000-
2	42	13	0.688	0.274	0.457	0.466	0.632	0.275 (8)	+-0+---00+-0++++0-;0-0+-----00+00-0++0
2	42	16	0.619	0.227	0.406	0.472	0.625	0.233 (8)	++++0-++00+-+0--00+;00-0+0---0+0+0000++
2	42	17	0.596	0.257	0.263	0.468	0.571	0.223 (8)	00----0+0+-0-000-0-0;+00+0-0+-00+-+---0
2	44	19	0.569	0.258	0.369	0.484	0.667	0.214 (8)	-00-0-+00+-----00000-0;--+0+0+++000-0-0-++0
2	46	5	0.884	0.202	0.794	0.378	0.667	0.391 (8)	+++-----+-----+0+-+0-+;0+-+-----0-+-----+0-+++
2	46	20	0.566	0.254	0.302	0.517	0.571	0.22 (8)	0+-+---0-00--0-000-+++;0-+0-00+0+0+00-0+0+0-
2	46	21	0.545	0.24	0.298	0.425	0.6	0.207 (8)	+00-+-0--00-----00++00-;+0-0000+0+0+0-0-0+++00
2	48	23	0.522	0.23	0.422	0.415	0.667	0.196 (8)	--00+0+-00+0-000-+-0-+;0-00+-----000+000-++000-0
2	50	21	0.581	0.235	0.433	0.402	0.544	0.242 (8)	0+0+-+0+0----0+++++0+---0;00-+0+000-0000-++00-000+
2	50	25	0.502	0.202	0.445	0.451	0.617	0.189 (8)	0-0-0-0+---00+0+-0-0+0-0;00+-000+00+0000-++0+---0
4	12	1	0.889	0.298	0.242	0.275	0.4	0.438 (3)	++-;---;--+;+0-
4	16	3	0.799	0.294	0.637	0.571	0.8	0.445 (3)	++++;-0-0;+---;+--0
4	16	5	0.683	0.306	0.364	0.64	0.667	0.415 (3)	+0-0;+-0;0---;+-0-
4	20	3	0.837	0.272	0.655	0.471	0.735	0.434*(4)	++0+;-0-++;++++-;+++0
4	20	5	0.743	0.304	0.752	0.719	0.8	0.405*(4)	0-00-;++++-;+++++;+-00-
4	20	9	0.553	0.278	0.766	0.47	0.667	0.291 (4)	+-00-;-0+0-;00000;-+---
4	24	3	0.862	0.253	0.667	0.408	0.889	0.419*(5)	-0-+0-;+---+;-+---0;+-----
4	24	7	0.704	0.31	0.424	0.506	0.8	0.368 (5)	++000+;++0-0;+++++;0+0-+-
4	24	10	0.584	0.279	0.333	0.577	0.667	0.288 (5)	+0000-;00-0+-;-----;0+0+0+
4	28	3	0.881	0.238	0.674	0.365	0.727	0.397 (6)	-----0;+-----;+-+0+-;--+0-++
4	28	6	0.779	0.267	0.203	0.446	0.75	0.368 (6)	000+0++;-+++++;-0-+---;+---0+-
4	28	7	0.745	0.274	0.819	0.568	0.869	0.346 (6)	+-0+++;+00-0++;+00-+0;+-----
4	28	9	0.676	0.248	0.526	0.628	0.694	0.307 (6)	++---+-;0-0---0;0+00-+0;+---00
4	28	10	0.642	0.321	0.206	0.614	0.667	0.294 (6)	+-+---+;0---000;-+00+0;0+-0+0+
4	28	11	0.607	0.238	0.604	0.604	0.75	0.262 (6)	-00+0-;000+00;-+-----;00+0-+-
4	28	12	0.573	0.237	0.43	0.462	0.8	0.228 (6)	--0-+;0+-+00-;0000000;-0-+---
4	28	13	0.538	0.247	0.578	0.531	0.75	0.216 (6)	+++0-0;000-00-;+0-0---;00+0+-0
4	28	14	0.503	0.25	0.438	0.53	0.875	0.189*(6)	+-+---+;00+0+0;0-0-000;0+0000-
4	32	3	0.895	0.224	0.68	0.333	0.769	0.42 (6)	+-0---0-;+++++---;+-----;+-----0
4	32	5	0.835	0.251	0.343	0.499	0.636	0.413 (6)	+0-0--0;+0+-----;+-----0;-+-----+
4	32	7	0.775	0.277	0.48	0.491	0.667	0.39 (6)	0+-+---;000-+0++;-+-----;0-+---0-
4	32	9	0.715	0.304	0.339	0.579	0.625	0.36 (6)	--0-+0;0-+---00;+-0-+---;0-+0+0+
4	32	10	0.685	0.297	0.407	0.563	0.75	0.341 (6)	0-+---000;0+++00+-;-+-----;0+++0-0+
4	32	11	0.655	0.273	0.327	0.635	0.667	0.32 (6)	---0000-;+-----+;00+0-0-;+-0-0-0+
4	32	13	0.594	0.294	0.394	0.563	0.559	0.279 (6)	0+-+0+++;+0-0+000;+-+000-;0+0-0--
4	32	14	0.564	0.277	0.253	0.509	0.625	0.257 (6)	0-00-0-;+0-+---;0+0+0-0-;0-+00-00
4	32	15	0.533	0.269	0.383	0.555	0.671	0.235 (6)	+0-+---0;00+00++;00+0-0-;00--0+00
4	36	1	0.96	0.173	0.305	0.143	0.706	0.369*(7)	--+---0+;-+-----+;-+-----;+---+---+
4	36	3	0.907	0.213	0.684	0.308	0.667	0.394 (7)	-----+;-0-+-----;+0-+-+---;+---+---0-

†The run size n of each design is $2m + 1$ (one center run), except for those marked with ‡, which have two center runs.

§ The number in parentheses is k for which $PEC_k^{SOE} = 1$; if $0.99 < PEC_k^{SOE} < 1$, PIC_k^{SOE} is followed by an asterisk (*).

When $m \geq 28$, a sample from $\binom{m}{k}$ projections on to k factors is selected to compute $(PEC_k^{SOE}, PIC_k^{SOE})$ (see Section 4).

Appendix: New OMARS designs with one, two and four circulant core†

r	m	s	D_{ME}	D_{ME+QE}	r_{QQ}	r_{QI}	r_{II}	$PIC_k^{SOE}§$	Generating vector(s)
4	36	5	0.853	0.232	0.786	0.454	0.692	0.383 (7)	++++-----;---0--0++;+++0--0+;+++0-+-
4	36	7	0.799	0.25	0.497	0.611	0.727	0.365 (7)	--0-+-+--;00-+0-00+;-----0;+++-----
4	36	9	0.746	0.274	0.288	0.59	0.7	0.341 (7)	++00+++0-;000-++++0;--++++-;0-+++0+-
4	36	10	0.719	0.322	0.198	0.603	0.667	0.328 (7)	0-+0+0+0+;-000+---+;---++++-;++0++0+-0
4	36	11	0.692	0.284	0.365	0.588	0.75	0.31 (7)	+++0-+-+;+-00-+-00;+0-+0+0;0---0-0+
4	36	13	0.638	0.323	0.293	0.6	0.714	0.277 (7)	--++++-;00-000+-;+0+0-0-0;---0000-+
4	36	14	0.611	0.295	0.373	0.586	0.667	0.257 (7)	+-----00;-0000+0-0;0++0++0+-;-+000+-
4	36	15	0.584	0.265	0.233	0.58	0.667	0.241 (7)	+0+-0--0;00----+-0;+000-000;0+-+0-0+
4	36	16	0.557	0.281	0.336	0.576	0.671	0.22 (7)	+00-++0-;0-00+0000;-0+0+++0;---+-000
4	36	17	0.53	0.259	0.427	0.524	0.667	0.202 (7)	0+00+00+0;+-----+0;+00+--00;0-0000-0-
4	36	18	0.503	0.24	0.535	0.523	0.667	0.184 (7)	00-+00-+0;0+-+0+-;-00000---;-00-000+0
4	40	3	0.916	0.203	0.687	0.288	0.941	0.351*(8)	--0----0+-;-+-----+;+-----+0-+;-----+--+
4	40	5	0.867	0.226	0.79	0.419	0.733	0.344 (8)	-0+++0-+-;-++0---+-;+-----0---;0+-----+
4	40	7	0.819	0.237	0.509	0.555	0.769	0.329 (8)	-00--00-+-;+0-+---+0;++++0---+-;-+-----
4	40	9	0.77	0.242	0.862	0.553	0.667	0.304 (8)	--+0-----;+0-----00-;-+0-+-+0-;00++++0--
4	40	10	0.746	0.323	0.286	0.612	0.727	0.298 (8)	0+0+0+0+-;+-----000;-----+---+;0-+0-0---
4	40	11	0.722	0.275	0.393	0.623	0.8	0.282 (8)	+-----+---+;-+0000-+-;-+0-+00-0;0-+-+0-0+
4	40	13	0.673	0.257	0.444	0.667	0.75	0.253 (8)	+000-0+0+;+-+00+0+-;+-----00+;+0+0-+0--
4	40	14	0.649	0.308	0.355	0.656	0.778	0.24 (8)	-00+0-0-00;+0+-+0+0+;+---000---;0+-0-----
4	40	15	0.625	0.28	0.477	0.566	0.75	0.225 (8)	+++00+++;-+0-0+0+0;0+0-0+0+0;+00+---+00
4	40	17	0.576	0.272	0.509	0.549	0.73	0.196 (8)	+00+-0+00+;+000000-+;+00-+0+0-0;++++0+-+-
4	40	18	0.551	0.29	0.393	0.546	0.6	0.182 (8)	+0+-0-++0+;00-0-00000;-+0+0-0+-;-0+-000++
4	40	19	0.527	0.266	0.484	0.545	0.8	0.168 (8)	+0-++++0;0+0+0-0+00;000++00+-+;0000-0+0+
4	40	20	0.502	0.232	0.407	0.497	0.8	0.152 (8)	++-+0+-0+;+0+0+000-0;+00-0000--;00+0-00-0+
4	44	1	0.967	0.156	0.31	0.128	0.667	0.359 (8)	-+-----+---;+++++---+---;-+-----+---;-+-----+0+
4	44	3	0.923	0.195	0.69	0.271	0.632	0.384 (8)	++0+++-----;+-----+---+;-0-+-+-----;+-----+0--
4	44	5	0.879	0.27	0.378	0.391	0.706	0.383 (8)	++++-----+;0-0--0---00;-+-----+---;+-----+---
4	44	6	0.857	0.221	0.279	0.45	0.728	0.376 (8)	0+-+0-+000;-----0+-+---;-+-----+---;+-----+---
4	44	7	0.835	0.225	0.84	0.512	0.645	0.365 (8)	+-----+0+++;-0+0-+++++;-000+---+---;-+0+-+-----
4	44	8	0.813	0.241	0.418	0.577	0.643	0.359 (8)	-+---0+-----;+0+0+---+000;+---0-+---+;0+-----+---
4	44	9	0.791	0.248	0.331	0.502	0.667	0.347 (8)	-+-----+---+;+0+---0+0-;0+0+0+0-0+;-+0-0-----
4	44	10	0.769	0.278	0.252	0.577	0.667	0.34 (8)	++-----+---+;000+---+0-0;0+-0-++0+-+;-+00+---+0-
4	44	11	0.747	0.277	0.29	0.604	0.696	0.326 (8)	0-+0+-+0---;-+-----+0+00;00+-+000+-+;+-----+---
4	44	13	0.702	0.282	0.256	0.583	0.696	0.299 (8)	0+---00+-+0;-+00-00+---+;-0+-+-----+;+0+0-0+0++
4	44	14	0.68	0.272	0.386	0.572	0.667	0.285 (8)	0+-0+0+0+;+00+0+000-;+-----+---+;0+000+++0-
4	44	15	0.658	0.268	0.485	0.563	0.667	0.271 (8)	00+0--0+0-;0+0---0-+-+;00+-----+0;0+00+0+-+
4	44	16	0.636	0.252	0.422	0.621	0.707	0.256 (8)	--0-0+0-00;0+---0000+-;0+-----+---+;0+00+0-0-0
4	44	17	0.614	0.278	0.413	0.579	0.65	0.243 (8)	0-+0+00-00+;+0000+++0+-;-+0-0+0---;+00+0+-+0
4	44	18	0.591	0.268	0.445	0.569	0.802	0.228 (8)	+0-0-+-+0-;+++000000+;+00-0-0+0--;-00-+00+++0
4	44	19	0.569	0.27	0.369	0.565	0.714	0.214 (8)	+-----+0+0;+0+0+0000-;0-00+-00-00;0+-0-00-0+-
4	44	20	0.547	0.254	0.457	0.521	0.772	0.198 (8)	-0+-+00+00+;00000000+0+;0+-----+00;-0-+-+00+++
4	44	21	0.524	0.264	0.37	0.52	0.676	0.186 (8)	+0000-----;0+0+0+0+---+;+0-00+0+0-;00-00000-+0
4	44	22	0.502	0.24	0.438	0.519	0.667	0.173 (8)	00+0--0+0+;000+000000-;-+0-0000+-;-+0+0+-+00
4	48	3	0.929	0.187	0.692	0.257	0.619	0.405 (8)	++-----+---+;+++0-----0--;-+-----+---+;-+-----+---+0

†The run size n of each design is $2m + 1$ (one center run), except for those marked with ‡, which have two center runs.

§ The number in parentheses is k for which $PEC_k^{SOE} = 1$; if $0.99 < PEC_k^{SOE} < 1$, PIC_k^{SOE} is followed by an asterisk (*).

When $m \geq 28$, a sample from $\binom{m}{k}$ projections on to k factors is selected to compute $(PEC_k^{SOE}, PIC_k^{SOE})$ (see Section 4).

Appendix: New OMARS designs with one, two and four circulant core†

r	m	s	D_{ME}	$D_{\text{ME+QE}}$	r_{QQ}	r_{QI}	r_{II}	$\text{PIC}_k^{\text{SOE}}\S$	Generating vector(s)
4	48	5	0.889	0.214	0.385	0.367	0.632	0.406 (8)	+0-+-+-----+;+++0-+0--0+;++++++-----;+++++---+0-
4	48	7	0.848	0.249	0.527	0.477	0.706	0.396 (8)	----0-+---0-;+-+---+-----+;0+---+---0-++;0-0-0-+-----
4	48	9	0.808	0.277	0.345	0.462	0.667	0.381 (8)	-----+---+;+0-+---0--0;--000-+-----;+--0+++0-0-
4	48	10	0.787	0.269	0.271	0.527	0.8	0.368 (8)	+++0+---+00;-----+-----;+---0--0+0-;00+---+0-+0-
4	48	11	0.767	0.278	0.254	0.596	0.714	0.358 (8)	-----+---+;+0+++00+0-;+0+++0-+0--;0+00-0++++-
4	48	12	0.747	0.248	0.353	0.579	0.815	0.343 (8)	+--0+--0+0+0;-----+-----;+0+00+0-+--;+00-++0-+0
4	48	13	0.726	0.285	0.334	0.523	0.667	0.333 (8)	--0--+0+---+;+0--0+0++++;0-0+0+0-0+;++000-++-++
4	48	14	0.706	0.267	0.213	0.602	0.667	0.321 (8)	+00-0+-----+0;00+0-0-+0+0;+-----+---+;++0--00-+00
4	48	15	0.686	0.313	0.336	0.525	0.727	0.309 (8)	--000-+-000;00000-+++++;0-0-0-+---0+;-----+-----
4	48	17	0.645	0.275	0.52	0.598	0.667	0.281 (8)	0++++-----+;0++-++-000+0;+00-0-+00+0;+0+0-+0++-00
4	48	18	0.625	0.273	0.476	0.591	0.674	0.266 (8)	0---+00-0-+;0-++000-0-00;--0-+00+---+;0-+++0+0+0-
4	48	19	0.604	0.289	0.4	0.589	0.75	0.254 (8)	++00--00+-00;--+0+-----0;0000+0-++0+;+++000+-+0-0
4	48	20	0.584	0.25	0.352	0.544	0.75	0.238 (8)	0-+---+00+0;+000-000-0-0;-----+0000-+;00--0-+0+---
4	48	21	0.563	0.294	0.253	0.506	0.617	0.228 (8)	+000-00+00-;0-0+0-0+0-0+;-00000-+++++;++0-0-+0+0+
4	48	22	0.543	0.262	0.492	0.504	0.667	0.212 (8)	--00-00--0+;+00-++00-0-;+0-0+0000+00;-000-0-+-----
4	48	23	0.522	0.243	0.422	0.503	0.8	0.199 (8)	-+0--00--0+;+0+---0-000;0+0-00++++00;+0-00-0000-0
4	48	24	0.502	0.271	0.34	0.537	0.8	0.184 (8)	0--0-+-00+00;000+00-00000;+---+0+00+0-;0-0---0+0+--

†The run size n of each design is $2m + 1$ (one center run), except for those marked with ‡, which have two center runs.

§ The number in parentheses is k for which $\text{PEC}_k^{\text{SOE}} = 1$; if $0.99 < \text{PEC}_k^{\text{SOE}} < 1$, $\text{PIC}_k^{\text{SOE}}$ is followed by an asterisk (*).

When $m \geq 28$, a sample from $\binom{m}{k}$ projections on to k factors is selected to compute $(\text{PEC}_k^{\text{SOE}}, \text{PIC}_k^{\text{SOE}})$ (see Section 4).