# Simple and efficient nonstandard finite difference schemes for an SIRC epidemic model of influenza A

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Abstract. In two well-known studies [Mathematics and Computers in Simulation 79(2008) 622-633] and [Mathematics and Computers in Simulation 182(2021) 397-410], some nonstandard finite difference (NSFD) schemes for an SIRC epidemic model of influenza A have been proposed. There have been attempts to prove that these NSFD schemes can preserve the positivity of the solutions, the invariance (conservation law) of the total population, equilibrium points and their asymptotic stability of the continuous-time model, for all finite values of the step size. Nevertheless, although the SIRC model possesses two equilibrium points, a unique disease-free equilibrium (DFE) point and a unique disease-endemic equilibrium (DEE) point, only the local asymptotic stability (LAS) of the DFE point has been established theoretically, whereas the LAS of the DEE point has only been confirmed through numerical simulations using some specific parameter sets.

In this work, we construct a new class of NSFD schemes for the SIRC epidemic model, for which the LAS of the equilibrium points of the constructed NSFD schemes is rigorously established from a theoretical perspective and validated through numerical experiments. These NSFD schemes are constructed based on a weighted approximation for linear terms and the renormalization of the denominator function. Thereafter, we give dynamic consistency thresholds that lead to easily-verified conditions, ensuring the NSFD schemes preserve all the qualitative dynamical properties of the continuous-time model, regardless of the values of the step size. In particular, thanks to the simple structure of the constructed NSFD schemes, their LAS can be easily established by the linearization method. Furthermore, they are capable of providing numerical approximations with higher-order accuracy compared to the existing NSFD schemes. Additionally, Richardson's extrapolation technique can be conveniently applied to increase the accuracy of the constructed NSFD schemes. Consequently, we obtain a new class of dynamically consistent NSFD schemes, which is not only simple but also efficient for numerical simulation of the SIRC model. Also, the constructed NSFD schemes improve those proposed in the two aforementioned studies in terms of both qualitative analysis and computational efficiency.

Lastly, numerical experiments are conducted to support the theoretical findings and demonstrate the advantages of the constructed NSFD schemes.

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## 1. Introduction

In an early work [**6**], Casagrandi et al. proposed a system of nonlinear ordinary differential equations to model the transmission mechanism of influenza A viruses. The mathematical model is represented by

$$\begin{aligned}
\dot{S} &= \mu(1-S) - \beta SI + \gamma C := f_1(S, I, R, C),\\
\dot{I} &= \beta SI + \sigma\beta CI - (\mu+\alpha)I := f_2(S, I, R, C),\\
\dot{R} &= (1-\sigma)\beta CI + \alpha I - (\mu+\delta)R := f_3(S, I, R, C),\\
\dot{C} &= \delta R - \beta CI - (\mu+\gamma)C := f_4(S, I, R, C).
\end{aligned} \tag{1.1}$$

In this model (see [**6**] and also [**17**])

- $S(t)$, $I(t)$, $R(t)$ and $C(t)$ are the proportions of the susceptible, infected, recovered and cross-immune individuals at time $t$, respectively;
- $\beta$ denotes the contact rate for the influenza disease, which is also called the rate of transmission for susceptibles to infected individuals;
- $\gamma^{-1}$ is the cross-immune period;
- $\alpha^{-1}$ represents for the infectious period;
- $\gamma^{-1}$ is the total immune period;
- $\sigma$ denotes the fraction of the exposed cross-immune individuals who are recruited per unit time into the infected sub-population.

Further details of the model (1.1) are presented in [**6**]. The mathematical analyses in [**6**] have shown that (1.1) possesses the following properties:
$(P_1)$ The positivity of the solutions: $S(t), I(t), R(t), C(t) \geq 0$ for all $t > 0$ whenever $S(0), I(0), R(0), C(0) \geq 0$.
$(P_2)$ The invariance (conservation law) of the total population: The total population $N(t) = S(t) + I(t) + R(t) + C(t)$ $(t \geq 0)$ satisfies

$$\dot{N} = \mu - \mu N, \tag{1.2}$$

consequently $S(t) + I(t) + R(t) + C(t) = 1$ for all $t \geq 0$.
$\mathbf{(}P_3\mathbf{)}$ The set of equilibrium points: A unique disease-free equilibrium (DFE) point $E_0 = (1,\ 0,\ 0,\ 0)$ exists for all values of the parameters while a unique (positive) disease-endemic equilibrium (DEE) point $E_* = (S_*,\ I_*,\ R_*,\ C_*)$ exists if and only if the basic reproduction number $\mathcal{R}_0$ is greater than 1, where

$$\mathcal{R}_0 = \frac{\beta}{\alpha+\mu}.$$

$(P_4)$ Asymptotic stability property of the DFE point: The DFE point is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.
$(P_5)$ Asymptotic stability property of the DEE point: The DEE point is locally asymptotically stable if and only if it exists, i.e., $\mathcal{R}_0 > 1$.

In [**17**] and [**19**], the Mickens' methodology [**20, 21, 22, 23, 24, 25**] has been applied to construct dynamically consistent nonstandard finite difference (NSFD) schemes for (1.1). More clearly, Jódar et al. in [**17**] introduced two NSFD schemes for the model (1.1). The first scheme is given by

$$\begin{aligned}
&\frac{S_{n+1}-S_n}{\phi(\Delta t)} = \mu - \mu S_{n+1} - \beta S_{n+1}I_n + \gamma C_n - \beta S_{n+1} + \beta S_n,\\
&\frac{I_{n+1}-I_n}{\phi(\Delta t)} = \beta S_n I_{n+1} + \sigma\beta C_n I_n - (\mu+\alpha)I_{n+1} + \beta I_n - \beta I_{n+1},\\
&\frac{C_{n+1}-C_n}{\phi(\Delta t)} = \delta(1 - C_n - S_n - I_n) - \beta C_{n+1}I_n - (\mu+\gamma)C_{n+1},\\
&R_{n+1} = 1 - S_{n+1} - I_{n+1} - C_{n+1},
\end{aligned} \tag{1.3}$$

where $\Delta t$ is the step size; $X_n$ is the intended approximation for $X(t_n) = X(t_0+n\Delta t)$ with $X \in \{S,\ I,\ R,\ C\}$, respectively; $\phi(\Delta t)$ is called a denominator function with the property that $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ as $\Delta t \to 0$.
The second NSFD scheme is based on the Mickens' techniques of conservation laws (see [**26, 27**]) and is given by

$$\begin{aligned}
&\frac{S_{n+1}-S_n}{\phi(\Delta t)} = \mu - \mu S_{n+1} - \beta S_{n+1}I_n + \gamma C_n,\\
&\frac{C_{n+1}-C_n}{\phi(\Delta t)} = \delta R_n - \beta C_{n+1}I_n - \mu C_{n+1} - \gamma C_n,\\
&\frac{I_{n+1}-I_n}{\phi(\Delta t)} = \beta S_{n+1}I_n + \sigma\beta C_{n+1}I_n - (\mu+\alpha)I_{n+1},\\
&\frac{R_{n+1}-R_n}{\phi(\Delta t)} = \beta C_{n+1}I_n - \sigma\beta C_{n+1}I_n + \alpha I_{n+1} - \mu R_{n+1} - \delta R_n.
\end{aligned} \tag{1.4}$$

Unlike the NSFD schemes (1.3) and (1.4), Khalsaraei et al. in [**19**] proposed a positive and elementary stable NSFD scheme of the form

$$\begin{aligned}
&\frac{S_{n+1}-S_n}{\phi(\Delta t)} = \mu - \mu\big(2S_{n+1} - S_n\big) - \beta S_{n+1}I_n + \gamma C_n,\\
&\frac{I_{n+1}-I_n}{\phi(\Delta t)} = \beta S_{n+1}I_n + \sigma\beta C_n I_n - \mu\big(2I_{n+1} - I_n\big) - \alpha I_{n+1},\\
&\frac{R_{n+1}-R_n}{\phi(\Delta t)} = (1-\sigma)\beta C_n I_n + \alpha I_n - \mu\big(2R_{n+1} - R_n\big) - \delta R_{n+1},\\
&\frac{C_{n+1}-C_n}{\phi(\Delta t)} = \delta R_{n+1} - \beta C_n I_n - \mu\big(2C_{n+1} - C_n\big) - \gamma C_n.
\end{aligned} \tag{1.5}$$

The positivity of the solutions, the invariance of the total population and the local asymptotic stability (LAS) of the DEE points of (1.3), (1.4) and (1.5) have been analyzed in [**17**] and [**19**]. A common feature of both works is that the stability analysis has only been partially completed for the LAS of the DEE point, whereas there has been no formal proof for the LAS of the DEE point, even though the numerical experiments indicate that it is locally stable for any chosen step-size $\Delta t > 0$.

Motivated and inspired by the above reason, we propose in this work a new class of NSFD schemes for the SIRC model (1.1), which is derived on a weighted approximation for linear terms and the renormalization of the denominator function. By a rigorous mathematical analysis, we give dynamic consistency thresholds that lead to easily-verified conditions under which the formulated NSFD schemes preserve Properties ($P_1$)-($P_5$) regardless of the chosen step sizes. More clearly, the constructed NSFD schemes are dynamically consistent with respect to the properties $(P_1)$-$(P_5)$ if the denominator function is chosen appropriately or the weight is sufficiently large. It is worth noting that thanks to the simple structure of the proposed NSFD schemes, their LAS can be easily established by the linearization method [**13**]. Consequently, the stability analysis conducted in [**17**] and [**19**] is improved.

By error analysis, we show that the constructed NSFD schemes are convergent of order 1 and give an error bound, in which the influence of the denominator function and the weight is analyzed. In fact, the denominator function and the weight can be considered as control parameters to manage the errors. It is shown by numerical experiments that the constructed NSFD schemes are capable of providing numerical approximations with higher-order accuracy compared to the NSFD schemes (1.3)-(1.5). Additionally, they can be easily combined with Richardson's extrapolation technique to produce highly accurate approximate solutions.

The idea of using weighted approximations for the linear terms have been used in [**17**] and in [**19**] to construct (1.3) and (1.5), respectively. However, the weighted approximation proposed in this work is more general, owing the fact that the weight can take arbitrary values rather than being fixed as in (1.3) and (1.5) (see (2.1)). Besides, the influence of the weight on the global errors of the constructed NSFD schemes is analyzed. A general approach for weighted approximations has been proposed by Roeger in [**29, 30**] to construct NSFD schemes for Lotka-Volterra systems. This approach was later applied by Dang and Hoang in [**8**] to design NSFD schemes for a general predator-prey system. In these works, both linear and nonlinear terms in the differential equation models are discretized using weighted approximations. However, this type of discretization leads to very complex NSFD schemes, which are therefore difficult to analyze their dynamical behaviour. On the other hand, as pointed out by Mickens and Dula in [**12**] that the NSFD schemes in [**8, 29, 30**] have not used the full machinery of the NSFD methodology to determine their particular discretizations of the counterpart differential equation models. In Section 2, we show that the weighted approximation for the linear terms makes the LAS analysis of the resulting NSFD schemes easier. As an important consequence, the obtained NSFD schemes are simple, but still ensure the dynamic consistency with respect to the continuous-time model.

In summary, we obtain a new class of dynamically consistent NSFD schemes that are not only simple but also efficient for numerical simulation of the SIRC model (1.1). They also improve the results in [**17**] and [**19**] in both qualitative analysis and computational efficiency aspects. On the other hand, the used approach can be used in constructing efficient NSFD schemes for a wide range of mathematical models, particularly those arising in epidemiology, and more generally, in real-world applications.

The plan of this work is as follows:
The new NSFD schemes are constructed in Section 2, in which their qualitative properties and error analysis are analyzed in detail. In Section 3, we conduct numerical simulations to support the theoretical findings and demonstrate the advantages of the proposed NSFD schemes. Some concluding remarks and discussions are provided in the last section.

## 2. Construction of the nonstandard finite difference schemes

**2.1. Formulation and dynamical analysis.** In this section, we construct dynamically consistent NSFD schemes for (1.1), which use a non-local approximation for the linear terms. These NSFD schemes are proposed in the following form

$$\begin{aligned}
\frac{S_{n+1}-S_n}{\phi(\Delta t)} &= \mu - \beta S_n I_n + \gamma C_n - \mu\big[\tau S_{n+1} + (1-\tau)S_n\big],\\
\frac{I_{n+1}-I_n}{\phi(\Delta t)} &= \beta S_n I_n + \sigma\beta C_n I_n - \alpha I_n - \mu\big[\tau I_{n+1} + (1-\tau)I_n\big],\\
\frac{R_{n+1}-R_n}{\phi(\Delta t)} &= (1-\sigma)\beta C_n I_n + \alpha I_n - \delta R_n - \mu\big[\tau R_{n+1} + (1-\tau)R_n\big],\\
\frac{C_{n+1}-C_n}{\phi(\Delta t)} &= \delta R_n - \beta C_n I_n - \gamma C_n - \mu\big[\tau C_{n+1} + (1-\tau)C_n\big],
\end{aligned} \tag{2.1}$$

where $\tau$ is a real number, which plays the role as a weight.

Remark 2.1. In (2.1), the linear term $\mu Y$ with $Y \in \{S, I, R, C\}$ is nonlocally approximated by $\big[\tau X_{n+1}+(1-\tau)X_n\big]$, whereas the nonlinear terms $SI$ and $CI$, and the other linear terms $Z$ ($Z \in \{S, I, R, C\}$ are all locally approximated by $S_nI_n$, $C_nI_n$ and $Z_n$, respectively. In general, local approximations for nonlinear terms may not guarantee the positivity of the resulting NSFD schemes; however, as pointed out below, the positivity of (2.1) is still guaranteed under suitable conditions on $\phi(\Delta t)$ and $\tau$, because the solutions generated by it remain bounded.

Our main objective at this stage is to analyze dynamics of the NSFD schemes of the form (2.1). For this purpose, we give the following hypothesis:

$$\tau \geq 0, \tag{2.2}$$

and

$$\phi(\Delta t) < \min\big\{\kappa_1,\ \ \kappa_2,\ \ \kappa_3\big\} \quad \text{for all} \quad \Delta t > 0, \tag{2.3}$$

where

$$\kappa_1 := \begin{cases} \dfrac{1}{\beta+\gamma+\mu-\mu\tau} & \text{if} \quad \beta+\gamma+\mu-\mu\tau > 0,\\[2ex] \infty \quad \text{if} \quad \beta+\gamma+\mu-\mu\tau \leq 0. \end{cases}$$

$$\kappa_2 := \begin{cases} \dfrac{1}{\alpha+\mu-\mu\tau} & \text{if} \quad \alpha+\mu-\mu\tau > 0,\\[2ex] \infty \quad \text{if} \quad \alpha+\mu-\mu\tau \leq 0. \end{cases}$$

$$\kappa_3 := \begin{cases} \dfrac{1}{\delta+\mu-\mu\tau} & \text{if} \quad \delta+\mu-\mu\tau > 0,\\[2ex] \infty \quad \text{if} \quad \delta+\mu-\mu\tau \leq 0. \end{cases}$$

From this point onward, we always assume that (2.2) and (2.3) are satisfied. Let us denote by $\mathcal{F}_C$ and $\mathcal{F}_D$ the sets of equilibrium points of (1.1) and (2.1), respectively.

THEOREM 2.2. *The following assertions holds for the NSFD schemes of the form* (2.1)*:*

*(i) The set* $\Omega$ *defined by*

$$\Omega = \Big\{(S, I, R, C) \in \mathbb{R}^4 | S, I, R, C \geq 0;\ \ S + I + R + C = 1\Big\} \tag{2.4}$$

*is a positively invariant set.*

*(ii)* $\mathcal{F}_D = \mathcal{F}_C$ *for all* $\Delta t > 0$.

PROOF. ***Proof of Part (i).*** We prove this part by mathematical induction. Assume that $(S_n, I_n, R_n, C_n) \in \Omega$. We denote by $N_n = S_n + I_n + R_n + C_n$ $(n \geq 0)$ the total population of (2.1). Then, it follows from (2.1) that

$$N_{n+1} = \frac{N_n + \phi\mu - \phi\mu(1-\tau)N_n}{1 + \phi\tau} = N_n + \frac{\phi\mu}{1 + \phi\mu}(1 - N_n), \tag{2.5}$$

which implies that $N_{n+1} = 1$ whenever $N_n = 1$.

Next, we rewrite (2.1) in the form

$$\begin{aligned}
S_{n+1} &= \frac{\phi\mu + S_n\Big[1 - \phi\beta I_n - \phi\mu(1-\tau)\Big] + \phi\gamma C_n}{1 + \phi\mu\tau},\\
I_{n+1} &= \frac{\phi\beta S_n I_n + \phi\sigma\beta C_n I_n + I_n\Big[1 - \phi\alpha - \phi\mu(1-\tau)\Big]}{1 + \phi\mu\tau},\\
R_{n+1} &= \frac{\phi(1-\sigma)\beta C_n I_n + \phi\alpha I_n + R_n\Big[1 - \phi\delta - \phi\mu(1-\tau)\Big]}{1 + \phi\mu\tau},\\
C_{n+1} &= \frac{\phi\delta R_n + C_n\Big[1 - \phi\beta I_n - \phi\gamma - \phi\mu(1-\tau)\Big]}{1 + \phi\mu\tau}.
\end{aligned} \tag{2.6}$$

From (2.3) and the fact that $S_n, I_n, R_n, C_n \in [0, 1]$, we have

$$\begin{aligned}
&1 - \phi\beta I_n - \phi\mu(1-\tau) \geq 1 - \phi(\beta + \mu - \tau) \geq 0,\\
&1 - \phi\alpha - \phi\mu(1-\tau) \geq 1 - \phi(\alpha + \mu - \tau) \geq 0,\\
&1 - \phi\delta - \phi\mu(1-\tau) \geq 1 - \phi(\delta + \mu - \tau) \geq 0,\\
&1 - \phi\beta I_n - \phi\gamma - \phi\mu(1-\tau) \geq 1 - \phi(\beta + \gamma + \mu - \tau) \geq 0.
\end{aligned}$$

Combining this estimate with (2.6) and (2.2) leads to $S_{n+1}, I_{n+1}, R_{n+1}, C_{n+1} \geq 0$. Thus, we conclude by mathematical induction that $S_n, I_n, R_n, C_n \geq 0$ and $S_n + I_n + R_n + C_n = 1$ whenever $S_0, I_0, R_0, C_0 \geq 0$ and $S_0 + I_0 + R_0 + C_0 = 1$. This is the desired conclusion. The proof is complete.

***Proof of Part (ii).*** It is easy to verify that (2.6) can be rewritten in the form

$$\begin{aligned}
S_{n+1} &= S_n + \frac{\phi}{1 + \phi\mu\tau} f_1(S_n, I_n, R_n, C_n),\\
I_{n+1} &= I_n + \frac{\phi}{1 + \phi\mu\tau} f_2(S_n, I_n, R_n, C_n),\\
R_{n+1} &= R_n + \frac{\phi}{1 + \phi\mu\tau} f_3(S_n, I_n, R_n, C_n),\\
C_{n+1} &= C_n + \frac{\phi}{1 + \phi\mu\tau} f_4(S_n, I_n, R_n, C_n),
\end{aligned} \tag{2.7}$$

where the functions $f_i$ $(i = 1, 2, 3, 4)$ are the right-hand side of (1.1). Hence, any equilibrium point of (2.1) is a solution of the system

$$f_i(S, I, R, C) = 0, \quad i = 1, 2, 3, 4.$$

This implies that the sets of equilibrium points of (1.1) and (2.1) are identical. The proof of this part is complete. □

We now apply the method in [**1**] to compute the basic reproduction number of the discrete-time model (2.1). First, we reorder the variables in (2.1) as $x_n = (I_n, R_n, C_n, S_n)$. Then, the DFE point $E_0$ is transformed to $x_0 = (0, 0, 0, 1)$. It follows from (2.6) that the Jacobian matrix of (2.1) evaluated at $x_0$ is given by

$$J = \begin{pmatrix} F + T & 0 \\ A & C \end{pmatrix},$$

where

$$F = \begin{pmatrix} \dfrac{\phi\beta}{1 + \phi\mu\tau} & 0 \\ 0 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} \dfrac{1 - \phi\alpha - \phi\mu(1 - \tau)}{1 + \phi\mu\tau} & 0 \\ 0 & \dfrac{1 - \phi\delta - \phi\mu(1 - \tau)}{1 + \phi\mu\tau} \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & \dfrac{\phi\delta}{1 + \phi\mu\tau} \\ \dfrac{-\phi\beta}{1 + \phi\mu\tau} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \dfrac{1 - \phi\gamma - \phi\mu(1 - \tau)}{1 + \phi\mu\tau} & 0 \\ \dfrac{\phi\gamma}{1 + \phi\mu\tau} & \dfrac{1 - \phi\mu(1 - \tau)}{1 + \phi\mu\tau} \end{pmatrix}.$$

It is easily verified that the submatrices $F$ and $T$ are non-negative, $F + T$ is irreducible, and $\rho(C), \rho(T) < 1$. Thus, the basic reproduction number of (2.1) is computed by

$$\mathcal{R}_0^D = \rho(F[\mathbb{I} - T]^{-1}) = \frac{\beta}{\alpha + \mu},$$

which equals to the basic reproduction number of (1.1).

As a direct consequence of [**1**, Theorem 2.1], we obtain:

LEMMA 2.3. *The DFE point of the discrete-time model* (2.1) *is locally asymptotically stable if* $\mathcal{R}_0^D < 1$ *and is unstable if* $\mathcal{R}_0^D > 1$.

Assume that $\mathcal{R}_0 > 1$. Then, the unique DEE point $E_*$ exists. We need to analyze the LAS of $E_*$ with respect to the NSFD model (2.1). For this purpose, let us denote by $J^C$ the Jacobian matrix of the system (1.1) evaluated at $E_*$. As proven in [**6**], $E_*$ is locally asymptotically stable and

$$Re\lambda^C < 0 \quad \text{for all} \quad \lambda^C \in Spec(J^C),$$

where $Spec(J^C)$ is the set of eigenvalues of $J^C$. For each eigenvalue $\lambda_i^C$ of $Spec(J^C)$ $(i = 1, 2, 3, 4)$, we define

$$\omega_i = \begin{cases} -\dfrac{2(Re\lambda_i^C)}{2(Re\lambda_i^C)\mu\tau + |\lambda_i^C|^2} & \text{if} \quad 2(Re\lambda_i^C)\mu\tau + |\lambda_i^C|^2 > 0, \\ \\ \infty & \text{if} \quad 2(Re\lambda_i^C)\mu\tau + |\lambda_i^C|^2 \leq 0. \end{cases}$$

THEOREM 2.4 (LAS of the DEE point). *Let $\phi(\Delta t)$ be a function that satisfies*

$$\phi(\Delta t) < \phi_{LAS} := \min_i \{\omega_i\} \quad \textit{for all} \quad \Delta t > 0. \tag{2.8}$$

*Then, the DEE point of the system* (2.1) *is locally asymptotically stable if it exists.*

PROOF. Let us denote by $J^D$ the Jacobian matrix of (2.1) evaluated at $E_*$. It follows from (2.6) that

$$J^D = \mathbb{I} + \frac{\phi}{1 + \phi\mu\tau} J^C,$$

which implies that

$$\begin{aligned} \det(z\mathbb{I} - J^D) &= \det\left(z\mathbb{I} - \mathbb{I} - \frac{\phi}{1 + \phi\mu\tau} J^C\right) \\ &= \left(\frac{\phi}{1 + \phi\mu\tau}\right)^4 \det\left[\frac{1 + \phi\mu\tau}{\phi}(z - 1)\mathbb{I} - J^C\right]. \end{aligned}$$

Therefore, $\lambda^D$ is an eigenvalue of $J^D$ if and only if $\lambda^C = \dfrac{1 + \phi\mu\tau}{\phi}(\lambda_D - 1)$ is an eigenvalue of $J^C$. Consequently,

$$\lambda^D = 1 + \frac{\phi}{1 + \phi\mu\tau}\lambda^C.$$

From this relation, we obtain

$$\begin{aligned} |\lambda^D|^2 - 1 &= \left(1 + \frac{\phi}{1 + \phi\mu\tau} Re\lambda^C\right)^2 + \left(\frac{\phi}{1 + \phi\mu\tau} Im\lambda^C\right)^2, \\ &= \frac{\phi}{1 + \phi\mu\tau}\left(2Re\lambda^C + \frac{\phi}{1 + \phi\mu\tau}|\lambda^C|^2\right), \end{aligned} \tag{2.9}$$

which implies that $|\lambda^D|^2 < 1$ if and only if

$$2Re\lambda^C + \frac{\phi}{1 + \phi\mu\tau}|\lambda^C|^2 < 0.$$

Thus, if (2.8) holds, then $|\lambda^D| < 1$ for all $\lambda^D \in Spec(J^D)$. Hence, we obtain the LAS of $E_*$ with respect to (2.1) by the linearization method (see [**13**]). The proof is complete. □

Summarizing the results in this section leads to the following statements for the dynamic consistency of the NSFD schemes of the form (2.1).

THEOREM 2.5 (Dynamically consistent NSFD schemes by renormalization of the denominator functions). *The following statements are true:*

(i) *Assume that $\mathcal{R}_0 < 1$ and* (2.2) *and* (2.3) *hold. Then, the NSFD schemes of the form* (2.1) *preserve Properties* $(P_1)$-$(P_5)$ *of the continuous-time model* (1.1) *for all finite values of the step size.*

*(ii) Assume that* $\mathcal{R}_0 > 1$ *and* (2.2), (2.3) *and* (2.8) *hold. Then, the NSFD schemes of the form* (2.1) *preserve Properties* $(P_1)$-$(P_5)$ *of the continuous-time model* (1.1) *for all finite values of the step size.*

The following theorem is proved similarly to Theorems 2.2 and 2.4 and Lemma 2.3.

THEOREM 2.6 (Influence of the weight). *Let* $\phi(\Delta t)$ *be any positive denominator function with the property that* $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ *as* $\Delta t > 0$. *The following assertions hold for the NSFD model* (2.1)*:*

*(i) Assume that* $\mathcal{R}_0 < 1$. *Then, the NSFD schemes of the form* (2.1) *preserve Properties* $(P_1)$-$(P_5)$ *of the continuous-time model* (1.1) *for all finite values of the step size if*

$$\tau \geq \tau_{DC} := \max\left\{\frac{\beta+\gamma+\mu}{\mu},\ \frac{\alpha+\mu}{\mu},\ \frac{\delta+\mu}{\mu}\right\}. \tag{2.10}$$

*(ii) Assume that* $\mathcal{R}_0 > 1$ *and* (2.10) *is satisfied. Then, the NSFD schemes of the form* (2.1) *preserve Properties* $(P_1)$-$(P_5)$ *of the continuous-time model* (1.1) *for all finite values of the step size if*

$$\tau \geq \tau_{LAS} := \max_{\lambda_i^C \in Spec(J^C)} \left\{\frac{|\lambda_i^C|}{2Re(\lambda_i^C)\mu}\right\}. \tag{2.11}$$

REMARK 2.7. (i) Since the total population is constant, (1.4) and (1.5) can be reduced to three-dimensional systems. Then, the LAS of DEE point can be analyzed based on the approaches used in [**11, 31**] and [**15**]. More clearly, one can determine a positive constant $\phi_S^*$ that plays the role as a stability threshold, such that the DEE point is locally asymptotically stable with respect to the NSFD schemes if $\phi(\Delta t) < \phi_S$ for all $\Delta t > 0$. However, calculating this threshold is not simple; therefore, it is more reasonable to use (2.1) since the primary aim is to construct a dynamically consistent discrete-time model for (1.1). As will be seen in the next section, (2.1) can provide approximations with higher accuracy compared to (1.3)-(1.5) (see Subsection 3.3).
(ii) The conditions imposed on $\phi(\Delta t)$ as stated in Theorem 2.5 can be represented in the form

$$\phi(\Delta t) < \phi_{DC} \quad \text{for all} \quad \Delta t > 0. \tag{2.12}$$

It is important to note that computing $\phi_{DC}$ is straightforward. One of the most suitable denominator function satisfying (2.12) is (see [**20, 21**])

$$\phi(\Delta t) = \frac{1-e^{-D\Delta t}}{D}, \quad D \geq \frac{1}{\phi_{DC}}. \tag{2.13}$$

(iii) The conditions imposed on $\tau$ as stated in Theorem 2.6 can be represented in the form

$$\tau \geq \tau_{DC}, \tag{2.14}$$

which implies that the denominator function $\phi(\Delta t)$ can be chosen arbitrarily if $\tau$ is sufficiently large. Note that determining the value of $\tau_{DC}$ is a simple task. A simple choice for $\phi(\Delta t)$ when $\tau \geq \tau_{DC}$ is $\phi(\Delta t) = \Delta t$. In a numerical example reported in the next section, we will show that the NSFD schemes using this denominator functions can provide better errors compared to those associated with other denominator functions (see Tables 6 and 7).

If $\tau = 0$, then (2.1) is simplified to

$$\begin{aligned}
\frac{S_{n+1} - S_n}{\phi(\Delta t)} &= \mu - \beta S_n I_n + \gamma C_n - \mu S_n,\\
\frac{I_{n+1} - I_n}{\phi(\Delta t)} &= \beta S_n I_n + \sigma\beta C_n I_n - \alpha I_n - \mu I_n,\\
\frac{R_{n+1} - R_n}{\phi(\Delta t)} &= (1-\sigma)\beta C_n I_n + \alpha I_n - \delta R_n - \mu R_n,\\
\frac{C_{n+1} - C_n}{\phi(\Delta t)} &= \delta R_n - \beta C_n I_n - \gamma C_n - \mu C_n.
\end{aligned} \tag{2.15}$$

This scheme is also known as the nonstandard explicit Euler scheme [**9, 11**]. As pointed out in [**17, 19**] that the standard Euler scheme cannot preserve the dynamical properties of the continuous-time model (1.1) for some given step sizes. However, by re-normalizing the denominator function $\phi(\Delta t)$, as presented in Theorem 2.5, the dynamic consistency of (2.15) is guaranteed.

**2.2. Convergence and error bounds.** Before ending this section, we analyze the convergence and error bounds for (2.1). Based on the error analysis techniques presented in [**7, 15, 16**], we can show that (2.1) is convergent of order 1. However, to give a detailed analysis of the influence of the weight $\tau$ and the denominator function $\phi(\Delta t)$ on the errors of (2.1), thereby determining optimal choices, we consider a special family of the denominator given by (2.13). Now, we denote

$$\begin{aligned}
&y(t) = \big(S(t),\, I(t),\, C(t),\, R(t)\big)^T,\\
&y_n = \big(S_n,\, I_n,\, C_n,\, R_n\big)^T,\\
&f(y(t)) = \big(f_1(y(t)),\, f_2(y(t)),\, f_3(y(t)),\, f_4(y(t))\big)^T.
\end{aligned}$$

Then, (1.1) can be rewritten in the form $\dot{y}(t) = f(y(t))$. Also, by using (2.7), one can represent (2.1) in the form

$$y_{n+1} = y_n + g(\Delta t) f(y_n), \tag{2.16}$$

where

$$g(\Delta t) = \frac{\phi(\Delta t)}{1 + \phi(\Delta t)\mu\tau}. \tag{2.17}$$

Lemma 2.8. *The function $g(\Delta t)$ defined in* (2.17) *satisfies*

$$g(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2) \quad as \quad \Delta t \to 0, \tag{2.18}$$

*and for $\Delta t > 0$*

$$|g''(\Delta t)| \le D + 2\mu\tau. \tag{2.19}$$

Proof. It is easy to verify that

$$g'(\Delta t) = \frac{\phi'(\Delta t)}{(1 + \phi(\Delta t)\mu\tau)^2},$$

which implies (2.18) due to the fact that $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ as $\Delta t \to 0$.

On the other hand, we have

$$
\begin{aligned}
|g''(\Delta t)| &= \left| \frac{(1+\phi(\Delta t)\mu\tau)^2\phi''(\Delta t) - 2(\phi'(\Delta t))^2\mu\tau(1+\phi(\Delta t)\mu\tau}{(1+\phi(\Delta t)\mu\tau)^4} \right| \\
&= \left| \frac{\phi''(\Delta t)}{(1+\phi(\Delta t)\mu\tau)^2} - \frac{2(\phi'(\Delta t))^2\mu\tau}{(1+\phi(\Delta t)\mu\tau)^3} \right| \\
&\leq \left| \frac{\phi''(\Delta t)}{(1+\phi(\Delta t)\mu\tau)^2} \right| + \left| \frac{2(\phi'(\Delta t))^2\mu\tau}{(1+\phi(\Delta t)\mu\tau)^3} \right| \\
&\leq |\phi''(\Delta t)| + 2(\phi'(\Delta t))^2\mu\tau \\
&= De^{-D\Delta t} + 2\mu\tau(e^{-2D\Delta t}) \leq D + 2\tau\mu.
\end{aligned}
$$

Therefore, (2.19) is confirmed. The proof is complete. □

As in [**2**], we define the global error of (2.1) by $e_n = y_n - y(t_n)$ for $n \geq 0$. Since $y(t), y_n \in \Omega$, where $\Omega$ is given in (2.4), it is valid to define

$$
L_1 := \max_{y \in \Omega} \left\| \frac{\partial f}{\partial y}(y) \right\|, \qquad L_2 := \max_{y \in \Omega} \left\| \frac{\partial f}{\partial y}(y) f(y) \right\|.
$$

Then, the following estimate hold

$$
\begin{aligned}
&\|f(y_1) - f(y_2)\| \leq L_1 \|y_1 - y_2\|, \quad y_1, y_2 \in \Omega, \\
&\|y''(t)\| \leq L_2, \qquad t > 0.
\end{aligned}
$$

THEOREM 2.9. *The NSFD model* (2.1) *is convergent of order* 1*. Furthermore, the following estimate holds for* $n \geq 0$

$$
\|e_n\| \leq \frac{L_2 + D + 2\tau\mu}{2L_1}(e^{t_{n+1}} - 1)\Delta t. \tag{2.20}
$$

PROOF. First, it follows from Taylor's expansion theorem and (2.16)-(2.18) that

$$
y_{n+1} = y_n + \Delta t f(y_n) + \frac{\Delta t^2}{2} g''(\xi_{\Delta t}), \quad 0 < \xi_{\Delta t} < \Delta t. \tag{2.21}
$$

On the other hand, applying Taylor's expansion theorem for the exact solution $y(t)$ gives

$$
y(t_{n+1}) = y(t_n) + \Delta t f(y(t_n)) + \frac{\Delta t^2}{2} y''(\xi_n), \quad \xi_n \in (t_n, t_{n+1}). \tag{2.22}
$$

By combining (2.21) and (2.22), we obtain

$$
\begin{aligned}
\|e_{n+1}\| &= \|y_{n+1} - y(t_{n+1})\| \\
&= \|(y_n - y(t_n)) + \Delta t[f(y_n) - f(y(t_n))] + \frac{\Delta t^2}{2}(g''(\xi_{\Delta t}) - y''(\xi_n))\| \\
&\leq \|y_n - y(t_n)\| + \Delta t L_1 \|y_n - y(t_n)\| + \frac{\Delta t^2}{2}(D + 2\mu\tau + L_2) \\
&= (1 + L_1\Delta t)\|e_n\| + \psi\Delta t^2, \quad \psi := \frac{D + 2\mu\tau + L_2}{2}.
\end{aligned}
$$

We deduce from this estimate that

$$\begin{aligned}\|e_{n+1}\| &\leq (1+L_1\Delta t)\|e_n\| + \psi\Delta t^2 \leq (1+L_1\Delta t)\big[(1+L_1\Delta t)\|e_{n-1}\| + \psi\Delta t^2\big] + \psi\Delta t^2 \\ &= (1+L_1\Delta t)^2\|e_{n-1}\| + \Big[1+(1+L_1\Delta t)\Big]\psi\Delta t^2 \\ &\leq \cdots \leq (1+L_1\Delta t)^n\|e_0\| + \psi\Delta t^2\sum_{j=0}^{n}(1+L_1\Delta t)^j.\end{aligned} \tag{2.23}$$

Since $e_0 = 0$, (2.23) is simplified to

$$\|e_{n+1}\| \leq \psi\Delta t^2\sum_{j=0}^{n}(1+L_1\Delta t)^j = \frac{\psi\Delta t}{L_1}\Big[(1+L_1\Delta t)^{n+1}-1\Big] \leq \frac{\psi\Delta t}{L_1}e^{L_1\Delta t(n+1)-1}. \tag{2.24}$$

Here, we have used the well-known inequality $e^z \geq 1+z$ for $z \geq 0$ to obtain the last estimate in (2.24). Thus, if follows from (2.24) that

$$\|e_{n+1}\| \leq \frac{\psi\Delta t}{L_1}(e^{L_1t_{n+1}}-1) = \frac{L_2+D+2\mu\tau}{2L_1}(e^{L_1t_{n+1}}-1)\Delta t.$$

Hence, the estimate (2.20) is proved. The proof is complete. □

Remark 2.10. (i) As shown in Theorem 2.9, the error bound is dependent on $D$ and $\tau$, which suggests that $D+2\mu\tau$ should be as small as possible. Note that $D$ is dependent of $\tau$. In Section 3, we will give some numerical examples to show the affects of $D$ and $\tau$ on the errors.
(ii) It is easy to verify that if $\phi(\Delta) = \Delta t$, then (2.20) is simplified to

$$\|e_n\| \leq \frac{L_2+2\tau\mu}{2L_1}(e^{t_{n+1}}-1)\Delta t.$$

This estimate is valid whenever $\tau \geq \tau_{DC}$.

## 3. Numerical experiments

In this section, we report some numerical experiments to support the theoretical findings. Also, the NSFD schemes of the form (2.1) will be compared with (1.3)-(1.5).

### 3.1. Selected parameters and implemented NSFD schemes.

In the numerical simulations reported below, the following sets of the parameters, which are taken from [**17**] but with time unit expressed in weeks instead of years, will be used:

Table 1. The parameters used in numerical examples

| Set | $\alpha$ | $\gamma$ | $\delta$ | $\mu$ | $\sigma$ | $\beta$ | $\mathcal{R}_0$ |
|---|---|---|---|---|---|---|---|
| 1 | $\left(\frac{365}{5}\right)\times\frac{7}{365}$ | $\left(\frac{1}{2}\right)\times\frac{7}{365}$ | $1\times\frac{7}{365}$ | $\left(\frac{1}{50}\right)\times\frac{7}{365}$ | 0.05 | $50\times\frac{7}{365}$ | 0.6847 |
| 2 | $\left(\frac{365}{5}\right)\times\frac{7}{365}$ | $\left(\frac{1}{2}\right)\times\frac{7}{365}$ | $1\times\frac{7}{365}$ | $\left(\frac{1}{50}\right)\times\frac{7}{365}$ | 0.05 | $200\times\frac{7}{365}$ | 2.7390 |

For the two sets of the parameters in Table 1, we determine the equilibrium points, the Jacobian matrices of (1.1) evaluated at the equilibrium points and their eigenvalues as in Table 2.

TABLE 2. Equilibrium point, Jacobian matrices and their eigenvalues

| Set | LAS equilibrium point | $J^C$ and $\sigma(J^C)$ |
|---|---|---|
| 1 | $E_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ | $J^C(E_0) = \begin{pmatrix} -0.0004 & -0.9589 & 0 & 0.0096 \\ 0 & -0.4415 & 0 & 0 \\ 0 & 1.4000 & -0.0196 & 0 \\ 0 & 0 & 0.0192 & -0.0100 \end{pmatrix}$ <br> $\sigma(J^C(E_0)) = \{-0.0004, -0.0100, -0.0196, -0.4415\}$ |
| 2 | $E_* = \begin{pmatrix} 0.3490 \\ 0.0025 \\ 0.3271 \\ 0.3214 \end{pmatrix}$ | $J^C(E_*) = \begin{pmatrix} -0.0027 & -1.3841 & 0 & 0.0096 \\ 0.0023 & -0.0001 & 0 & 0.0001 \\ 0 & 1.7070 & -0.0196 & 0.0022 \\ 0 & -0.3232 & 0.0192 & -0.0123 \end{pmatrix}$ <br> $\sigma(J^C(E_*)) = \{-0.0017 \pm 0.0582i, \ -0.0309, \ -0.0004\}$ |

For each fixed value $\tau$, we denote by "$NSFD^\tau$" the resulting NSFD scheme obtained from (2.1) and by ${\phi_{DC}}^\tau$ and $\tau_{DC}$ the corresponding dynamic consistency thresholds, which are determined from Theorems 2.4 and 2.6, respectively. Also, the corresponding denominator function is given by

$$\phi^\tau(\Delta t) = \frac{1 - e^{-D^\tau \Delta t}}{D^\tau}, \quad D^\tau > \frac{1}{\phi^\tau_{DC}}.$$

**3.2. Numerical dynamics of the NSFD schemes.** In this example, we investigate dynamics of the NSFD schemes of the form (2.1) using the parameters listed in Table 1. The dynamic consistency thresholds for the used NSFD schemes, which correspond to some specific values of $\tau$, are computed as in the Table 3.

TABLE 3. The dynamic consistency thresholds for some values of $\tau$

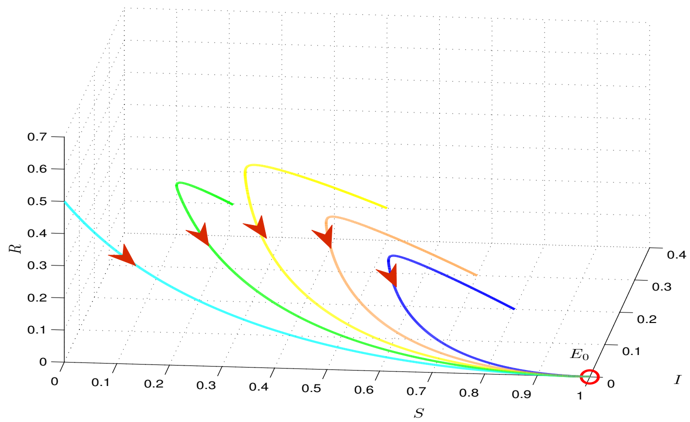
| Set | $\phi^0_{DC}(\tau = 0)$ | $1/(\phi^0_{DC})$ | $\phi^1_{DC}(\tau = 1)$ | $1/(\phi^1_{DC})$ | $\phi^5_{DC}(\tau = 5)$ | $1/(\phi^5_{DC})$ | $\phi^{10}_{DC}(\tau = 10)$ | $1/(\phi^{10}_{DC})$ | $\phi^{100}_{DC}(\tau = 100)$ | $1/(\phi^{100}_{DC})$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.7141 | 1.4004 | 0.7143 | 1.4000 | 0.7151 | 1.3985 | 0.7161 | 1.3965 | 0.7342 | 1.3620 |
| 2 | 0.5187 | 1.9278 | 0.5188 | 1.9274 | 0.5192 | 1.9259 | 0.5198 | 1.9239 | 0.5293 | 1.8894 |

Also, we determine the dynamic consistency thresholds for the weight $\tau$ as in Table 4. Because the denominator function can be chosen arbitrarily, we take $\phi(\Delta t) = \Delta t$.

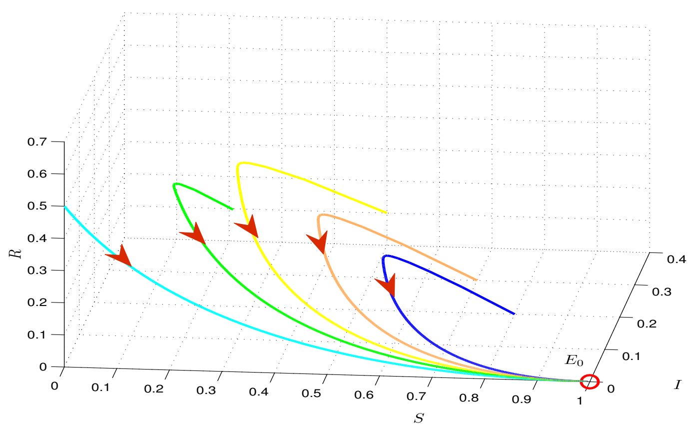
TABLE 4. The dynamic consistency thresholds for the weight $\tau$

| Set of parameters | $\tau_{DC}$ | Denominator function |
|---|---|---|
| 1 | 3651 | $\phi(\Delta t) = \Delta t$ |
| 2 | 5026 | $\phi(\Delta t) = \Delta t$ |

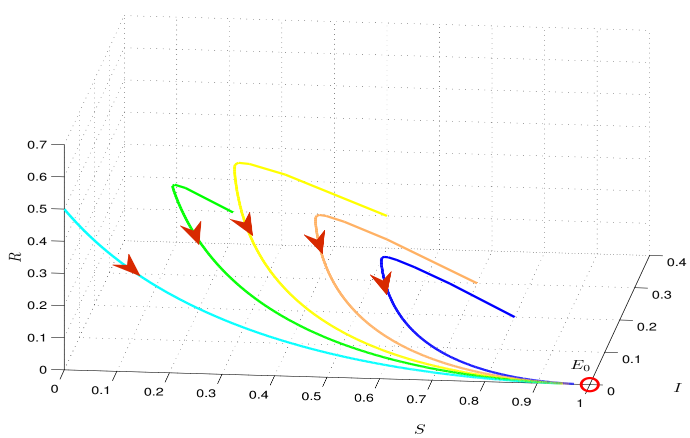
Phase spaces of (1.1) corresponding to some specific sets of initial data, which are generated by (2.1) using three different step sizes $\Delta t \in \{10^{-2}, 1, 2\}$, are depicted in Figures 1-4. From these figures, we see that the used NSFD schemes preserve the dynamical properties of the continuous-time model regardless of chosen step sizes, and their numerical dynamics are independent of the step sizes. This is consistent with the theoretical findings presented in Section 2. Hence, the advantage of (2.1) compared to standard numerical schemes it that is simple and efficient to simulate dynamics of the continuous-time model over long time periods. In the next subsection, we will show that (2.1) can provide better errors compared to the NSFD schemes (1.3)-(1.5).




(A) $\Delta t = 0.01$

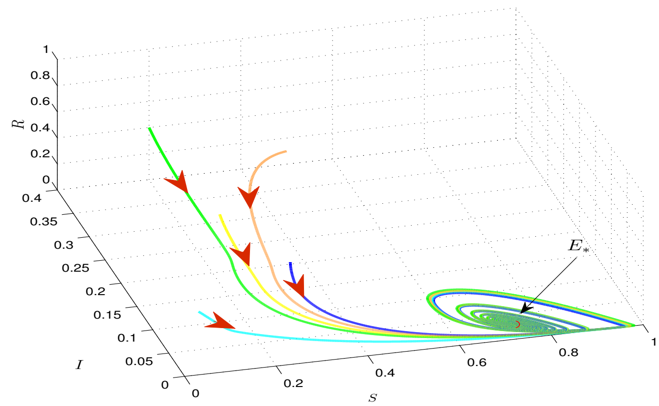



(B) $\Delta t = 1.0$

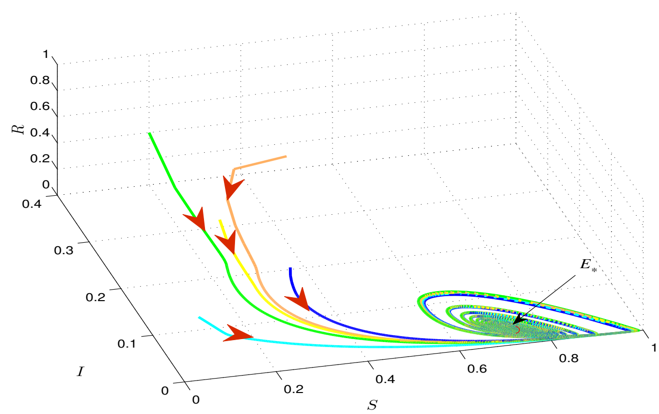



(C) $\Delta t = 2.0$

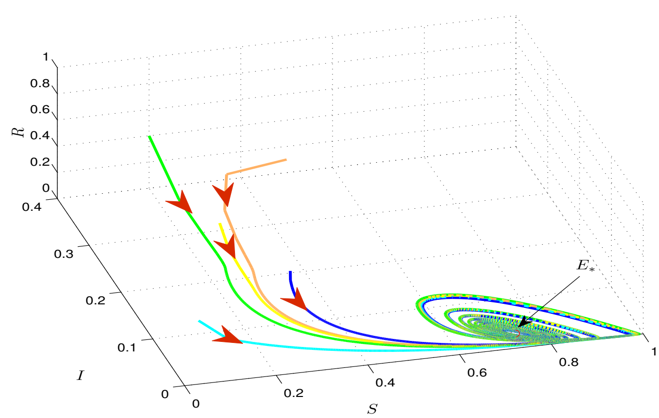
FIGURE 1. The phase spaces of (1.1) generated by the NSFD scheme $NSFD^0(\tau = 0)$, with $\phi(\Delta t) = \dfrac{1 - e^{-1.45\Delta t}}{1.45}$, for Set 1 of the parameters and $t \in [0,\, 10^3]$.




(A) $\Delta t = 0.01$

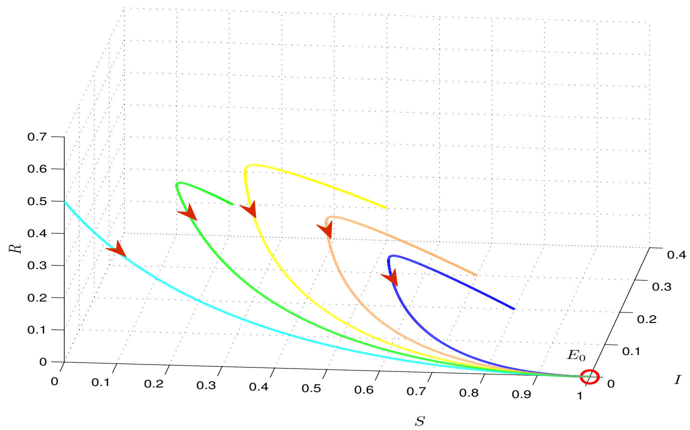



(B) $\Delta t = 1.0$

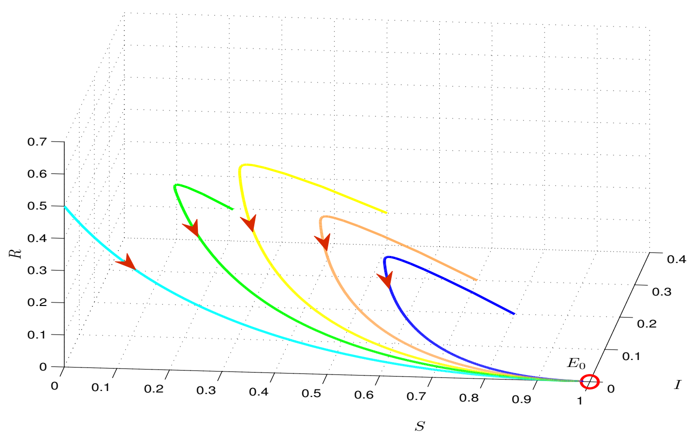



(C) $\Delta t = 2.0$

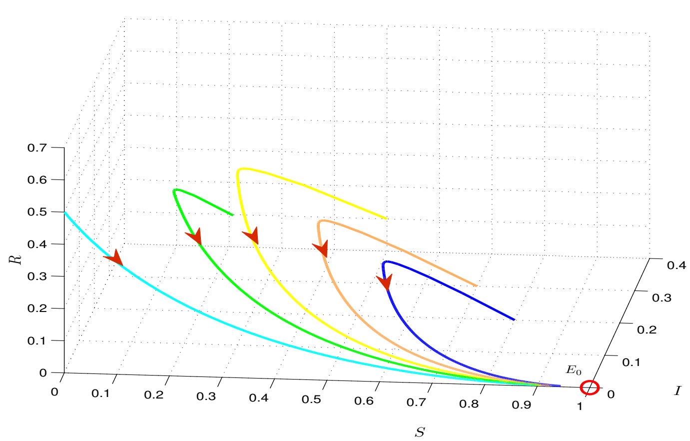
FIGURE 2. The phase spaces of (1.1) generated by the NSFD scheme $NSFD^0(\tau = 0)$, with $\phi(\Delta t) = \dfrac{1 - e^{-1.95\Delta t}}{1.95}$, for Set 2 of the parameters and $t \in [0,\, 10^5]$.




(A) $\Delta t = 0.01$

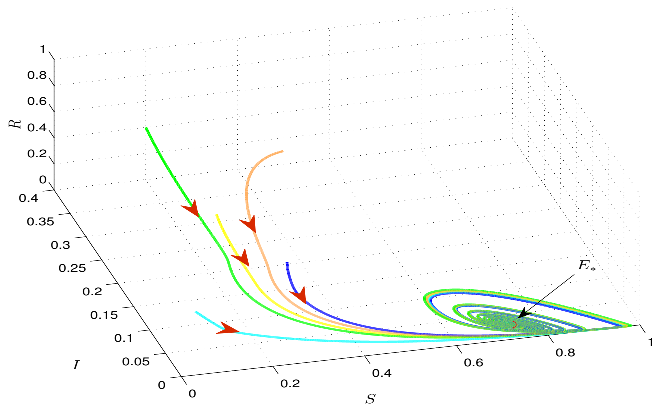



(B) RK2 scheme using $\Delta t = 1.0$

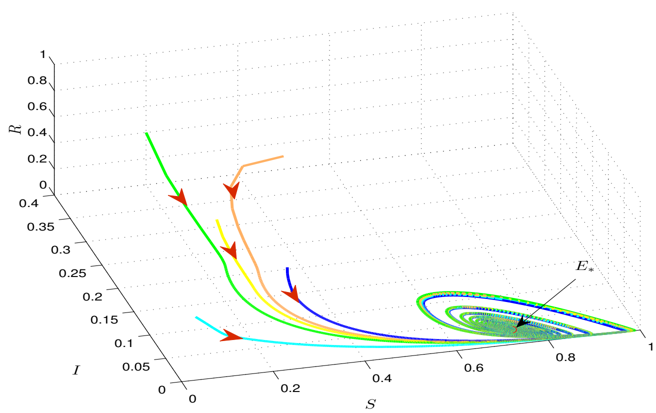



(C) $\Delta t = 1.0$

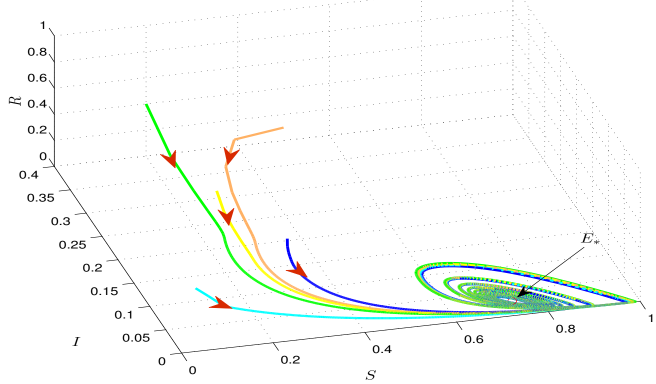
FIGURE 3. The phase spaces of (1.1) generated by the NSFD scheme $NSFD^{3651}(\tau = 3651)$, with $\phi(\Delta t) = \Delta t$, for Set 1 of the parameters and $t \in [0, 10^3]$.




(A) $\Delta t = 0.01$




(B) $\Delta t = 1.0$




(C) $\Delta t = 2.0$

FIGURE 4. The phase spaces of (1.1) generated by the NSFD scheme $NSFD^{5026}(\tau = 5026)$, with $\phi(\Delta t) = \Delta t$, for Set 2 of the parameters and $t \in [0,\, 10^5]$.

**3.3. Error Analysis and comparison of numerical schemes.** The aim of this subsection is to conduct detailed error analyses, in which errors generated by the NSFD schemes (2.1) and (1.3)-(1.5) are estimated and compared. Also, the influence of the denominator function and weight on the errors are analyzed.

For the above purpose, we consider the model (1.1) with the parameters given in Set 1 of Table 1 and initial data $(S(0), I(0), R(0), C(0)) = (0.25, 0.25, 0.25, 0.25)$. Because the exact solution cannot be determined, we admit a numerical approximation, which are generated by the classical four-stage Runge-Kutta method (see [**2**]) using $\Delta t = 10^6$, as a reference solution. Then, we observe the global error estimated at the time $T = 10$ that is computed by

$$\text{error}(\Delta t) = |S(T) - S_N| + |I(t) - I_N| + |R(T) - R_N| + |C(T) - C_N|, \;\; T = 10, \;\; N = \frac{10}{\Delta t}.$$

Besides, rate of convergence (ROC) is estimated by (see [**2**])

$$\text{rate} = \log_{\left(\frac{\Delta t_2}{\Delta t_1}\right)}\left(\frac{\text{error}(\Delta t_2)}{\text{error}(\Delta t_1)}\right).$$

We first consider the global errors generated by the NSFD schemes $NSFD^{\tau}$ with $\tau \in \{0, 1, 5, 10, 100\}$. The results are reported in Table 5. In this table, the denominator functions are given by

$$\phi(\Delta t) = \frac{1 - e^{-D^{\tau}\Delta t}}{D^{\tau}}, \quad D^{\tau} = \frac{1}{\phi_{DC}^{\tau}},$$

where $D^{\tau} = \dfrac{1}{\phi_{DC}^{\tau}}$ is determined as in Table 1.

TABLE 5. The errors and rates of convergence of $NSFD^{\tau}$ with $\tau \in \{0,\ 1\,5\,10,\ 100\}$

| $\Delta t$ | error $NSFD^0$ | ROC | error $NSFD^1$ | ROC | error $NSFD^5$ | $ROC$ | error $NSFD^{10}$ | ROC | error $NSFD^{100}$ | ROC |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.0002e-001 | | 1.0002e-001 | | 1.0005e-001 | | 1.0008e-001 | | 1.0062e-001 | |
| $10^{-1}$ | 1.3504e-002 | 0.8696 | 1.3506e-002 | 0.8696 | 1.3519e-002 | 0.8693 | 1.3534e-002 | 0.8689 | 1.3813e-002 | 0.8624 |
| $10^{-2}$ | 1.3927e-003 | 0.9866 | 1.3931e-003 | 0.9866 | 1.3946e-003 | 0.9865 | 1.3963e-003 | 0.9865 | 1.4290e-003 | 0.9853 |
| $10^{-3}$ | 1.3970e-004 | 0.9987 | 1.3974e-004 | 0.9987 | 1.3989e-004 | 0.9986 | 1.4007e-004 | 0.9986 | 1.4339e-004 | 0.9985 |
| $10^{-4}$ | 1.3975e-005 | 0.9999 | 1.3978e-005 | 0.9999 | 1.3993e-005 | 0.9999 | 1.4011e-005 | 0.9999 | 1.4344e-005 | 0.9999 |
| $10^{-5}$ | 1.3975e-006 | 1.0000 | 1.3979e-006 | 1.0000 | 1.3994e-006 | 1.0000 | 1.4012e-006 | 1.0000 | 1.4345e-006 | 1.0000 |
| $10^{-6}$ | 1.3976e-007 | 1.0000 | 1.3978e-007 | 1.0000 | 1.3993e-007 | 1.0000 | 1.4012e-007 | 1.0000 | 1.4344e-007 | 1.0000 |

Now, we consider (2.1) for large values of $\tau$. More clearly, as computed in Table 4, $\phi(\Delta t)$ can be chosen arbitrarily whenever $\tau \geq \tau_{DC} = 3651$. Hence, we take $\phi(\Delta t) = \Delta t$. The errors and ROC of the $NSFD^{\tau}$ with $\tau \geq 3651$ are given in Table 6.

TABLE 6. The errors and rates of convergence of $NSFD^{\tau}$ with large values of $\tau$

| $\Delta t$ | Error $\tau = 3651$ | ROC | Error $\tau = 4000$ | ROC | Error $\tau = 5000$ | ROC |
|---|---|---|---|---|---|---|
| 1 | 1.2361e-001 | | 1.2791e-001 | | 1.3779e-001 | |
| $10^{-1}$ | 2.4482e-002 | 0.7032 | 2.6491e-002 | 0.6838 | 3.2018e-002 | 0.6338 |
| $10^{-2}$ | 2.7149e-003 | 0.9551 | 2.9664e-003 | 0.9509 | 3.6836e-003 | 0.9391 |
| $10^{-3}$ | 2.7448e-004 | 0.9952 | 3.0023e-004 | 0.9948 | 3.7399e-004 | 0.9934 |
| $10^{-4}$ | 2.7478e-005 | 0.9995 | 3.0060e-005 | 0.9995 | 3.7456e-005 | 0.9993 |
| $10^{-5}$ | 2.7481e-006 | 1.0000 | 3.0063e-006 | 1.0000 | 3.7462e-006 | 0.9999 |
| $10^{-6}$ | 2.7482e-007 | 1.0000 | 3.0064e-007 | 1.0000 | 3.7463e-007 | 1.0000 |

From Tables 5 and 6, we see that the used NSFD schemes are all convergence of order 1 and the errors becomes smaller as the value of $\tau$ decreases. In the computation in Table 7 below, we fix the value of $\tau$ to 3651 but change from $\phi(\Delta t) = \Delta t$ to $\phi(\Delta t) = 1 - e^{-\Delta t}$ (as in [**17**]) and $\phi(\Delta t) = \tanh(\Delta t)$ (as in [**19**]). It is clear that the NSFD scheme associated with $\phi(\Delta t) = t$ provides the better errors.

TABLE 7. The error and rates of convergence of $NSFD^{3651}$ with $\phi(\Delta t) = 1 - e^{-\Delta t}$ and $\phi(\Delta t) = \tanh(\Delta t)$

| $\Delta t$ | $\phi(\Delta t) = 1 - e^{-\Delta t}$ | ROC | $\phi(\Delta t) = \tanh(\Delta t)$ | ROC |
|---|---|---|---|---|
| 1 | 1.3915e-001 | | 1.3293e-001 | |
| $10^{-1}$ | 3.1891e-002 | 0.6398 | 2.4986e-002 | 0.7259 |
| $10^{-2}$ | 3.6527e-003 | 0.9411 | 2.7211e-003 | 0.9629 |
| $10^{-3}$ | 3.7066e-004 | 0.9936 | 2.7454e-004 | 0.9961 |
| $10^{-4}$ | 3.7120e-005 | 0.9994 | 2.7479e-005 | 0.9996 |
| $10^{-5}$ | 3.7126e-006 | 0.9999 | 2.7481e-006 | 1.0000 |
| $10^{-6}$ | 3.7127e-007 | 1.0000 | 2.7482e-007 | 1.0000 |

Before ending this subsection, we consider the errors generated by the NSFD schemes (1.3) and (1.4) with $\phi(\Delta t) = 1 - e^{-\Delta t}$ [**17**] and the NSFD scheme (1.5) with $\phi(\Delta t) = \tanh(\Delta t)$ [**19**]. From Table 8, we see that (1.3)-(1.5) are all convergent of order 1; however, the NSFD schemes $NSFD^{\tau}$ in the form (2.1) yield the better errors. Also, (2.1) is simpler because it can be represented in the matrix form (2.16)-(2.17). In the next section, (2.1) will be combined with Richardson's extrapolation technique to improve its accuracy.

Table 8. The errors and ROC of the NSFD schemes constructed in [**17**] and [**19**]

| $\Delta t$ | NSFD (1.3) [**17**] | ROC | NSFD (1.4) [**17**] | ROC | NSFD (1.5) [**19**] | ROC |
|---|---|---|---|---|---|---|
| 1 | 2.1209e-001 | | 1.9229e-001 | | 1.8934e-001 | |
| $10^{-1}$ | 3.6003e-002 | 0.7702 | 2.8134e-002 | 0.8347 | 2.0219e-002 | 0.9715 |
| $10^{-2}$ | 3.8783e-003 | 0.9677 | 2.9390e-003 | 0.9810 | 1.9957e-003 | 1.0057 |
| $10^{-3}$ | 3.9092e-004 | 0.9966 | 2.9521e-004 | 0.9981 | 1.9926e-004 | 1.0007 |
| $10^{-4}$ | 3.9124e-005 | 0.9997 | 2.9535e-005 | 0.9998 | 1.9923e-005 | 1.0001 |
| $10^{-5}$ | 3.9127e-006 | 1.0000 | 2.9536e-006 | 1.0000 | 1.9923e-006 | 1.0000 |
| $10^{-6}$ | 3.9128e-007 | 1.0000 | 2.9556e-007 | 0.9997 | 1.9914e-007 | 1.0002 |

**3.4. NSFD schemes combined with Richardson's extrapolation technique.** As shown in Subsection 3.3, the NSFD schemes of the form (2.1) are convergent of order 1. In this subsection, we combine (2.1) with Richardson's extrapolation method (see [**3, 5, 18, 28**]) to improve its errors. This approach has been used in [**14**] to improve numerical approximations of population models, which include differential equation models of phytoplankton-nutrient interaction under nutrient recycling and whooping cough in the human population.

Let us denote by $y_n^{\Delta t}$ and $y_n^{\Delta t/2}$ $(n \geq 1)$ the numerical approximations generated by the first-order NSFD model (2.1) with the step sizes $\Delta t$ and $\Delta t/2$, respectively. Then, we define (see [**4**])

$$z_n^{\Delta t} := 2y_n^{\Delta t/2} - y_n^{\Delta t}. \tag{3.1}$$

Then, $\{z_n^{\Delta t}\}$ provides an $\mathcal{O}(\Delta t^2)$ approximate formula for the solutions of (1.1).

By a similar way, we can obtain an $\mathcal{O}(\Delta t^3)$ approximate formula by defining

$$w_n^{\Delta t} := \frac{4z_n^{\Delta t/2} - z_n^{\Delta t}}{3}. \tag{3.2}$$

Generally, it is possible to obtain higher-order approximate formulas by combining lower-order formulas [**4**].

We next examine the continuous-time model (1.1) with Set 1 of the parameters given in Table 1. In the following computations, we apply (3.1) and (3.2) for $(NSFD^1,\ \phi(\Delta t) = (1 - e^{-1.45\Delta t})/1.45)$ and $(NSFD^{3651},\ \phi(\Delta t) = \Delta t)$ to obtain second-order and third-order extrapolated schemes. Table 9 and 10 report the global errors at $T = 10$, which are generated by these resulting extrapolated NSFD schemes. From these tables, we see that the approximate solutions generated by the underlying first-order NSFD schemes are improved.

TABLE 9. The global errors and rates of convergence of the 2nd extrapolated NSFD schemes

| $\Delta t$ | 2nd extrapolated NSFD ($NSFD^1$, $\phi(\Delta t) = (1 - e^{-1.45\Delta t})/1.45$) | ROC | 2nd extrapolated NSFD ($NSFD^{3651}$, $\phi(\Delta t) = \Delta t$) | ROC |
|---|---|---|---|---|
| 1 | 1.9082e-002 | | 4.5969e-002 | |
| 0.5 | 5.5863e-003 | 1.7723 | 1.9296e-002 | 1.2524 |
| 0.25 | 1.5086e-003 | 1.8887 | 6.6992e-003 | 1.5262 |
| 0.2 | 9.8045e-004 | 1.9311 | 4.6145e-003 | 1.6706 |
| $10^{-1}$ | 2.5275e-004 | 1.9557 | 1.3493e-003 | 1.7740 |
| $10^{-2}$ | 2.5982e-006 | 1.9880 | 1.5736e-005 | 1.9332 |
| $10^{-3}$ | 2.6053e-008 | 1.9988 | 1.5992e-007 | 1.9930 |
| $10^{-4}$ | 2.6085e-010 | 1.9995 | 1.6018e-009 | 1.9993 |

TABLE 10. The global errors and rates of convergence of the 3rd extrapolated NSFD schemes

| $\Delta t$ | 3rd extrapolated NSFD ($NSFD^1$, $\phi(\Delta t) = (1 - e^{-1.45\Delta t})/1.45$) | ROC | 3rd extrapolated NSFD ($NSFD^{3651}$, $\phi(\Delta t) = \Delta t$) | ROC |
|---|---|---|---|---|
| 1 | 1.0900e-003 | | 1.0454e-002 | |
| 0.5 | 1.4950e-004 | 2.8661 | 2.5055e-003 | 2.0609 |
| 0.25 | 1.9690e-005 | 2.9246 | 4.6699e-004 | 2.4236 |
| 0.2 | 1.0191e-005 | 2.9516 | 2.6114e-004 | 2.6048 |
| $10^{-1}$ | 1.3017e-006 | 2.9688 | 3.9337e-005 | 2.7309 |
| $10^{-2}$ | 1.3270e-009 | 2.9917 | 4.7155e-008 | 2.9213 |
| $10^{-3}$ | 1.3443e-012 | 2.9944 | 4.8101e-011 | 2.9914 |
| $10^{-4}$ | 6.7436e-013 | 0.2996 | 1.6585e-013 | 2.4624 |

## 4. Concluding remarks and discussions

In this work, we have proposed and analyzed a new class of simple and efficient NSFD schemes for an SIRC epidemic model of influenza A, which was first constructed in [**6**]. The constructed NSFD schemes have used a weighted approximation for the linear terms and the renormalization of the denominator function. By a rigorous mathematical analysis, we have determined the dynamic consistency thresholds that lead to easily-verified conditions ensuring that the NSFD schemes preserve all qualitative dynamical properties of the continuous-time model regardless of the chosen step sizes (Theorems 2.2-2.6). In particular, thanks to the simple structure of the proposed NSFD schemes, their LAS can be easily established by the linearization method. As a consequence, we have obtained a new class of dynamically consistent NSFD schemes, which is not only simple but also efficient for numerical simulation of the SIRC model (1.1). They have also improved the results in [**17**] and [**19**] in terms of both qualitative analysis and computational efficiency. More clearly, the constructed NSFD schemes are capable of providing numerical approximations with higher-order accuracy compared to the NSFD schemes proposed in [**17**] and [**19**]. Additionally, they can be easily combined with Richardson's extrapolation technique to produce highly accurate approximate solutions. On the other hand, the used approach can be used in constructing efficient NSFD schemes for a wide range of mathematical models, particularly those arising in epidemiology, and more generally, in real-world applications.

The dynamic consistency of the proposed NSFD schemes are guaranteed by the renormalization of denominator functions or sufficiently large weights. By the error and convergence analysis in Section 3 and the numerical experiments in Section 4, we identify the influence of the denominator function $\phi(\Delta t)$ and the weigh $\tau$ on the global errors. In fact, $\phi(\Delta t)$ and $\tau$ can be considered as control parameters to manage the errors.

In the near future, we will extend the obtained results to construct effective NSFD schemes for mathematical models of infectious diseases. In particular, higher-order NSFD schemes for the SIRC model (1.1) will be also studied.

**Availability of supporting data:** The authors declare that the data supporting the findings of this study are available within the article.

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