Some projective distance inequalities for subvarieties of complex projective spaces and their applications

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Abstract

We establish a type of Lojasiewicz inequality for the Fubini-Study distance in the projective space $P^n(\mathbb{C})$ and give its applications to Nevanlinna theory.
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1 Introduction

The Lojasiewicz inequality [15] gives an upper bound for the Euclidean distance of a point to the nearest zero of a given real analytic function. Let $f : U \to \mathbb{R}$ be a real analytic function on an open set U in $\mathbb{R}^n$. If the zero locus Z of f is not empty, then for any compact set K in U, there exist positive constants α and C such that, for all $x \in K$

$$\mathrm{dist}(x, Z)^\alpha \leqslant C|f(x)|.$$

Each complex analytic set $Z \subset \mathbb{C}^n$ defining by complex analytic functions $f_1, \ldots, f_k$ can be viewed as a real analytic set in $\mathbb{R}^{2n}$ defined by $f := |f_1|^2 + \cdots + |f_k|^2$. Thus the Lojasiewicz inequality applies to complex analytic sets too.

If the defining equations f_i are polynomials one would like to estimate the exponent α in terms of degrees of the polynomials. The problem of determining the sharpness of the Łojasiewicz exponent is a hard problem and has an answer only in the case of two variables [5, 9, 13].

If the set K is not compact, the classical Lojasiewicz inequality does not necessarily hold. There are various versions of the classical Łojasiewicz inequality on non-compact domains. For example, Kurdyka and Spodzieja [14] showed that for each polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d, there exists a positive constant c such that

$$c\left(\frac{\operatorname{dist}(x, Z)}{1+\|x\|^2}\right)^{d(6d-3)^{n-1}} \leqslant |f(x)| \text{ for all } x \in \mathbb{R}^n.$$

The cases of algebraic sets in $\mathbb{R}^n$ and in $\mathbb{C}^n$ have been intensively studied, for example, in [1, 6, 8, 14] for the real case, and in [2, 3, 4, 11, 17] for the complex case. It seems that the difference in method and result between these cases mainly comes from the fact that a polynomial has more complex zeros than real ones.

In this paper we shall examine the case of algebraic sets in the complex projective space $P^n(\mathbb{C})$ with the Fubini-Study distance. As a corollary, we also obtain that the complement of the $\epsilon-$neighborhood of a hypersurface in $P^n(\mathbb{C})$ does not contain any non-constant entire curves. It is worth noticing that, the connection between small and big Picard theorems, Montel's theorem and hyperbolicity has been seen. In the dimensional one case, for three distinct points a, b, c in $P^1(\mathbb{C})$, we have that every holomorphic map $f : \mathbb{C} \to P^1(\mathbb{C}) \setminus \{a, b, c\}$ is constant (Small Picard Theorem); every holomorphic map g from a punctured disk $\triangle \setminus \{0\}$ into $P^1(\mathbb{C}) \setminus \{a, b, c\}$ can be extended to a holomorphic map from $\triangle$ into $P^1(\mathbb{C})$ (Big Picard Theorem); and the space of holomorphic maps $Hol(\triangle, P^1(\mathbb{C}) \setminus \{a, b, c\})$ is relatively compact in $Hol(\triangle, P^1(\mathbb{C}))$ (Montel's Theorem). In the general case, the pair $(P^1(\mathbb{C}), P^1(\mathbb{C}) \setminus \{a, b, c\})$ is replaced by (X, Y), where Y is a submanifold of a complex manifold X. The relation between the above theorems and Kobayashi hyperbolicity is as follows: the big Picard theorem is true if X is hyperbolic and $Y = X$ or Y is relatively compact and hyperbolically embedded in X; the true of Montel's theorem for the pair (X, Y) is equivalent to that Y is hyperbolically embedded in X. In the case $Y = X$ is a compact manifold, then all three theorems are equivalent to the hyperbolicity of X.

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2 Preliminaries

We define an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$ by declaring that two non-zero vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{C}^{n+1}$ are equivalent if there exists a non-zero complex scalar c such that $\overrightarrow{u} = c\overrightarrow{v}$. The set of all such equivalence classes is denoted by $P^n(\mathbb{C})$ and is called the complex projective space of dimension n. The class containing a non-zeo vector $(\omega_0, \ldots, \omega_n) \in \mathbb{C}^{n+1}$ is called a point in $P^n(\mathbb{C})$ and denoted by $(\omega_0 : \cdots : \omega_n)$.

The Fubini-Study metric is given in homogeneous coordinates $\omega = (\omega_0 : \cdots : \omega_n)$ by

$$ds^2 = \frac{\langle d\omega, d\omega\rangle\langle \omega, \omega\rangle - |\langle \omega, d\omega\rangle|^2}{\langle \omega, \omega\rangle^2}$$

where $\langle \cdot, \cdot \rangle$ stands for the standard Hermitian product in $\mathbb{C}^{n+1}$.

If $\overrightarrow{u}$ and $\overrightarrow{v}$ are two unit vectors in $\mathbb{C}^{n+1}$ representing points U, V in $P^n(\mathbb{C})$ then the Fubini-Study distance $d_{FS}(U, V)$ between the two these points is

$$d_{FS}(U, V) = \|\overrightarrow{u} \wedge \overrightarrow{v}\|.$$

For each positive constant ϵ and each non-empty subset $S \subset P^n(\mathbb{C})$, the $\epsilon-$neighborhood of S, denoted by S_ϵ, is the set of points whose Fubini-Study distance to S is less than ϵ.

Let f be a holomorphic mapping of $\mathbb{C}$ into $P^n(\mathbb{C})$, with a reduced representation $f := (f_0 : \cdots : f_n)$. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log \|f(e^{i\theta})\| d\theta, \quad r > 1,$$

where $\|f\| := \max\{|f_0|, \ldots, |f_n|\}$.

Let ν be a divisor on $\mathbb{C}$. The counting function of ν is defined by

$$N_\nu(r) := \int_1^r \log \frac{\sum_{|z|<t} \nu(z)}{t} dt, \quad r > 1.$$

For a meromorphic function φ ($\not\equiv 0, \not\equiv \infty$) denote by $(\varphi)_0$ the zero divisor of φ, and set $N_\varphi(r) := N_{(\varphi)_0}(r)$. We have the following Jensen's formula for the counting function:

$$N_\varphi(r) - N_{\frac{1}{\varphi}}(r) = \frac{1}{2\pi}\int_0^{2\pi} \log \left|\varphi(re^{i\theta})\right| d\theta + O(1).$$

Let Q be a homogeneous polynomial in $\mathbb{C}[x_0, \dots, x_n]$. If $Q(f) := Q(f_0, \dots, f_n) \not\equiv 0$, the counting function $N_f(r, Q)$ of f for Q is defined by

$$N_f(r, Q) := N_{Q(f)}(r).$$

3 Estimates of the distance from a point two a subvariety

We begin with the linear case, which will be used later as a lemma in the non-linear case. The following formula was showed in [10] for the case of hyperplanes.

Theorem 3.1. *Let $p(a_0 : \cdots : a_n)$ be a point in $P^n(\mathbb{C})$ and α be a subspace of $P^n(\mathbb{C})$ of dimension m, generated by $m+1$ points $p_1(p_{00} : \cdots : p_{0n}), \dots, p_1(p_{m0} : \cdots : p_{mn})$. Then*

$$d_{FS}(p, \alpha) = \frac{\|\overrightarrow{p} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|}{\|\overrightarrow{p}\| \cdot \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|}, \tag{2.1}$$

where $\overrightarrow{p} = (a_0, \dots, a_n)$ and $\overrightarrow{p_i} = (p_{i0}, \dots, p_{in})$ are vectors in $\mathbb{C}^{n+1}$.

Proof. By changing the coordinates if necessary, we may assume that

$$\alpha = \{(\omega_0 : \cdots : \omega_n) \in P^n(\mathbb{C}) : \omega_{m+1} = \cdots = \omega_n = 0\}.$$

Then $\overrightarrow{p_i} = (p_{i0}, \dots, p_{im}, 0, \dots, 0)$. Denote by $(\overrightarrow{e_0}, \dots, \overrightarrow{e_n})$ the standard orthonormal basis of $\mathbb{C}^{n+1}$. Each point $\omega(\omega_0 : \cdots : \omega_m : 0 \cdots : 0) \in \alpha$ ($|\omega_0|^2 + \cdots + |\omega_m|^2 = 1$) corresponds to an unit vector $\overrightarrow{\omega} = (\omega_0, \dots, \omega_m, 0, \dots, 0) \in$

$\mathbb{C}^{n+1}$, and we have

$$\begin{aligned}
d_{FS}(p,\omega) &= \frac{1}{\|\overrightarrow{p}\|}\|\overrightarrow{p}\wedge\overrightarrow{\omega}\| \\
&= \frac{1}{\|\overrightarrow{p}\|}\|\,(a_0\overrightarrow{e_0}+\cdots a_n\overrightarrow{e_n})\wedge(\omega_0\overrightarrow{e_0}+\cdots+\omega_m\overrightarrow{e_m})\,\| \\
&= \frac{1}{\|\overrightarrow{p}\|}\Big\|\sum_{i=0}^{m} a_{m+1}\omega_i\overrightarrow{e_{m+1}}\wedge\overrightarrow{e_i}+\cdots+\sum_{i=0}^{m} a_n\omega_i\overrightarrow{e_n}\wedge\overrightarrow{e_i} \\
&\qquad\qquad + \sum_{0\leqslant s<k\leqslant m}(a_s\omega_k-a_k\omega_s)\overrightarrow{e_s}\wedge\overrightarrow{e_k}\Big\| \\
&= \frac{1}{\|\overrightarrow{p}\|}\left(\sum_{m+1\leqslant t\leqslant n, 0\leqslant i\leqslant m}|a_t\omega_i|^2+\sum_{0\leqslant s<k\leqslant m}|a_s\omega_k-a_k\omega_s|^2\right)^{1/2} \qquad (2.2) \\
&\geq \frac{1}{\|\overrightarrow{p}\|}\left(\sum_{m+1\leqslant t\leqslant n, 0\leqslant i\leqslant m}|a_t\omega_i|^2\right)^{1/2} \\
&= \frac{1}{\|\overrightarrow{p}\|}\left(|a_{m+1}|^2+\cdots+|a_n|^2\right)^{1/2}\left(|\omega_0|^2+\cdots+|\omega_m|^2\right)^{1/2} \\
&= \frac{1}{\|\overrightarrow{p}\|}\left(|a_{m+1}|^2+\cdots+|a_n|^2\right)^{1/2}.
\end{aligned}$$

Now we take $\omega=(a_0:\cdots:a_m:0:\cdots:0)\in\alpha$ and

$$\overrightarrow{\omega}=(\frac{a_0}{(|a_0|^2+\cdots+|a_m|^2)^{1/2}},\ldots,\frac{a_m}{(|a_0|^2+\cdots+|a_m|^2)^{1/2}},0,\ldots,0)\in\mathbb{C}^{n+1}.$$

Then, by (2.2) we have

$$\begin{aligned}
d_{FS}(p,\omega) &= \frac{1}{\|\overrightarrow{p}\|}\left(\sum_{m+1\leqslant t\leqslant n, 0\leqslant i\leqslant m}\frac{|a_t a_i|^2}{\sum_{s=m+1}^{n}|a_s|^2}\right)^{1/2} \\
&= \frac{1}{\|\overrightarrow{p}\|}\left(|a_{m+1}|^2+\cdots+|a_n|^2\right)^{1/2}.
\end{aligned}$$

Therefore

$$d_{FS}(p,\alpha)=\frac{1}{\|\overrightarrow{p}\|}\left(|a_{m+1}|^2+\cdots+|a_n|^2\right)^{1/2}. \qquad (2.3)$$

We have

$$\overrightarrow{p} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m} = \left(\sum_{i=0}^{n} a_i \overrightarrow{e_i}\right) \wedge \left(\sum_{i=0}^{m} p_{0i} \overrightarrow{e_i}\right) \wedge \cdots \wedge \left(\sum_{i=0}^{m} p_{mi} \overrightarrow{e_i}\right)$$
$$= \sum_{i=m+1}^{n} a_i \det(p_{st})_{0\leqslant s,t\leqslant m} \overrightarrow{e_i} \wedge \overrightarrow{e_0} \wedge \cdots \wedge \overrightarrow{e_m}$$

Therefore

$$\|\overrightarrow{p} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\| = |\det(p_{st})_{0\leqslant s,t\leqslant m}| \left(|a_{m+1}|^2 + \cdots + |a_n|^2\right)^{1/2}$$
$$= \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\| \cdot \left(|a_{m+1}|^2 + \cdots + |a_n|^2\right)^{1/2}.$$

Combining with (2.3), we have

$$d_{FS}(p, \alpha) = \frac{\|\overrightarrow{p} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|}{\|\overrightarrow{p}\| \cdot \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|},$$

□

Remark 3.2. *If α is a hyperplane generated by a linear form $L(x_0, \ldots, x_n) = b_0x_0 + \cdots + b_nx_n$, then (2.3) can be written as*

$$d_{FS}(p, \alpha) = \frac{|L(p)|}{\|p\| \cdot (|b_0|^2 + \cdots + |b_n|^2)^{1/2}}.$$

Corollary 3.3. *Let α be a subspace of dimension m ($0 \leqslant m \leqslant n-1$) in $P^n(\mathbb{C})$, and let ρ be a positive constant. Let f be an entire curve f in $P^n(\mathbb{C}) \setminus \alpha_\rho$. Then the following assertions hold:*

a) If $m < n-1$, then for each set of $(m+1)$ independent points $p_0, \ldots, p_m$ in α,

$$T_{f\wedge p_0\wedge\cdots\wedge p_m}(r) = T_f(r) + O(1).$$

b) If $m = n-1$ then f is a constant curve.

Proof. Assume that α is generated by $m+1$ points $p_0(p_{00} : \cdots : p_{0n}), \ldots, p_m(p_{m0} : \cdots : p_{mn})$. We do with a reduced representation $(f_0 : \cdots : f_n)$ of f. Set $\overrightarrow{p_i} = (p_{i0}, \ldots, p_{in}) \in \mathbb{C}^{n+1}$, and for each $z \in \mathbb{C}$, set $\overrightarrow{f(z)} = (f_0(z), \ldots, f_n(z)) \in \mathbb{C}^{n+1}$.

Case 1: $m < n-1$. By Theorem 3.1, we have

$$\rho \leqslant d_{FS}(f(z),\alpha) = \frac{\|\overrightarrow{f(z)} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|}{\|\overrightarrow{f(z)}\| \cdot \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|}.$$

Therefore

$$\log\|\overrightarrow{f(z)} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\| \geq \log\|\overrightarrow{f(z)}\| + \log\left(\rho \cdot \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|\right).$$

Then by integrating, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log\|\overrightarrow{f(re^{i\theta})} \wedge \overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\| d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\| d\theta + O(1).$$

Then

$$T_{f\wedge p_0\wedge\cdots\wedge p_m}(r) \geq T_f(r) + O(1). \tag{2.4}$$

On the other hand,

$$\|f(z) \wedge p_0 \wedge \cdots \wedge p_m\| \leqslant \|f(z)\| \cdot \|p_0\| \cdots \|p_m\| \binom{n+1}{m+1}^{1/2}.$$

Then by integrating, we get

$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \log\|f(re^{i\theta}) \wedge p_0 \wedge \cdots \wedge p_m\| d\theta \leqslant \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \log\|f(e^{ri\theta})\| d\theta + O(1).$$

Then

$$T_{f\wedge p_0\wedge\cdots\wedge p_m}(r) \leqslant T_f(r) + O(1).$$

Combining with (2.4), we have

$$T_{f\wedge p_0\wedge\cdots\wedge p_m}(r) = T_f(r) + O(1).$$

Case 2: $m = n-1$. Denote by $\det\left(\overrightarrow{f(z)}, \overrightarrow{p_0}, \ldots, \overrightarrow{p_m}\right)$ the determinant of the matrix of coordinates of vectors $\overrightarrow{f(z)}, \overrightarrow{p_0}, \ldots, \overrightarrow{p_m}$. By Theorem 3.1 we have

$$\left|\det\left(\overrightarrow{f(z)}, \overrightarrow{p_0}, \ldots, \overrightarrow{p_m}\right)\right| \geq \rho\|\overrightarrow{f(z)}\| \cdot \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|.$$

Then

$$\log \left|\det\left(\overrightarrow{f(z)}, \overrightarrow{p_0}, \ldots, \overrightarrow{p_m}\right)\right| \geq \log \|f(z)\| + \log \rho \|\overrightarrow{p_0} \wedge \cdots \wedge \overrightarrow{p_m}\|.$$

Therefore, by integrating and by using Jensen's Lemma, we get that

$$N_f(r, \alpha) = N_{\det\left(\overrightarrow{f(z)}, \overrightarrow{p_0}, \ldots, \overrightarrow{p_m}\right)}(r) \geq T_f(r) + O(1).$$

On the other hand, $N_f(r, \alpha) = 0$. Hence, f is a constant curve.

We have completed the proof of Corollary 3.3 □

The following example shows that the assertion b) in Corollary 3.3 is not valid to the case where $m < n - 1$.

Example 1: $\alpha := \{(\omega_0 : \cdots : \omega_n) \in P^n(\mathbb{C}) : \omega_{m+1} = \cdots = \omega_n = 0\}$, and $f = (0 : \cdots : 0 : f_{m+1} : \cdots : f_n)$ is an arbitrary nonconstant curve in the subspace $\beta := \{(\omega_0 : \cdots : \omega_n) \in P^n(\mathbb{C}) : \omega_0 = \cdots = \omega_m = 0\}$. It is easy to see that $d_{FS}(p_\alpha, p_\beta) = 1$ for all points $p_\alpha \in \alpha$, $p_\beta \in \beta$, hence, $d_{FS}(\alpha, \beta) = 1$ and $d_{FS}(f(z), \alpha) = 1$, for all $z \in \mathbb{C}$.

For $m + 1$ points $a_0, \ldots, a_m$ in $P^n(\mathbb{C})$, we denote by $d_{FS}(a_0, \ldots, a_n)$ the minimum of the Fubini-Study distances from each point to the subspace generated by these m other points.

Lemma 3.4 ([10], Corollary 14). *Let $P_0(\omega_{00} : \cdots : \omega_{0n}), \ldots, P_n = (\omega_{n0} : \cdots : \omega_{nn})$ be $n + 1$ independent points in $P^n(\mathbb{C})$. Then*

$$d^n_{FS}(P_0, \ldots, P_n) \leqslant \frac{|\det(P_0, \ldots, P_n)|}{\|P_0\| \cdots \|P_n\|} \leqslant d_{FS}(P_0, \ldots, P_n),$$

where $\|P_j\| = (|\omega_{j0}|^2 + \cdots + |\omega_{jn}|^2)^{\frac{1}{2}}$ and $\det(P_0, \ldots, P_n) := \det(\omega_{ji})_{0 \leqslant i,j \leqslant n}$.

Corollary 3.5. *Let $f^0, \ldots, f^m$ be $(m+1)$ entire curves in $P^n(\mathbb{C})$, and let α be a subspace of dimension $(n - m - 1)$ in $P^n(\mathbb{C})$ $(n \geq 2, 1 \leqslant m < n - 1)$. Assume that there is a positive constant ρ such that for all $z \in \mathbb{C}$,*

a) $d_{FS}(f^0(z), \ldots, f^m(z)) \geq \rho$;

b) The Fubini-Study distance between α and $\langle f^0(z), \ldots, f^m(z) \rangle$ is not less than ρ.

Then $f^0, \ldots, f^m$ are constant curves.

Proof. Assume that α is generated by $(n-m)$ independent points $p_i(p_{i0} : \cdots : p_{in}), i = m+1, \ldots, n$. For each $z \in \mathbb{C}$, it is clear that $f^0(z), \ldots, f^m(z), p_{m+1}, \ldots, p_n$ are independent.

Claim: There exists a positive constant ϵ such that

$$d_{FS}(f^0(z), \ldots, f^m(z), p_{m+1}, \ldots, p_n) \geq \epsilon,$$

for all $z \in \mathbb{C}$.

Indeed, otherwise there is a subsequence $\{z_k\} \subset \mathbb{C}$, such that

$$\lim_{k\to\infty} d_{FS}(f^0(z_k), \ldots, f^m(z_k), p_{m+1}, \ldots, p_n) = 0.$$

We may assume (by replacing by subsequences if necessary) that each sequence $\{f^i(z_k\}$ $(i = 0, \ldots, m)$ converges to a point p_i in $P^n(\mathbb{C})$. Then

$$d_{FS}(p_0, \ldots, p_m) = \lim_{k\to\infty} d_{FS}(f^0(z_k), \ldots, f^m(z_k)) \geq \rho.$$

Therefore, $p_0, \ldots, p_m$ are independent. Furthermore, by assumption b), distance between α and $\langle p_0, \ldots, p_m \rangle$ is not less than ρ. Therefore, α and $\langle p_0, \ldots, p_m \rangle$ have no common point. Therefore, $p_0, \ldots, p_{n+1}$ are dependent. This contradicts to the fact that

$$d_{FS}(p_0, \ldots, p_{n+1}) = \lim_{k\to\infty} d_{FS}(f^0(z_k), \ldots, f^m(z_k), p_{m+1}, \ldots, p_{n+1}) = 0.$$

Let $(f_0^i : \cdots : f_n^i)$ be a reduced representation of f^i $(i = 0, \ldots, m)$. Set

$$D(z) = \begin{vmatrix} f_0^0(z) & f_1^0(z) & \ldots & f_n^0(z) \\ \cdot & \cdot & \cdots & \cdot \\ f_0^m(z) & f_1^m(z) & \cdots & f_n^m(z) \\ p_{(m+1)0} & p_{(m+1)1} & \cdots & p_{(m+1)n} \\ \cdot & \cdot & \cdots & \cdot \\ p_{n0} & p_{n1} & \cdots & p_{nn} \end{vmatrix}.$$

From the claim and by Lemma 3.4, we have

$$\frac{|D(z)|}{\|f^0(z)\| \cdots \|f^m(z)\| \cdot \|p_{m+1}\| \cdots \|p_n\|} \geq \epsilon^n.$$

Then

$$\log |D(z)| \geq \log \|f^0(z)\| + \cdots + \log \|f^m(z)\| + \log \|p_{m+1}\| + \cdots + \log \|p_n\| + n \log \epsilon.$$

Hence, by integrating and by using Jensen's Lemma we get that

$$0 = N_D(r) \geq T_{f^0}(r) + \cdots + T_{f^m}(r) + O(1).$$

This implies that $f^0, \ldots, f^m$ are constant curves. $\square$

The following example points out the necessity of assumption a) in Corollary 3.5.

Example 2: $\alpha := \{(\omega_0 : \cdots : \omega_n) \in P^n(\mathbb{C}) : \omega_0 = \cdots = \omega_m = 0\}$, and $f^0, \ldots, f^m$ are arbitrary nonconstant curves in $\beta := \{(\omega_0 : \cdots : \omega_n) \in P^n(\mathbb{C}) : \omega_{m+1} = \cdots = \omega_n = 0\}$. We have $d_{FS}(\alpha, \beta) = 1$ and $\langle f^0(z), \ldots, f^n(z) \rangle \subset \beta$, for all $z \in \mathbb{C}$. Hence, the distance between α and $\langle f^0(z), \ldots, f^m(z) \rangle$ is not less than 1.

The following example shows that condition b) cannot be replaced by the condition b'): The Fubini-Study distance from each point $f^0(z), \ldots, f^m(z)$ to α is not less than a constant positive ρ, for all $z \in \mathbb{C}$.

Example 3: $\alpha := \{(\omega_0 : \cdots : \omega_n) \in P^n(\mathbb{C}) : \omega_0 = \cdots = \omega_m = 0\}$, and

$$\begin{aligned}
f^0(z) \equiv& (1 : 0 : 0 : \cdots : 0),\\
f^1(z) \equiv& (0 : 1 : 0 : \cdots : 0),\\
&\cdots\\
f^{m-1}(z) \equiv& (0 : \cdots : 0 : 1 : \quad 0 \quad : \cdots : 0),\\
f^m(z) =& (z : \cdots : z : z : 1 + z : z : 0 : \cdots : 0),\\
&S(1 : \cdots : 1 : 1 : \quad 1 \quad : 0 : 0 : \cdots : 0),\\
&T(0 : \cdots : 0 : 0 : \quad 1 \quad : 0 : 0 : \cdots : 0).
\end{aligned}$$

Denote by Δ the straight line passing through two points S and T. Then $\Im(f^m) = \Delta \backslash \{S\}$. For each point $M \in \Delta$, it is clear that $M, f^0(z), \ldots, f^{m-1}(z)$ are projective independent, and hence, $d_{FS}(f^0(z), \ldots, f^{m-1}(z), M) > 0$. On the other hand, Δ is compact, and $f^0, \ldots, f^{m-1}$ are constant. Hence, there is a positive constant ρ such that $d_{FS}(f^0(z), \ldots, f^{m-1}(z), M) > \rho$, for all $M \in \Delta$. This implies that $f^0, \ldots, f^m$ satisfy condition a).

We have $\Delta \cap \alpha = \varnothing$. Hence, $d_{FS}(f^m(z), \alpha) \geq d_{FS}(\Delta, \alpha) > 0$. Therefore, the Fubini-Study distance from each point $f^0(z), \ldots, f^m(z)$ to α is not less than $\delta := \min\{d_{FS}(\Delta, \alpha), d_{FS}(f^0(z), \alpha), \ldots, d_{FS}(f^{m-1}(z), \alpha)\} > 0$. Then condition b') is satisfied.

Theorem 3.6. *Let D be an irreducible hypersurface of degree d in $P^n(\mathbb{C})$, defined by a homogeneous polynomial $Q \in \mathbb{C}[x_0, \dots, x_n]$, $\deg Q = d$. Then there exists a positive constant c such that*

$$d^d_{FS}(p, D) \leqslant \frac{|Q(p)|}{c \cdot \|p\|^d} \tag{2.5}$$

for all point $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C})$, where $\|p\| = |a_0|^2 + \cdots + |a_n|^2)^{1/2}$, $Q(p) = Q(a_0, \dots, a_n)$.

Proof. For a generic point S, each straight line Δ passing through S will meet D at d points $P_1, \dots, P_d$ include multiplicites. We fix a such point $S \notin D$. We may assume (by changing the coordinates if necessary) that $S(1 : 0 : \cdots : 0)$. Since $S \notin D$, the coefficient c of x_0^d in Q is different from 0. Now we consider a generic point $p(a_0 : \cdots : a_n) \neq S(1 : 0 : \cdots : 0)$. Then the straight line Δ passing through S and p meets D at d points $P_1(a_0 + z_1 : a_1 : \dots, a_n), \dots, P_1(a_0 + z_d : a_1 : \cdots : a_n)$ (with multiplicites), where $z_1, \dots, z_d$ are d roots of the polynomial $R(z) := Q(a_0 + z, a_1, \cdots, a_n)$. We can write $R(z) = c(z - z_1) \cdots (z - z_d)$. Denote by $(\overrightarrow{e_0}, \dots, \overrightarrow{e_n})$ the standard orthonormal basis of $\mathbb{C}^{n+1}$. Set $\overrightarrow{p} = (a_0, \dots, a_n)$, $\overrightarrow{P_j} = (a_0 + z_j, a_1, \dots, a_n) \in \mathbb{C}^{n+1}$, $j = 1, \dots, d$. Then $\overrightarrow{P_j} = \overrightarrow{p} + z_j \overrightarrow{e_0}$. For $j = 1, \dots, d$, we have

$$\begin{aligned}
d_{FS}(p, D) \leqslant d_{FS}(p, P_j) &= \frac{\|\overrightarrow{p} \wedge \overrightarrow{P_j}\|}{\|\overrightarrow{p}\| \cdot \|\overrightarrow{P_j}} \\
&= \frac{\|\overrightarrow{p} \wedge z_j \overrightarrow{e_0}\|}{\|\overrightarrow{p}\| \cdot \|\overrightarrow{P_j}\|} \\
&= \frac{\|z_j a_1 \overrightarrow{e_1} \wedge \overrightarrow{e_0} + \cdot + z_j a_n \overrightarrow{e_n} \wedge \overrightarrow{e_0}\|}{\|\overrightarrow{p}\| \cdot \|\overrightarrow{P_j}\|} \\
&= \frac{|z_j|(|a_1|^2 + \cdots + |a_n|^2)^{1/2}}{\|p\|(|a_0 + z_j|^2 + |a_1|^2 + \cdots + |a_n|^2)^{1/2}} \\
&\leqslant \frac{|z_j|}{\|p\|}.
\end{aligned}$$

Then

$$d^d_{FS}(p, D) \leqslant \prod_{j=1}^{d} d_{FS}(p, P_j) \leqslant \frac{|z_1| \cdots |z_d|}{\|p\|^d} = \frac{|R(0)|}{|c| \cdot \|p\|^d} = \frac{|Q(p)|}{|c| \cdot \|p\|^d}. \tag{2.6}$$

for all generic point $p \neq S$. Then by the continuity, (2.5) holds for all p. $\square$

Corollary 3.7. *Let D be a hypersurface in $P^n(\mathbb{C})$, and let ϵ be a positive constant. Then every entire curve f in $P^n(\mathbb{C}) \setminus D_\epsilon$ is constant.*

Proof. Let f be an entire curve in $P^n(\mathbb{C}) \setminus D_\epsilon$, with a reduced representation $f = (f_0 : \cdots : f_n)$. Assume that D is defined by a homogeneous polynomial $Q \in \mathbb{C}[x_0, \ldots, x_n]$, $\deg Q = \deg D$. By the assumption and by Lemma 3.6, there exists a positive constant c such that

$$\epsilon^{\deg Q} \leqslant d_{FS}^{\deg Q}(f(z), D) \leqslant \frac{|Q(f(z))|}{c \cdot \|f(z)\|^{\deg Q}}$$

for all $z \in \mathbb{C}$. Hence

$$\deg Q \log \|f(z)\| \leqslant \log |Q(f(z))| + O(1).$$

Hence, by integrating and by using Jensen's Lemma, we get that

$$\begin{aligned} \deg Q \cdot T_f(r) &= \deg Q \cdot \left(\frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \|f(e^{i\theta})\| d\theta \right) \\ &\leqslant \frac{1}{2\pi} \int_0^{2\pi} \log |Q(f(re^{i\theta})| d\theta + O(1) \\ &= N_f(r, Q) + O(1). \end{aligned}$$

On the other hand, $\Im f \cap D = \varnothing$. Hence,

$$\deg Q \cdot T_f(r) \leqslant N_f(r, Q) + O(1) = O(1).$$

This implies that f is constant. □

Lemma 3.8. *Let ϵ be a positive constant, let A be a point and H be a hyperplane in $P^n(\mathbb{C})$, $A \notin H$. There exists a positive constant c (depending on A, H, ϵ) such that for two arbitrary distinct points p, q in $P^n(\mathbb{C})$, if the Fubini-Study distance from A to the straight line pq (passing through p and q) is not less than ϵ, then $d_{FS}(p, q) \leqslant c \cdot d_{FS}(p', q')$, where p', q' are, respectively, images of p, q by the central projection from A onto the hyperplane H.*

Proof. We may assume (by changing the coordinates if necessary) that $A(1 : 0 : \cdots : 0)$. Assume that H has equation

$$H : x_0 - a_1 x_1 - \cdots - a_n x_n = 0.$$

Assume that $p(p_0 : \cdots : p_n)$ and $q(q_0 : \cdots : q_n)$, where $|p_0|^2 + \cdots + |p_n|^2 = |q_0|^2 + \cdots + |q_n|^2 = 1$. Then $p'(a_1p_1 + \cdots + a_np_n : p_1 : \cdots : p_n), q'(a_1q_1 + \cdots + a_nq_n : q_1 : \cdots : q_n)$. Denote by $(\overrightarrow{e_0}, \ldots, \overrightarrow{e_n})$ the standard orthonormal basis of $\mathbb{C}^{n+1}$. Set $\overrightarrow{p} = (p_0, \ldots, p_n)$, $\overrightarrow{q} = (q_0, \ldots, q_n) \in \mathbb{C}^{n+1}$.
We have

$$d_{FS}(p, q) = \left(\sum_{0 \leqslant i < j \leqslant n} |p_i q_j - p_j q_i|^2 \right)^{1/2} \tag{2.7}$$

By the assumption and by Theorem 3.1, we have

$$\begin{aligned} \epsilon \leqslant d_{FS}(A, pq) &= \frac{\|\overrightarrow{e_0} \wedge \overrightarrow{p} \wedge \overrightarrow{q}\|}{\|\overrightarrow{p} \wedge \overrightarrow{q}\|} \\ &= \frac{\left(\sum_{1 \leqslant i < j \leqslant n} |p_i q_j - p_j q_i|^2\right)^{1/2}}{\left(\sum_{0 \leqslant i < j \leqslant n} |p_i q_j - p_j q_i|^2\right)^{1/2}} \end{aligned} \tag{2.8}$$

We have

$$\begin{aligned} d_{FS}(p', q') &= \frac{\|\left((\sum_{k=1}^n a_k p_k)\overrightarrow{e_0} + \sum_{i=1}^n p_i \overrightarrow{e_i}\right) \wedge \left((\sum_{s=1}^n a_s q_s)\overrightarrow{e_0} + \sum_{j=1}^n q_j \overrightarrow{e_j}\right)\|}{\|(\sum_{k=1}^n a_k p_k)\overrightarrow{e_0} + \sum_{i=1}^n p_i \overrightarrow{e_i}\| \cdot \|(\sum_{s=1}^n a_s q_s)\overrightarrow{e_0} + \sum_{j=1}^n q_j \overrightarrow{e_j}\|} \\ &\geq \frac{\left(\sum_{1 \leqslant i < j \leqslant n} |p_i q_j - p_j q_i|^2\right)^{1/2}}{\left(|\sum_{i=1}^n a_i p_i|^2 + \sum_{s=1}^n |x_s|^2\right)^{1/2} \cdot \left(|\sum_{i=1}^n a_i q_i|^2 + \sum_{s=1}^n |p_s|^2\right)^{1/2}} \\ &\geq \frac{\left(\sum_{1 \leqslant i < j \leqslant n} |p_i q_j - p_j q_i|^2\right)^{1/2}}{(1 + |a_1|^2 + \cdots + |a_n|^2)\left(\sum_{s=1}^n |p_s|^2\right)^{1/2} \cdot \left(\sum_{s=1}^n |q_s|^2\right)^{1/2}} \\ &\geq \frac{\left(\sum_{1 \leqslant i < j \leqslant n} |p_i q_j - p_j q_i|^2\right)^{1/2}}{1 + |a_1|^2 + \cdots + |a_n|^2}. \end{aligned}$$

Combining with (2.7) and (2.8), we get that

$$d_{FS}(p', q') \geq \frac{\epsilon}{1 + |a_1|^2 + \cdots + |a_n|^2} d_{FS}(p, q).$$

This completes the proof of Lemma 3.8. $\square$

Lemma 3.9. *Let $V \subset P^n(\mathbb{C})$ be an irreducible subvariety of degree d. Then there exist finitely many homogenous polynomials $Q_1, \ldots, Q_m$ in $\mathbb{C}[x_0, \ldots, x_n]$ of degree at most d, vanishing on V and a positive constant ρ such that*

$$d^d_{PS}(p, V) \leqslant \max\left\{\frac{|Q_j(p)|}{\rho \cdot \|p\|^{\deg Q_j}} \ : \ 1 \leqslant j \leqslant m\right\} \tag{2.9}$$

for all point $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C})$, where $\|p\| = |a_0|^2 + \cdots + |a_n|^2)^{1/2}$, $Q_j(p) = Q_j(a_0, \ldots, a_n)$.

Proof. We consider a generic projection $\mathcal{P}$ defined by a subspace $L_1 \subset P^n(\mathbb{C})$, from $P^n(\mathbb{C}) \backslash L_1$ into a subspace L_2, where $\dim L_1 = n - \dim V - 1$, $L_1 \cap V = \varnothing$, $\dim L_2 = \dim V$, $L_1 \cap L_2 = \varnothing$. For each point $p \in P^n(\mathbb{C}) \setminus L_1$, denote by $\langle L_1, p \rangle$ the subspace generated by $L_1 \cup \{p\}$. Then $\mathcal{P}(p)$ is the intersection point of $\langle L_1, p \rangle$ and L_2. The restriction $\mathcal{P}_V : V \to L_2$ is finite of degree d and surjective.

Claim: Then there exist finitely many homogenous polynomials $Q_1, \ldots, Q_m$ in $\mathbb{C}[x_0, \ldots, x_n]$ of degree at most d, vanishing on V and a positive constant c such that for all generic point $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C}) \setminus (L_1 \cup V)$,

$$\min\{d^d_{FS}(p, P_j) : 1 \leqslant j \leqslant s\} \leqslant \max\left\{\frac{|Q_j(p)|}{c \cdot \|p\|^{\deg Q_j}} \ : \ 1 \leqslant j \leqslant m\right\}, \tag{2.10}$$

where $\{P_1, \ldots, P_s\} := \langle L_1, p \rangle \cap V = \mathcal{P}_V^{-1}(\mathcal{P}(p))$ (as sets), $s \leqslant d$.

We prove the claim by induction on the codimension of V. If $\mathrm{codim} V = 1$, then V is an irreducible hypersurface generated by a single homogeneous polynomial $Q \in \mathbb{C}[x_0, \ldots, x_n]$ of degree d. In this case, $s = d$ and by (2.6), there is a positive constant c such that

$$\begin{aligned}\min\{d^d_{FS}(p, P_j) : \ 1 \leqslant j \leqslant d\} &\leqslant \prod_{j=1}^{d} d_{FS}(p, P_j) \\ &\leqslant \frac{|Q(p)|}{c \cdot \|p\|^d}.\end{aligned}$$

all point $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C}) \setminus L_1$.

In the case where $\mathrm{codim} V > 1$, we fix $d + 1$ generic points $A_0, \ldots, A_d$ in L_1, and a hyperplane H containing L_2, but not passing through any point A_i.

We now prove that there is a positive constant ϵ such that for each straight line Δ having nonempty intersection with V, there exists at most one point A_i $(0 \leqslant i \leqslant d)$ such that $d_{FS}(A_i, \Delta_n) \leqslant \epsilon$. Indeed, otherwise, there exist two points A_{i_2}, A_{i_2} $(0 \leqslant i_1 < i_2 \leqslant d)$ such that for each positive integer n, there is a straight line Δ_n satisfying $\Delta_n \cap V \neq \varnothing$ and $d_{FS}(A_{i_1}, \Delta_n) < \frac{1}{n}, d_{FS}(A_{i_2}, \Delta_n) < \frac{1}{n}$. For each n, take a point $X_n \in \Delta \cap V$. Then $\{\Delta_n\}$ converges to the straight line $A_{i_1}A_{i_2}$ (passing through A_{i_1}, A_{i_2}). On the other hand, Δ_n meets V for all n, hence, $A_{i_1}A_{i_2}$ also meets V. This is impossible by the fact that $L_1 \cap V = \varnothing$ and $A_{j_1}A_{j_2} \subset L_1$.
By the above argument, there exists a positive constant ϵ, such that for each point P_j $(1 \leqslant j \leqslant s)$, there exists at most one point $A_{j'}$ $(0 \leqslant j' \leqslant d)$ which satisfies $d_{FS}(A_{j'}, pP_j) < \epsilon$. On the other hand, $s < d+1$, hence, there exists $i_0 \in \{0, \ldots, d\}$ (depending on p) such that

$$d_{FS}(A_{i_0}, pP_j) \geq \epsilon, \tag{2.11}$$

for all $j \in \{1, \ldots, s\}$.
Let $L_1' = L_1 \cap H$ and denote by $V', p', P_1', \ldots, P_s'$ the images of $V, p, P_1, \ldots, P_s$, respectively, by the central projection from A_{i_0} onto hyperplane H. In subspace $H \equiv P^{n-1}(\mathbb{C})$, we have $\{P_1', \ldots, P_s'\} = \langle L_1', p' \rangle \cap V'$. We may assume that $H : x_0 = 0$, $A_{i_0}(\alpha_0 : \cdots : \alpha_n)$, $\alpha_0 \neq 0$. Then the central projection from A_{i_0} onto hyperplane $H \equiv P^{n-1}(\mathbb{C})$ sends each point $X(x_0 : \cdots : x_n) \in P^n(\mathbb{C}) \setminus \{A_0\}$ to the point $X'(\omega_1 - \frac{\alpha_1}{\alpha_0}\omega_0 : \cdots : \omega_n - \frac{\alpha_n}{\alpha_0}\omega_0) \in P^{n-1}(\mathbb{C})$.

By induction hypothesis, there are m homogeneous polynomials $Q_1', \ldots, Q_m'$ in $\mathbb{C}[x_1, \ldots, x_n]$ of degree at most d, vanishing on V' such that

$$\min\{d_{FS}^d(p', P_i') : 1 \leqslant i \leqslant s\} \leqslant \max\left\{\frac{|Q_j'(p')|}{\|p'\|^{\deg Q_j'}} \;:\; 1 \leqslant j \leqslant m\right\}, \tag{2.12}$$

for all $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C}) \setminus (L_1 \cup V)$, and hence, $p' = (a_1 - \frac{\alpha_1}{\alpha_0}a_0 : \cdots : a_n - \frac{\alpha_n}{\alpha_0}a_0) \in P^{n-1}(\mathbb{C}) \setminus (L_1' \cup V')$.

Denote by $B_{FS}(A_{i_0}, \epsilon)$ the open ball of radius ϵ, centered at A_{i_0}. Let $h : P^n(\mathbb{C}) \to \mathbb{R}$ be the continue function which sends each point $X(\omega_0 : \cdots : \omega_n)$ to the value

$$\frac{\left(|\omega_1 - \frac{\alpha_1}{\alpha_0}\omega_0|^2 + \cdots + |\omega_n - \frac{\alpha_n}{\alpha_0}\omega_0|^2\right)^{1/2}}{(|\omega_0|^2 + \cdots + |\omega_n|^2)^{1/2}}.$$

It is clear that $h(X) = 0$ if and only if $X \equiv A_{i_0}$. Therefore

$$c_1 := \min\{h(X) : X \in P^n(\mathbb{C}) \setminus B_{FS}(A_{i_0}, \epsilon)\} > 0.$$

On the other hand, $d_{FS}(A_{i_0}, p) \geq d_{FS}(A_{i_0}, pP_j) \geq \epsilon$. Hence,

$$\frac{\left(|a_1 - \frac{\alpha_1}{\alpha_0}a_0|^2 + \cdots + |a_n - \frac{\alpha_n}{\alpha_0}a_0|^2\right)^{1/2}}{\left(|a_0|^2 + \cdots + |a_n|^2\right)^{1/2}} = h(p) \geq c_1. \tag{2.13}$$

Set $Q_j(x_0, \ldots, x_n) = Q'_j(x_1 - \frac{\alpha_1}{\alpha_0}x_0, \ldots, x_n - \frac{\alpha_n}{\alpha_0}x_0) \in \mathbb{C}[x_0, \ldots, x_n]$. Since Q'_j vanish on V', we have that Q_j vanish on V. Therefore, by (2.13)

$$\frac{|Q'_j(p')|}{\|p'\|^{\deg Q'_j}} = \frac{|Q_j(p)|}{\|p'\|^{\deg Q_j}} \leqslant \frac{|Q_j(p)|}{(c_1 \cdot \|p\|)^{\deg Q_j}} \tag{2.14}$$

for all $j \in \{1, \ldots, m\}$.
By Lemma 3.8 and by (2.11), there exists a positive constant c_2 such that

$$d_{FS}(p, P_i) \leqslant c_2 \cdot d_{FS}(p', P'_i), \tag{2.15}$$

for all $i \in \{1, \ldots, s\}$.
From (2.12), (2.14), (2.15), we get the claim.
By the claim and by the fact that $d^d_{FS}(p, V) = \min\{d^d_{FS}(p, P_j) : 1 \leqslant j \leqslant s\}$, we get (2.9) for all $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C}) \setminus (L_1 \cup V)$. Hence by the continuity, (2.9) holds for all $p \in P^n(\mathbb{C})$. $\square$

Let $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be a set of m homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$. Denote by d_j the degree of Q_j, and assume that $d_1 \geq d_2 \geq \cdots \geq d_m$. Let $\mathcal{M}$ denote the maximal homogeneous ideal; that is, $\mathcal{M} = (x_0, \ldots, x_n)$. Set

$$N_{\mathcal{Q}} := \min_{(e_0, \ldots, e_\mu, \mathcal{P}_1, \ldots, \mathcal{P}_s)} \{e_0 + e_1 \deg \mathcal{P}_1 + \cdots + e_s \deg \mathcal{P}_s\}$$

where the minimum is taken over all set $\{e_0, \ldots, e_s, \mathcal{P}_1, \ldots, \mathcal{P}_s\}$ satisfying that e_0 is a nonnegative integer, $e_1 \ldots, e_s$ are positive integers, $\mathcal{P}_1, \ldots, P_s$ are homogeneous prime ideals containing the homogenous ideals $\mathcal{I} = (Q_1, \ldots, Q_m)$, and

$$\mathcal{M}^{e_0}\mathcal{P}_1^{e_1} \cdots \mathcal{P}_s^{e_s} \subset (Q_1, \ldots, Q_m).$$

- If all d_j are different from 2, then by ([12], Theorem 1.5), we have

$$\frac{N_{\mathcal{Q}}}{\deg \sqrt{\mathcal{I}}} \leqslant \begin{cases} d_1 \cdots d_m & \text{if } m \leqslant n \\ d_1 \cdots d_{n-1} \cdot d_m & \text{if } m > n > 1 \\ d_1 + d_m - 1 & \text{if } m > n = 1 \end{cases}$$

Removing the assumption that all d_j are different from 2, by ([16], Corollary 1.4), we have

$$\frac{N_{\mathcal{Q}}}{\deg \sqrt{\mathcal{I}}} \leqslant \begin{cases} d_1 \cdots d_m & \text{if } m \leqslant n+1 \\ d_1 \cdots d_n \cdot d_m & \text{if } m > n+1 \end{cases}$$

- If all d_j are not less than 3, then by ([2], Theorem 1 and Remark 2), we have that $N_{\mathcal{Q}} \leqslant d_1 \cdots d_{\min\{n,m\}-1} \cdot d_m$.

We would like to note that a "primary decomposition" version of the Nullstellensatz also has been given by Ein and Lazarsfeld [7].

Theorem 3.10. *Let $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be a set of m homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ such that the common zero set $V \subset P^n(\mathbb{C})$ of these polynomials is nonempty, $\deg V = d$. Then there is a positive integer $N \leqslant \mathcal{N}_{\mathcal{Q}}$ and a positive constant c such that*

$$d_{FS}^N(p, V) \leqslant \max\{\frac{|Q_j(p)|}{c \cdot \|p\|^{\deg Q_J}} : 1 \leqslant j \leqslant m\}$$

for all $p(a_0 : \cdots : a_n) \in P^n(\mathbb{C})$, where $\|p\| = |a_0|^2 + \cdots + |a_n|^2)^{1/2}$, $Q_j(p) = Q_j(a_0, \ldots, a_n)$.

Proof. Let e_0 be a nonnegative integer, $e_1, \ldots, e_s$ be positive integers, and let $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be homogeneous prime ideals containing the homogenous ideals $\mathcal{I} = (Q_1, \ldots, Q_m)$, such that $e_0 + e_1 \deg \mathcal{P}_1 + \cdots + e_s \deg \mathcal{P}_s = \mathcal{N}_{\mathcal{Q}}$ and

$$\mathcal{M}^{e_0} \mathcal{P}_1^{e_1} \cdots \mathcal{P}_s^{e_s} \subset (Q_1, \ldots, Q_m). \tag{2.16}$$

Denote by Z_i the irreducible variety defined by $\mathcal{P}_i$. We have $V = Z_1 \cup \cdots \cup Z_s$. By Lemma 3.9, for each $i \in \{1, \ldots, s\}$, there exist finitely many homogenous polynomials $Q_{i1}, \ldots, Q_{im_i}$ in $\mathbb{C}[x_0, \ldots, x_n]$ of degree at most $\deg Z_i$ ($= \deg \mathcal{P}_i$), vanishing on Z_i and a positive constant C_i such that

$$d_{PS}^{\deg \mathcal{P}_i}(p, Z_i) \leqslant \max \left\{ \frac{|Q_{ik_i}(p)|}{C_i \cdot \|p\|^{\deg Q_{ik_i}}} \ : \ 1 \leqslant k_i \leqslant m_i \right\}$$

for all $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C})$. Therefore, there exists a positive constant C such that

$$\begin{aligned} d_{PS}^{e_1 \deg \mathcal{P}_1 + \cdots + e_s \deg \mathcal{P}_s}(p, V) &\leqslant \prod_{i=1}^s d_{PS}^{e_i \deg \mathcal{P}_i}(p, Z_i) \\ &\leqslant \max \left\{ \frac{\prod_{i=1}^s |Q_{ik_i}^{e_i}|}{C \cdot \|p\|^{\sum_{i=1}^s e_i \deg Q_{ik_i}}} \ : \ 1 \leqslant k_1 \leqslant m_1, \ldots, 1 \leqslant k_s \leqslant m_s \right\} \end{aligned} \tag{2.17}$$

for all $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C})$. For each $\kappa = (k_1, \ldots, k_s)$ $(1 \leqslant k_i \leqslant m_i)$ and each $\ell \in \{0, \ldots, n\}$, by Lemma 2.16, we have

$$x_\ell^{e_0} \cdot \prod_{i=1}^{s} Q_{ik_i}^{e_i} \in (Q_1, \ldots, Q_m).$$

Therefore, we can write

$$x_\ell^{e_0} \prod_{i=1}^{s} Q_{ik_i}^{e_i} = \sum_{j=1}^{m} P_{\kappa\ell j} Q_j,$$

where $P_{\kappa\ell j} \in \mathbb{C}[x_0, \ldots, x_n]$ is a homogeneous polynomial of $\deg P_{\kappa\ell j} = e_0 + e_1 \deg Q_{1k_1} + \cdots + e_s \deg Q_{sk_s} - \deg Q_j$.
On the other hand, for each $j \in \{1, \ldots, m\}$, there is a positive constant C'_j such that

$$\frac{|P_{\kappa\ell j}(p)|}{\|p\|^{e_0 + e_1 \deg Q_{1k_1} + \cdots + e_s \deg Q_{sk_s} - \deg Q_j}} \leqslant C'_j,$$

for all $p = (a_0 : \cdots : a_n) \in P^n(\mathbb{C})$.
Hence,

$$|a_\ell|^{e_0} \left| \prod_{i=1}^{s} Q_{ik_i}^{e_i}(p) \right| \leqslant \sum_{j=1}^{m} C'_j \cdot \|p\|^{e_0 + e_1 \deg Q_{1k_1} + \cdots + e_s \deg Q_{sk_s} - \deg Q_j} |Q_j(p)|.$$

for all $\ell \in \{0, \ldots, n\}$.
This implies that

$$\left| \prod_{i=1}^{s} Q_{ik_i}^{e_i}(p) \right| \leqslant (n+1)^{e_0} \sum_{j=1}^{m} C'_j \cdot \|p\|^{e_1 \deg Q_{1k_1} + \cdots + e_s \deg Q_{sk_s} - \deg Q_j} |Q_j(p)|.$$

Then

$$\left| \frac{\prod_{i=1}^{s} Q_{ik_i}^{e_i}(p)}{C \cdot \|p\|^{\sum_{i=1}^{s} e_i \deg Q_{ik_i}}} \right| \leqslant \sum_{j=1}^{m} \frac{(n+1)^{e_0} C'_j \cdot |Q_j(p)|}{C \cdot \|p\|^{\deg Q_j}}.$$

Combining with (2.17), there exists a positive constant c such that

$$\begin{aligned} d_{PS}^{e_1 \deg \mathcal{P}_1 + \cdots + e_s \deg \mathcal{P}_s}(p, V) &\leqslant \sum_{j=1}^{m} \frac{(n+1)^{e_0} C'_j \cdot |Q_j(p)|}{C \cdot \|p\|^{\deg Q_j}} \\ &\leqslant \max\{\frac{|Q_j(p)|}{c \cdot \|p\|^{\deg Q_j}} : 1 \leqslant j \leqslant m\}. \end{aligned}$$

□

Conflict of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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