Optimal approximation and sampling recovery in measured-based function spaces

Dinh Dũng[a]

[a]Information Technology Institute, Vietnam National University, Hanoi
144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
Email: dinhzung@gmail.com

August 18, 2025

Abstract

We studied optimal linear approximations in terms of Kolmogorov, linear and sampling n-widths, of functions with mixed smoothness on $\mathbb{R}^d$, endowed with a measure μ. We proved the right convergence rates of these n-widths of the μ-measure-based function classes with Sobolev mixed smoothness $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ in the μ-measure-based Lebesgue space $L_q(\mathbb{R}^d;\mu)$ for some cases of p, q satisfying the condition $1 \le q \le p \le \infty$. The underlying measure μ is defined via a density function of tensor-product exponential weight. We introduced a novel method for constructing linear algorithms which achieve the convergence rates of the Kolmogorov and linear n-widths. The right convergence rates of the sampling n-widths are established through non-constructive methods.

Keywords and Phrases: Optimal linear approximation; Sampling recovery; Widths; Measure-based Sobolev spaces with mixed smoothness; Convergence rate.

MSC (2020): 41A46; 41A25; 41A63; 41A81; 65D05.

1 Introduction

The aim of this paper is to study optimal linear approximations in terms of of Kolmogorov, linear and sampling n-widths for functions with mixed smoothness on $\mathbb{R}^d$, endowed with an exponential positive measure. In particular, we emphasize the right convergence rates for classes with Sobolev mixed smoothness and constructive methods of optimal linear approximation and sampling recovery.

We first introduce measure-based Sobolev spaces of mixed smoothness of functions on $\mathbb{R}^d$. Let

$$w(\boldsymbol{x}) := \bigotimes_{i=1}^d w(x_i), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{1.1}$$

be the tensor product of d copies of the generating univariate exponential weight

$$w(x) := \exp\left(-a|x|^\lambda + b\right), \tag{1.2}$$

where

$$\lambda > 0, \quad a > 0, \quad b \in \mathbb{R}.$$

In what follows, we fix the weight w and hence the parameters λ, a, b.

Let Ω be a Lebesgue-measurable set on $\mathbb{R}^d$. Let μ be the positive measure on Ω defined by

$$\mu(A) := \int_A w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

for any measurable subset A in Ω, i.e., the weight w is the density function of μ. With an abuse, we also write μ in the tensor product from as:

$$\mu(\boldsymbol{x}) := \bigotimes_{i=1}^{d} \mu(x_i), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{1.3}$$

Let $1 \le q \le \infty$. We denote by $L_q(\Omega;\mu)$ the μ-measure-based Lebesgue space of all measurable functions f on Ω such that the norm

$$\|f\|_{L_q(\Omega;\mu)} := \left(\int_\Omega |f(\boldsymbol{x})|^q \mathrm{d}\mu(\boldsymbol{x})\right)^{1/q} = \left(\int_\Omega |f(\boldsymbol{x})|^q w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\right)^{1/q} \tag{1.4}$$

for $1 \le p < \infty$, and assuming f is continuous on Ω,

$$\|f\|_{L_\infty(\Omega;\mu)} := \|f\|_{C(\Omega)} := \sup_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})| \tag{1.5}$$

is finite.

For $r \in \mathbb{N}$ and $1 \le p \le \infty$, the μ-measure-based Sobolev space $W^r_p(\Omega;\mu)$ of mixed smoothness r is defined as the normed space of all functions $f \in L_p(\Omega;\mu)$ such that the weak partial derivative $D^{\boldsymbol{k}} f$ belongs to $L_p(\Omega;\mu)$ for every $\boldsymbol{k} \in \mathbb{N}^d_0$ satisfying the inequality $|\boldsymbol{k}|_\infty \le r$. The norm of a function f in this space is defined by

$$\|f\|_{W^r_p(\Omega;\mu)} := \left(\sum_{|\boldsymbol{k}|_\infty \le r} \|D^{\boldsymbol{k}} f\|^p_{L_p(\Omega;\mu)}\right)^{1/p}. \tag{1.6}$$

The well-known Gaussian-measure-measured spaces $L_p(\mathbb{R}^d;\gamma)$ and $W^r_p(\mathbb{R}^d;\gamma)$ are used in many applications. Here the standard Gaussian measure γ is defined via the density function $w_{\mathrm{g}}(\boldsymbol{x}) := (2\pi)^{-d/2}\exp(-|\boldsymbol{x}|_2^2/2)$.

Next, we introduce concepts of various n-widths, characterizations of optimal linear approximations and sampling recovery of functions.

Let $n \in \mathbb{N}$ and let X be a normed space and Φ a central symmetric compact set in X. Then the Kolmogorov n-width of Φ is defined by

$$d_n(\Phi, X) = \inf_{M_n \in \mathcal{M}(X)} \sup_{f \in \Phi} \inf_{g \in L_n} \|f - M_n(f)\|_X,$$

where $\mathcal{M}_n(X)$ denotes the set of all operators M_n in X such that $M_n(X)$ is a linear subspace in X of dimension at most n. The linear n-width of the set Φ which is defined by

$$\lambda_n(\Phi, X) := \inf_{A_n \in \mathcal{A}_n(X)} \sup_{f \in \Phi} \|f - A_n(f)\|_X,$$

where $\mathcal{A}_n(X)$ denotes the set of all linear operators A_n in X of rank at most n.

The concepts of Kolmogorov n-widths and linear n-widths are related to linear approximation. Namely, $d_n(\Phi, X)$ characterizes the optimal approximation of elements from X by linear subspaces of dimension at most n, and $\lambda_n(\Phi, X)$ by linear methods of rank at most n.

Let X be a normed space of functions on Ω. Given sample points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \Omega$, we consider the approximate recovery of a continuous function f on Ω from their values $f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_k)$ by a linear sampling algorithm S_k on Ω of the form

$$S_k(f) := \sum_{i=1}^{k} f(\boldsymbol{x}_i) h_i, \tag{1.7}$$

where $h_1, \ldots, h_k$ are given continuous functions on Ω. For convenience, we assume that some of the sample points $\boldsymbol{x}_i$ may coincide. The approximation error is measured by the norm $\|f - S_k(f)\|_X$. Denote by $\mathcal{S}_n(\Omega)$ the family of all linear sampling algorithms S_k of the form (1.7) with $k \le n$.

Let $\Phi \subset X$ be a set of continuous functions on Ω. To study the optimality of linear sampling algorithms from $\mathcal{S}_n(\Omega)$ for Φ and their convergence rates we use the (linear) sampling n-width

$$\varrho_n(\Phi, X) := \inf_{S_n \in \mathcal{S}_n(\Omega)} \sup_{f \in \Phi} \|f - S_n(f)\|_X. \tag{1.8}$$

Obviously, we have the inequalities

$$d_n(\Phi, X) \le \lambda_n(\Phi, X) \le \varrho_n(\Phi, X). \tag{1.9}$$

A substantial strand of research is devoted to the problem of optimal unweighted linear approximations and sampling recovery for functions with mixed smoothness on compact domains. A central issue is the determination of optimal convergence rates for various n-widths in linear approximation and sampling recovery of such functions, with particular emphasis on the right convergence rate of these n-widths. For a comprehensive overview and bibliography, see, for example, [3, 9, 17, 20, 5].

Furthermore, the problem of optimal linear approximation and sampling recovery in terms of linear, Kolmogorov and sampling n-widths, of functions on $\mathbb{R}^d$ equipped with standard Gaussian measure has been investigated in [8, 6, 19]. In that context, we have established in [8], in a constructive manner, the right convergence rate of the Kolmogorov and linear n-widths for $1 \le q < p < \infty$,

$$d_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_q(\mathbb{R}^d;\gamma)\big) \asymp \lambda_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_q(\mathbb{R}^d;\gamma)\big) \asymp n^{-r}(\log n)^{r(d-1)}, \qquad (1.10)$$

and

$$d_n\big(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)\big) \asymp \lambda_n\big(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)\big) \asymp n^{-r/2}(\log n)^{r(d-1)/2}. \qquad (1.11)$$

Here and in what follows, for a normed space X of functions on Ω, the boldface $\boldsymbol{X}$ denotes the unit ball in X. In that context, for $2 < p \le \infty$, we have established in [8], in a non-constructive manner, the right convergence rate of the sampling n-widths

$$\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)\big) \asymp n^{-r}(\log n)^{r(d-1)}, \qquad (1.12)$$

and

$$\varrho_n\big(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)\big) \asymp n^{-r/2}(\log n)^{r(d-1)/2} \quad (r \ge 2). \qquad (1.13)$$

In the present work, we extend and generalize the results (1.10) and (1.11) as well as (1.12) and (1.13) associated with the standard Gaussian measure γ to the measure μ with density function of tensor-product exponential weight w.

The main results of the present paper are the right convergence rates of Kolmogorov, linear and sampling n-widths of $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ in the space $L_q(\mathbb{R}^d;\mu)$ in some particular cases of p, q satisfying the condition $1 \le q < p \le \infty$. More precisely, we prove in constructive manner, for $1 \le q < p \le \infty$,

$$d_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big) \asymp n^{-r}(\log n)^{r(d-1)}, \qquad (1.14)$$

and for $1 \le q < p < \infty$,

$$\lambda_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big) \asymp n^{-r}(\log n)^{r(d-1)}. \qquad (1.15)$$

and in a non-constructive manner, for $1 \le q \le 2 < p \le \infty$,

$$\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big) \asymp n^{-r}(\log n)^{r(d-1)}. \qquad (1.16)$$

The linear approximation algorithms that achieve the upper bounds in (1.14) and (1.15) are constructed through a process of *assembling* linear algorithms which are designed for the related Sobolev spaces on the integer-shifted d-cubes which cover $\mathbb{R}^d$. This is a novel method for constructing linear algorithms for approximation of functions on $\mathbb{R}^d$ endowed with measure. It crucially differs from classical methods of weighted polynomial approximation functions based on orthonormal polynomial expansions, see, e.g., [13] for a survey and bibliography on weighted polynomial approximation of function on $\mathbb{R}$.

The convergence rate (1.8) is established in a non-constructive way by using (1.14) and a result on sampling n-widths in the space $L_2(\mathbb{R}^d;\mu)$ proven in [11]. Concerning constructive way, in [7], some sparse-grid linear sampling algorithms which achieve the worse upper bound $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big) \ll n^{-r}(\log n)^{(r+1/2)(d-1)}$ for $1 < q < p < \infty$, have been constructed.

Notice that the right convergence rates of the n-widths in (1.14)–(1.8) coincide with those of the same unweighted n-widths for functions defined on a compact domain (see, e.g., [9, 5]).

It is worth emphasizing that in the results (1.14)–(1.8), the primary parameter λ – the most influential factor shaping the properties of the associated weight w and measure μ – is treated merely as *a positive number*. This substantially distinguishes our framework from the classical theory of weighted approximation (see, e.g., [15], [14], [13]), where it is typically assumed that $\lambda > 1$ in the weight w (Freud-type weight).

The problem of determining the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_p, L_p(\mathbb{T}^d))$ for $1 \le p \le \infty$ has long remained open (see Outstanding open problem 1.4 in [9, Page 12]). From recent results of [11] on inequality between the sampling and Kolmogorov n-widths in reproducing kernel Hilbert spaces (RKHS) one can solve in a non-constructive manner this problem in the case $p = 2$. This result allowed also to establish the right convergence rate (1.13). In the present paper, extending the result (1.13) to the measure μ generated from the Freud-type weight

$$w(x) := \exp\left(-ax^4 + b\right), \ a > 0, \ b \in \mathbb{R}, \tag{1.17}$$

we prove the right convergence rate

$$\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp n^{-3r/4}(\log n)^{3r(d-1)/4}, \tag{1.18}$$

for $1 \le q \le 2$. A key role playing in the proof of this result is a RKHS structure of the space $W^r_2(\mathbb{R}^d;\mu)$, which is derived from some old results [1, 2, 12] on properties of the orthonormal polynomials associated with the weight w^2.

The paper is organized as follows. In Section 2, we prove the convergence rates of the linear and Kolmogorov n-widths in (1.14) and (1.15). In Section 3, we prove the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $1 \le q \le 2 < p \le \infty$ and $1 \le q \le p = 2$.

Notation. Denote $\boldsymbol{x} =: (x_1, ..., x_d)$ for $\boldsymbol{x} \in \mathbb{R}^d$; $\mathbf{1} := (1, ..., 1) \in \mathbb{R}^d$; for $0 < \lambda < \infty$ and $\boldsymbol{x} \in \mathbb{R}^d$, $|\boldsymbol{x}|_\lambda := \left(\sum_{j=1}^d |x_j|^\lambda\right)^{1/\lambda}$ and $|\boldsymbol{x}|_\infty := \max_{1\le j\le d} |x_j|$. We use letter C to denote general positive constants which may take different values. For the quantities $A_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ independent of $n, f, \boldsymbol{k}$ such that $A_n(f, \boldsymbol{k}) \le CB_n(f, \boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f, \boldsymbol{k}) \gg B_n(f, \boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f, \boldsymbol{k}) \asymp B_n(f, \boldsymbol{k})$ if $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k}) \ll A_n(f, \boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G. For a Banach space X, denote by the boldface $\boldsymbol{X}$ the unit ball in X.

2 Optimal linear approximations

In this section, we prove the convergence rates in (1.14) and (1.15). We develop linear sampling algorithms that attain the upper bounds for these convergence rates by assembling a collection of linear methods, each tailored to the corresponding Sobolev spaces defined on the unit d-cubes shifted by integers to cover $\mathbb{R}^d$. Notably, established linear schemes such as Smolyak-type algorithms based on hyperbolic-cross trigonometric approximations have been explicitly constructed for periodic functions with Sobolev mixed smoothness (see, for example, [9, Section 4]). Adapting these constructions to generate linear algorithms for functions on $\mathbb{R}^d$ with Sobolev smoothness requires modifying and extending the underlying framework to fit the non-periodic measure-based setting.

Denote by $\tilde{C}(\mathbb{I}^d)$, $\tilde{L}_q(\mathbb{I}^d)$ and $\tilde{W}^r_p(\mathbb{I}^d)$ the subspaces of $C(\mathbb{I}^d)$, $L_q(\mathbb{I}^d)$ and $W^r_p(\mathbb{I}^d)$, respectively, of all functions f on the d-unit cube $\mathbb{I}^d := [0,1]^d$, which can be extended to the whole $\mathbb{R}^d$ as 1-periodic functions in each variable (denoted again by f). Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$. We use $\delta_n(F,X)$ to denote either $d_n(F,X)$ or $\lambda_n(F,X)$, $\mathcal{F}_n(X)$ to denote either $\mathcal{M}_n(X)$ or $\mathcal{F}_n(X)$, and $F_n \in \mathcal{F}_n(X)$ to denote elements $M_n \in \mathcal{M}_n(X)$ or $A_n \in \mathcal{F}_n(X)$, respectively. Let $F_n \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d)\right)$. Assume it holds that

$$\|f - F_n(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \le C n^{-\alpha} (\log n)^{\beta} \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{2.1}$$

Then based on F_n, we will construct an operator belonging to $\mathcal{F}_n\left(L_q(\mathbb{R}^d;\mu)\right)$, which approximates $f \in W^r_p(\mathbb{R}^d;\mu)$ with the same error bound as in (2.1) for the approximation error measured in the norm of $L_q(\mathbb{R}^d;\mu)$. Such an operator will be constructed by assembling operators which are designed for the related Sobolev spaces on the integer-shifted d-cubes which cover $\mathbb{R}^d$. Let us present this construction.

Fix a number $\theta > 0$ and put $\mathbb{I}^d_\theta := [-\theta, 1+\theta]^d$. Denote by $\tilde{C}(\mathbb{I}^d_\theta)$, $\tilde{L}_q(\mathbb{I}^d_\theta)$ and $\tilde{W}^r_p(\mathbb{I}^d_\theta)$ the subspaces of $C(\mathbb{I}^d_\theta)$, $L_q(\mathbb{I}^d_\theta)$ and $W^r_p(\mathbb{I}^d_\theta)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as $(1+2\theta)$-periodic functions in each variable (denoted again by f). A sampling algorithm $F_n \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d)\right)$ induces the operator $F_{\theta,n} \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d_\theta)\right)$ defined for a function $f \in \tilde{C}(\mathbb{I}^d_\theta)$ by

$$F_{\theta,n}(f)(\boldsymbol{x}) := F_n(f(\boldsymbol{x}/(1+2\theta) + \theta\boldsymbol{1})), \quad \boldsymbol{x} \in \mathbb{I}^d_\theta. \tag{2.2}$$

From (2.1) it follows that

$$\|f - F_{\theta,n}(f)\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \le C n^{-\alpha} (\log n)^{\beta} \|f\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d_\theta).$$

We define for $n \in \mathbb{N}$,

$$m_n := \left(\delta^{-1}\alpha \log n\right)^{1/\lambda}, \tag{2.3}$$

and for $\boldsymbol{k} \in \mathbb{Z}^d$,

$$n_{\boldsymbol{k}} := \begin{cases} \lfloor \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|_\lambda^\lambda} + 1 \rfloor & \text{if } |\boldsymbol{k}|_\lambda < m_n, \\ 0 & \text{if } |\boldsymbol{k}|_\lambda \ge m_n, \end{cases} \tag{2.4}$$

where an appropriate fixed value of parameter $\delta > 0$ will be chosen below,

$$\varrho^{-1} := V_\lambda^d \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha} s^\lambda} < \infty,$$

and V_λ^d denote the volume of the set

$$B_\lambda^d := \left\{ \boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}|_\lambda \le 1 \right\}.$$

We write $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \boldsymbol{k} + \mathbb{I}^d_\theta$ for $\boldsymbol{k} \in \mathbb{Z}^d$, and denote by $f_{\theta,\boldsymbol{k}}$ the restriction of f on $\mathbb{I}^d_{\theta,\boldsymbol{k}}$ for a function f on $\mathbb{R}^d$.

It is well-known that one can constructively define a unit partition $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ such that

(i) $\varphi_{\boldsymbol{k}} \in C_0^\infty(\mathbb{R}^d)$ and $0 \le \varphi_{\boldsymbol{k}}(\boldsymbol{x}) \le 1, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \boldsymbol{k} \in \mathbb{Z}^d$;

(ii) $\operatorname{supp} \varphi_{\boldsymbol{k}}$ are contained in the interior of $\mathbb{I}^d_{\theta,\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$;

(iii) $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_{\boldsymbol{k}}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^d$;

(iv) $\|\varphi_{\boldsymbol{k}}\|_{W_p^r(\mathbb{I}^d_{\theta,\boldsymbol{k}})} \le C_{r,d,\theta}, \quad \boldsymbol{k} \in \mathbb{Z}^d$,

(see, e.g., [18, Chapter VI, 1.3]).

We define the $(1+2\theta)$-periodic functions $\tilde{f}_{\theta,\boldsymbol{k}}$ on $\mathbb{I}^d_\theta$ for $\boldsymbol{k} \in \mathbb{Z}^d$ by

$$\tilde{f}_{\theta,\boldsymbol{k}} := f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}).$$

For $n \in \mathbb{N}$, taking the sequence $(n_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ given as in (2.4) and satisfying the condition

$$\sum_{|\boldsymbol{k}|_\lambda < m_n} n_{\boldsymbol{k}} \le n,$$

we define the linear sampling algorithm $F^\mu_{\theta,n} \in \mathcal{F}_n\left(L_q(\mathbb{R}^d;\mu)\right)$ generated from F_n by

$$\left(F^\mu_{\theta,n} f\right)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|<m_n} \left(F_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\boldsymbol{x} - \boldsymbol{k}). \tag{2.5}$$

where $F_{\theta,n_{\boldsymbol{k}}} \in \mathcal{F}_{n_{\boldsymbol{k}}}\left(\tilde{L}_q(\mathbb{I}^d_\theta)\right)$ are defined by (2.2).

Theorem 2.1 *Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$, $\theta > 0$. Assume that for any $n \in \mathbb{N}$, there is an operator $F_n \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d)\right)$ such that the convergence rate (2.1) holds. Then for any $n \in \mathbb{N}$, based on this operator, one can construct an operator $F^\mu_{\theta,n} \in \mathcal{F}_n\left(L_q(\mathbb{R}^d;\mu)\right)$ of the form (2.5) so that*

$$\|f - F^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \le Cn^{-\alpha}(\log n)^\beta \|f\|_{W_p^r(\mathbb{R}^d;\mu)}, \quad f \in W_p^r(\mathbb{R}^d;\mu). \tag{2.6}$$

Proof. This theorem can be established in a manner analogous to [7, Theorem 2.1], with a few necessary modifications. For completeness, we present the proof. We auxiliarily present a function in $W_p^r(\mathbb{R}^d;\mu)$ as a sum of functions on $\mathbb{R}^d$ having support contained in integer translations of the d-cube $\mathbb{I}_\theta^d$. A suitable sampling algorithm for for $W_p^r(\mathbb{R}^d;\mu)$ can be constructed as the sum of integer-translated dilations of F_n. From the items (ii) and (iii) in the definition of unite partition it follows that

$$f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}, \tag{2.7}$$

where $f_{\theta,\boldsymbol{k}}$ denotes the restriction of f to $\mathbb{I}_{\theta,\boldsymbol{k}}^d$. Hence we obtain

$$\begin{aligned}\|f - F_{\theta,n}^{\mu}(f)\|_{L_q(\mathbb{R}^d;\mu)} \le &\sum_{|\boldsymbol{k}|_\lambda < m_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}_{\theta,\boldsymbol{k}}^d;\mu)} \\ &+ \sum_{|\boldsymbol{k}|_\lambda \ge m_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}_{\theta,\boldsymbol{k}}^d;\mu)}.\end{aligned} \tag{2.8}$$

With the fixed θ, there exists a constant C depending p, λ, a, θ only such that $w^{-1/p}(\boldsymbol{x}) \le w^{-1/p}(\boldsymbol{k})$ for every $\boldsymbol{x} \in \mathbb{I}_{\theta,\boldsymbol{k}}^d$. Consequently,

$$\|f_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}_p^r(\mathbb{I}_\theta^d)} \le C w^{-1/p}(\boldsymbol{k})\|f\|_{W_p^r(\mathbb{I}_{\theta,\boldsymbol{k}}^d;\mu)} \le C w^{-1/p}(\boldsymbol{k})\|f\|_{W_p^r(\mathbb{R}^d;\mu)}. \tag{2.9}$$

Because $W_p^r(\mathbb{I}^d)$ is a multiplication algebra (see [16, Theorem 3.16]), from (2.9) and property (iv) of the unit partition $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, we derive $\tilde{f}_{\theta,\boldsymbol{k}} := f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \tilde{W}_p^r(\mathbb{I}_\theta^d)$, and

$$\begin{aligned}\|\tilde{f}_{\theta,\boldsymbol{k}}\|_{\tilde{W}_p^r(\mathbb{I}_\theta^d)} &\le C\|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}_p^r(\mathbb{I}_\theta^d)} \cdot \|\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}_p^r(\mathbb{I}_\theta^d)} \\ &\le C w^{-1/p}(\boldsymbol{k})\|f\|_{W_p^r(\mathbb{R}^d;\mu)}.\end{aligned} \tag{2.10}$$

Analogously,

$$\left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}_{\theta,\boldsymbol{k}}^d;\mu)} \le C w^{1/q}(\boldsymbol{k}) \left\|\tilde{f}_{\theta,\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)\right\|_{\tilde{L}_q(\mathbb{I}_\theta^d)}. \tag{2.11}$$

Since $q < p$, from the definition of the weight w in (1.2) it follows that there are numbers C and $0 < \delta' < a\,(1/q - 1/p)$ such that

$$w^{1/q-1/p}(\boldsymbol{k}) \le C e^{-\delta'|\boldsymbol{k}|_\lambda^\lambda}, \quad \boldsymbol{k} \in \mathbb{Z}^d. \tag{2.12}$$

We select a number δ in (2.3), satisfying the condition

$$\delta \max\,(1, a/\alpha) < \delta'. \tag{2.13}$$

Firstly, with this selection of δ, let us verify that $F_{\theta,n}^{\mu} \in \mathcal{F}_n\left(L_q(\mathbb{R}^d;\mu)\right)$. Indeed, putting

$$B_\lambda^d(s) := \left\{\boldsymbol{x} \in \mathbb{R}^d : \, |\boldsymbol{x}|_\lambda \le s\right\},$$

and denoting by $V_\lambda^d(s)$ the volume of $B_\lambda^d(s)$, we have

$$\begin{aligned} m \le \sum_{|\boldsymbol{k}|_\lambda < m_n} n_{\boldsymbol{k}} &\le \sum_{|\boldsymbol{k}|=1}^{\lfloor m_n \rfloor} \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|_\lambda^\lambda} \le n\varrho \sum_{s=0}^{\lfloor m_n \rfloor} \sum_{\boldsymbol{k}\in B_\lambda^d(s)} e^{-\frac{a\delta}{\alpha}s^\lambda} \\ &\ll n\varrho \sum_{s=0}^{\lfloor m_n \rfloor} V_\lambda^d(s) e^{-\frac{a\delta}{\alpha}s^\lambda} \ll n\varrho\, V_\lambda^d \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha}s^\lambda} \le n. \end{aligned} \tag{2.14}$$

Secondly, we establish the upper bound (2.6). By (2.4), (2.1) (2.10) and (2.11) we deduce the estimates

$$\begin{aligned} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll w^{1/q}(\boldsymbol{k}) \left\| \tilde{f}_{\theta,\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)\right\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \\ &\ll w^{1/q}(\boldsymbol{k}) n_{\boldsymbol{k}}^{-\alpha}(\log n_{\boldsymbol{k}})^\beta \|f(\cdot+\boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot+\boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \\ &\ll w^{1/q}(\boldsymbol{k}) w^{-1/p}(\boldsymbol{k}) n^{-\alpha}(\log n)^\beta e^{\delta\frac{a}{\alpha}|\boldsymbol{k}|_\lambda^\lambda} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \end{aligned}$$

where the numbers $n_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$, are defined as in (2.4). Hence, by (2.12) we get

$$\left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \ll e^{-\varepsilon|\boldsymbol{k}|_\lambda^\lambda} n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)},$$

where $\varepsilon := \delta' - \delta a/\alpha > 0$ due to (2.13). This in a similar manner as (2.14) implies that

$$\begin{aligned} \sum_{|\boldsymbol{k}|_\lambda < m_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - F_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll \sum_{|\boldsymbol{k}|_\lambda < m_n} e^{-\varepsilon|\boldsymbol{k}|_\lambda^\lambda} n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\ &\le n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}| < m_n} e^{-\varepsilon|\boldsymbol{k}|_\lambda^\lambda} \\ &\ll n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \end{aligned}$$

Again in an analogous manner as (2.14) we have by (2.3), (2.12) and the inequality $\delta'/\delta > 1$ provided with (2.13),

$$\begin{aligned} \sum_{|\boldsymbol{k}|_\lambda \ge m_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll \sum_{|\boldsymbol{k}| \ge m_n} w^{1/q}(\boldsymbol{k}) w^{-1/p}(\boldsymbol{k}) \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\ &\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}|_\lambda \ge m_n} e^{-\delta'|\boldsymbol{k}|_\lambda^\lambda} \\ &\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{s \ge \lfloor m_n \rfloor} V_\lambda^d(s) e^{-\delta' s^\lambda} \\ &\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} V_\lambda^d \sum_{s \ge \lfloor m_n \rfloor} s^d e^{-\delta' s^\lambda} \\ &\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} m_n^d e^{-\delta' m_n^\lambda} \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha}s^\lambda} \\ &\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} (\log n)^{d/\lambda} e^{-\delta'\alpha \log n/\delta} \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha}s^\lambda} \\ &\ll n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \end{aligned}$$

From the last two estimates and (2.8) we prove (2.6). ∎

Lemma 2.2 *Let* $1 \le q < p < \infty$ *For every* $n \in \mathbb{N}$, *one can construct an operator* $F_n \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d)\right)$ *such that*

$$\delta_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \asymp \sup_{f \in \tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d)} \|f - F_n(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \asymp n^{-r}(\log n)^{(d-1)r}. \tag{2.15}$$

Moreover, the statement still holds true if $\delta_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) = d_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d))$ *and* $p = \infty$.

Such an operator $F_n \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d)\right)$ in this lemma can be constructed via Smolyak algorithms based on hyperbolic cross approximations. For detail on the proof of Lemma 2.2 and the hyperbolic cross approximation see, e.g., in [9, Section 4].

Theorem 2.3 *Let* $1 \le q < p < \infty$, $\theta > 0$. *Then for any* $n \in \mathbb{N}$, *based on the operator* $F_n \in \mathcal{F}_n\left(\tilde{L}_q(\mathbb{I}^d)\right)$ *in Lemma 2.2, one can construct the operator* $F^\mu_{\theta,n} \in \mathcal{F}_n\left(L_q(\mathbb{R}^d;\mu)\right)$ *as in (2.5) so that there holds the right convergence rate*

$$\delta_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \|f - F^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \asymp n^{-r}(\log n)^{(d-1)r}. \tag{2.16}$$

Moreover, the statement still holds true if $\delta_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) = d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ *and* $p = \infty$.

Proof. The upper bounds in (2.16) follow from Lemma 2.2 and Theorem 2.1 with $\alpha = r$ and $\beta = (d-1)r$. Let us prove the lower bound in (2.16). If f is a 1-periodic function on $\mathbb{R}^d$ and $f \in \tilde{W}^r_p(\mathbb{I}^d)$, then one can immediately derive that

$$\|f\|_{W^r_p(\mathbb{R}^d;\mu)} \ll \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}$$

for $1 < p \le \infty$. On the other hand,

$$\|f\|_{\tilde{L}_q(\mathbb{I}^d)} \ll \|f\|_{L_q(\mathbb{R}^d;\mu)}.$$

Hence we get by Lemma 2.2

$$\delta_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \gg \delta_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \gg n^{-r}(\log n)^{r(d-1)}.$$

∎

3 Convergence rate of sampling widths

In this section, we prove the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $d \ge 2$ and $1 < q \le 2 < p < \infty$. We also prove the RKHS structure of the space $W^r_2(\mathbb{R}^d;\mu)$, and the right convergence rate of the sampling n-widths $\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\mu), L_p(\mathbb{R}^d;\mu))$ $(d \ge 2)$ for $1 < q \le p = 2$ in the particular case of measure μ when w is the univariate Freud-type weight given by (1.17).

Assumption 3.1 *Let F be a class of complex-valued functions on the measurable set $\Omega \subset \mathbb{R}^d$. We say that F satisfies Assumption 3.1, if there is a metric on F such that F is continuously embedded into the separable space with measure $L_2(\Omega;\mu)$, and for each $\boldsymbol{x} \in \Omega$, the evaluation functional $f \mapsto f(\boldsymbol{x})$ is continuous on F.*

The following lemma is a consequence of [11, Corollary 4].

Lemma 3.2 *Assume that F satisfies Assumption 3.1 and that*

$$d_n(F, L_2(\Omega;\mu)) \ \ll \ n^{-\alpha} \log^{-\beta} n \tag{3.1}$$

for some $\alpha > 1/2$ and $\beta \in \mathbb{R}$. Then

$$\varrho_n(F, L_2(\Omega;\mu)) \ \ll \ n^{-\alpha} \log^{-\beta} n. \tag{3.2}$$

Theorem 3.3 *Let $r \in \mathbb{N}$ and $1 \le q \le 2 < p \le \infty$. Then there holds the right convergence rate*

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp n^{-r}(\log n)^{(d-1)r}. \tag{3.3}$$

Proof. The lower bound in (3.3) is implied from the inequalities (1.9) and Theorem 2.3 . By the norm inequality $\|\cdot\|_{L_q(\mathbb{R}^d;\mu)} \ll \|\cdot\|_{L_2(\mathbb{R}^d;\mu)}$ for $1 \le q \le 2$, it is sufficient to prove the upper bound in (3.3) for $q = 2$. By (2.16) we have that

$$d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \ll n^{-r}(\log n)^{(d-1)r}. \tag{3.4}$$

Notice that the separable normed space $W^r_p(\mathbb{R}^d;\mu)$ is continuously embedded into $L_2(\mathbb{R}^d;\mu)$, and the evaluation functional $f \mapsto f(\boldsymbol{x})$ is continuous on the space $W^r_p(\mathbb{R}^d;\mu)$ for each $\boldsymbol{x} \in \mathbb{R}^d$. This means that the set $\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)$ satisfies Assumption 3.1. By Lemma 3.2 and (3.4) we prove the upper bound:

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \ll d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \ll n^{-r}(\log n)^{(d-1)r}.$$

□

Let $(\phi_m)_{m\in\mathbb{N}_0}$ be the sequence of orthonormal polynomials with respect to the univariate Freud-type weight

$$v(x) := w^2(x) = \exp\left(-2a|x|^\lambda + 2b\right). \tag{3.5}$$

For every multi-degree $\boldsymbol{k} \in \mathbb{N}^d_0$, the d-variate polynomial $\phi_{\boldsymbol{k}}$, we define

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{d} \phi_{k_j}(x_j), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

The polynomials $\{\phi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}^d_0}$ constitute an orthonormal basis of the Hilbert space $L_2(\mathbb{R}^d;\mu)$, and every $f \in L_2(\mathbb{R}^d;\mu)$ can be represented by the polynomial series

$$f = \sum_{\boldsymbol{k}\in\mathbb{N}^d_0} \hat{f}(\boldsymbol{k})\phi_{\boldsymbol{k}} \ \text{ with } \ \hat{f}(\boldsymbol{k}) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \phi_{\boldsymbol{k}}(\boldsymbol{x}) w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \tag{3.6}$$

converging in the norm of $L_2(\mathbb{R}^d;\mu)$. Moreover, there holds Parseval's identity

$$\|f\|^2_{L_2(\mathbb{R}^d;\mu)} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} |\hat{f}(\boldsymbol{k})|^2.$$

For $r>0$ and $\boldsymbol{k}\in\mathbb{N}_0^d$, we define

$$\rho_{\lambda,r,\boldsymbol{k}} := \prod_{j=1}^d (k_j+1)^{r_\lambda}.$$

Denote by $\mathcal{H}^{r_\lambda}(\mathbb{R}^d)$ the space of all functions $f\in L_2(\mathbb{R}^d;\mu)$ represented by the series (3.6) for which the norm

$$\|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^d} |\rho_{\lambda,r,\boldsymbol{k}}\hat{f}(\boldsymbol{k})|^2\right)^{1/2}$$

is finite. Notice that for $r>\frac{\lambda}{2(\lambda-1)}$, we have $r_\lambda>1/2$ and therefore, $\mathcal{H}^{r_\lambda}(\mathbb{R}^d)$ is a separable RKHS with the reproducing kernel

$$K(\boldsymbol{x},\boldsymbol{y}) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \rho^{-2}_{\lambda,r,\boldsymbol{k}}\phi_{\boldsymbol{k}}(\boldsymbol{x})\phi_{\boldsymbol{k}}(\boldsymbol{y}). \tag{3.7}$$

Theorem 3.4 *We have for any $\lambda>1$ and $r>\frac{\lambda}{2(\lambda-1)}$,*

$$\varrho_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{3.8}$$

Proof. The proof of theorem is similar to the proof of [8, Theorem 3.5]. For completeness, we shortly perform it. We need the following result on Kolmogorov widths $d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu))$ (see, e.g., [9, page 45] for the definition of Kolmogorov widths) which can be proven in the same manner as the proof of [8, (3.18)]. We have for $r>0$,

$$d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{3.9}$$

The lower bound of (3.8) follows from (3.9) and the inequality

$$\varrho_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ge d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)).$$

We check the upper bound of (3.8). By (3.9) we get

$$d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}. \tag{3.10}$$

From the orthonormality of the system $\{\phi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{N}_0^d}$ it is easy to see that $K(\boldsymbol{x},\boldsymbol{y})$ satisfies the finite trace assumption

$$\int_{\mathbb{R}^d} K(\boldsymbol{x},\boldsymbol{x})w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \ < \ \infty. \tag{3.11}$$

Hence by (3.10) and [11, Corollary 2] we obtain

$$\varrho_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ll d_n(\boldsymbol{\mathcal{H}}^{r_\lambda}(\mathbb{R}^d), L_2(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}.$$

□

Lemma 3.5 *Let λ be an even integer. Then we have the inequality*

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \ll \|f\|_{\mathcal{H}^{r\lambda}(\mathbb{R})}, \quad f \in W_2^r(\mathbb{R};\mu). \tag{3.12}$$

Proof. We will use the following representation of the derivative of the polynomials ϕ_m for $m \in \mathbb{N}$, which was proven in [1, Lemma 3]:

$$\phi_m' = \sum_{k=m-\lambda+1}^{m-1} a_{m,k}\phi_k, \tag{3.13}$$

where

$$a_{m,k} := \lambda \int_{\mathbb{R}} \phi_m(x)\phi_k(x)x^{\lambda-1}w(x)\mathrm{d}x \tag{3.14}$$

satisfying the inequalities

$$|a_{m,k}| \le Cm^{1-1/\lambda} \tag{3.15}$$

for some positive constant C independent of m, k.

We first prove the lemma for $r = 1$. Given $f \in W_2^1(\mathbb{R};\mu)$, we denote $g := f'$. From Parseval's identity and the equality (3.13) we have

$$g = \sum_{m\in\mathbb{N}_0} \hat{f}(m)\phi_m' = \sum_{m\in\mathbb{N}_0} \hat{f}(m) \sum_{k=m-\lambda+1}^{m-1} a_{m,k}\phi_k = \sum_{k\in\mathbb{N}} \phi_k \sum_{m=k+1}^{k+\lambda-1} a_{m,k}\hat{f}(m), \tag{3.16}$$

and consequently, for every $k \in \mathbb{N}$,

$$\hat{g}(k) = \sum_{m=k+1}^{k+\lambda-1} a_{m,k}\hat{f}(m). \tag{3.17}$$

Hence, by (3.15)

$$|\hat{g}(k)|^2 \le (\lambda-1)\sum_{m=k+1}^{k+\lambda-1} |a_{m,k}\hat{f}(m)|^2 \le C(\lambda-1)\sum_{m=k+1}^{k+\lambda-1} |m^{1-1/\lambda}\hat{f}(m)|^2. \tag{3.18}$$

This and Parseval's identity yield

$$\begin{aligned}
\|g\|^2_{L_2(\mathbb{R};\mu)} &= \sum_{k\in\mathbb{N}_0} |\hat{g}(k)|^2 \le C(\lambda-1)\sum_{k\in\mathbb{N}_0}\sum_{m=k+1}^{k+\lambda-1} |m^{1-1/\lambda}\hat{f}(m)|^2 \\
&\le C_\lambda \sum_{k\in\mathbb{N}_0} |k^{1-1/\lambda}|\hat{f}(k)|^2 \\
&\ll \sum_{k\in\mathbb{N}_0} |\rho_{\lambda,1,k}\hat{f}(k)|^2 = \|f\|^2_{\mathcal{H}^1(\mathbb{R})},
\end{aligned} \tag{3.19}$$

which implies

$$\|f\|_{W_2^1(\mathbb{R};\mu)} = \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{L_2(\mathbb{R};\mu)} \ll \|f\|_{\mathcal{H}^1(\mathbb{R})}\,. \tag{3.20}$$

This proves the lemma in the case $r = 1$. In the general case it can be obtained by induction on r. Assuming that (3.12) is true for $r-1$, we prove it for r. Again, given $f \in W_2^r(\mathbb{R};\mu)$, we denote $g := f' \in W_2^{r-1}(\mathbb{R};\mu)$. From the induction assumption, in a way similar to (3.19) we derive

$$\begin{aligned}\|g\|^2_{W_2^{r-1}(\mathbb{R};\mu)} \asymp \|g\|_{\mathcal{H}^{r-1}(\mathbb{R})} &= \sum_{k\in\mathbb{N}_0} |\rho_{\lambda,r-1,k}\hat{g}_k|^2 \\ &\ll \sum_{k\in\mathbb{N}_0} |k^{1-1/\lambda}\rho_{\lambda,r-1,k}\hat{f}(k)|^2 \le \|f\|^2_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,.\end{aligned} \tag{3.21}$$

Hence,

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \asymp \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{W_2^{r-1}(\mathbb{R};\mu)} \le \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,. \tag{3.22}$$

□

We show the equivalence between the norm of the space $W_2^r(\mathbb{R};\mu)$ and the norm of $H_w^{r_\lambda}(\mathbb{R}^d)$ in the case $\lambda = 4$ in the weight (1.2) by proving the inequality inverse to (3.12). In order to achieve this, it is necessary to employ certain properties of the polynomials ϕ_m for this particular case. Denote by $\gamma_m > 0$ the leading coefficient of the polynomial ϕ_m, i.e., $\phi_m(x) := \gamma_m x^m + \varphi$ for some $\varphi \in \mathcal{P}_{m-1}$. We put $\alpha_m := \gamma_{m-1}/\gamma_m$ for $m \in \mathbb{N}$. Then we have the following equalities for $\lambda = 4$.

(i)

$$\phi'_m = \frac{m}{\alpha_m}\,\phi_{m-1} + 4a\,\alpha_m\alpha_{m-1}\alpha_{m-2}\,\phi_{m-3}. \tag{3.23}$$

(ii)

$$4a\,\alpha_m^2(\alpha_{m+1}^2 + \alpha_m^2 + \alpha_{m-1}^2) = m. \tag{3.24}$$

(iii)

$$\lim_{m\to\infty}\left(\frac{12}{m}\right)^{1/4}\alpha_m = 1. \tag{3.25}$$

Here the parameter a is the same as in (1.2). The claims (i) and (ii) were proven in [2], the claim (iii) in [12].

Theorem 3.6 *Let $\lambda = 4$. Then we have the norm equivalence*

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,, \quad f \in W_2^r(\mathbb{R};\mu). \tag{3.26}$$

Proof. Without loss of generality we can assume $a = 1$. Otherwise, we can achieve this by changing variable $y = a^{1/4}x$ and considering an equivalent norm of $W_2^r(\mathbb{R};\mu)$. By the inequality (3.12) of Lemma 3.5, to prove the theorem it is sufficient to show the inverse inequality

$$\|f\|_{W_2^r(\mathbb{R};\mu)} \gg \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})}\,, \quad f \in W_2^r(\mathbb{R};\mu). \tag{3.27}$$

We first prove this inequality for $r = 1$. Given $f \in W_2^1(\mathbb{R};\mu)$, we denote $g := f'$. Put

$$b_k := \frac{k}{\alpha_k}, \quad c_k := 4\,\alpha_k\alpha_{k-1}\alpha_{k-2},$$

where recall, a is the parameter in the definition (1.2) of the generating univariate weight w. For any fixed $m_0 \in \mathbb{Z}$, from the equality $\lim_{m\to\infty} \frac{m+m_0}{m} = 1$, (3.24) and (3.25) it follows that

$$\lim_{m\to\infty} \left(\frac{12}{m}\right)^{3/4} b_{m+m_0} = 12, \quad \lim_{m\to\infty} \left(\frac{12}{m}\right)^{3/4} c_{m+m_0} = 4, \tag{3.28}$$

and

$$b_{m+m_0} \asymp c_{m+m_0} \asymp m^{3/4}, \quad k \in \mathbb{N}_0. \tag{3.29}$$

By using the equality (3.23) and

$$g = \sum_{m\in\mathbb{N}_0} \hat{f}(m)\phi'_m, \tag{3.30}$$

we have for every $k \in \mathbb{N}_0$,

$$\hat{g}(k) = b_{k+1}\hat{f}(k+1) + c_{k+3}\hat{f}(k+3), \quad k \in \mathbb{N}_0. \tag{3.31}$$

By (3.24) there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$,

$$\left(\frac{12}{k}\right)^{3/4} b_k \ge 9, \quad \left(\frac{12}{k}\right)^{3/4} c_k \le 6, \tag{3.32}$$

Hence by Parseval's identity and (3.29) we obtain

$$\begin{aligned}
\|g\|_{L_2(\mathbb{R};\mu)} &\ge \left(\sum_{k\ge1} |b_k\hat{f}(k)|^2\right)^{1/2} - \left(\sum_{k\ge3} |c_k\hat{f}(k)|^2\right)^{1/2} \\
&\ge \left(\sum_{k>k_0} |b_k\hat{f}(k)|^2\right)^{1/2} - \left(\sum_{k>k_0} |c_k\hat{f}(k)|^2\right)^{1/2} - \left(\sum_{k\le k_0} |c_k\hat{f}(k)|^2\right)^{1/2} \\
&\ge 3(12)^{-3/4}\left(\sum_{k>k_0} |k^{3/4}\hat{f}(k)|^2\right)^{1/2} - \max_{0\le k\le k_0}|c_k| \left(\sum_{k\le k_0} |\hat{f}(k)|^2\right)^{1/2} \\
&\ge 3(12)^{-3/4}\left(\sum_{k\in\mathbb{N}_0} |k^{3/4}\hat{f}(k)|^2\right)^{1/2} \\
&- 3(12)^{-3/4}k_0^{3/4}\left(\sum_{k\le k_0} |\hat{f}(k)|^2\right)^{1/2} - \max_{0\le k\le k_0}|c_k| \left(\sum_{k\le k_0} |\hat{f}(k)|^2\right)^{1/2} \\
&\ge C_1\,\|f\|_{\mathcal{H}^1(\mathbb{R})} - C_2\,\|f\|_{L_2(\mathbb{R};\mu)},
\end{aligned} \tag{3.33}$$

where

$$C_1 := 3(12)^{-3/4} > 0, \quad C_2 := 3(12)^{-3/4}k_0^{3/4} + \max_{0\le k\le k_0}|c_k| > 0.$$

This yields that

$$\begin{aligned} \|f\|_{W_2^1(\mathbb{R};\mu)} &\asymp C_2 \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{L_2(\mathbb{R};\mu)} \\ &\ge C_2 \|f\|_{L_2(\mathbb{R};\mu)} + C_1 \|f\|_{\mathcal{H}^1(\mathbb{R})} - C_2 \|f\|_{L_2(\mathbb{R};\mu)} = C_1 \|f\|_{\mathcal{H}^1_w(\mathbb{R})} . \end{aligned} \tag{3.34}$$

This and (3.20) prove the theorem in the case $r = 1$. In the general case it can be established by induction on r. Assuming that (3.27) is true for $r-1$, we prove it for r. Again, given $f \in W_2^r(\mathbb{R};\mu)$, we denote $g := f' \in W_2^{r-1}(\mathbb{R};\mu)$. From the induction assumption and (3.31), in a way similar to (3.33) we derive

$$\begin{aligned} \|g\|_{W_2^{r-1}(\mathbb{R};\mu)} &\asymp \|g\|_{\mathcal{H}^{r-1}(\mathbb{R})} = \left(\sum_{k\in\mathbb{N}_0} \rho_{\lambda,r-1,k} |\hat{g}(k)|^2 \right)^{1/2} \\ &\ge \left(\sum_{k\ge 1} \rho_{\lambda,r-1,k} |b_k \hat{f}(k)|^2 \right)^{1/2} - \left(\sum_{k\ge 3} \rho_{\lambda,r-1,k} |c_k \hat{f}(k)|^2 \right)^{1/2} \\ &\ge 3(12)^{-3/4} \left(\sum_{k\in\mathbb{N}_0} \rho_{\lambda,r-1,k} |k^{3/4} \hat{f}(k)|^2 \right)^{1/2} \\ &\quad - 3(12)^{-3/4} k_0^{3/4} \rho_{\lambda,r-1,k_0} \left(\sum_{k\le k_0} |\hat{f}(k)|^2 \right)^{1/2} \\ &\quad - \rho_{\lambda,r-1,k_0} \max_{0\le k\le k_0} |c_k| \left(\sum_{k\le k_0} |\hat{f}(k)|^2 \right)^{1/2} \\ &\ge C_1 \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})} - C_2 \rho_{\lambda,r-1,k_0} \|f\|_{L_2(\mathbb{R};\mu)} , \end{aligned} \tag{3.35}$$

where k_0, C_1 and C_2 are the same constants as in (3.33). Hence, similarly to (3.34) we obtain that

$$\begin{aligned} \|f\|_{W_2^r(\mathbb{R};\mu)} &\asymp C_2 \rho_{\lambda,r-1,k_0} \|f\|_{L_2(\mathbb{R};\mu)} + \|g\|_{W_2^{r-1}(\mathbb{R};\mu)} \\ &\ge C_2 \rho_{\lambda,r-1,k_0} \|f\|_{L_2(\mathbb{R};\mu)} + C_1 \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})} - C_2 \rho_{\lambda,r-1,k_0} \|f\|_{L_2(\mathbb{R};\mu)} \\ &= C_1 \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R})} . \end{aligned} \tag{3.36}$$

∎

For $\boldsymbol{x} \in \mathbb{R}^d$ and $e \subset \{1,...,d\}$, let $\boldsymbol{x}^e \in \mathbb{R}^{|e|}$ be defined by $(x^e)_i := x_i$, and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$ by $(\bar{x}^e)_i := x_i$, $i \in \{1,...,d\} \setminus e$. With an abuse we write $(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e) = \boldsymbol{x}$. For the proof of the following lemma, see [4, Lemma 3.2].

Lemma 3.7 *Let $1 \le p \le \infty$, $e \subset \{1,...,d\}$ and $\boldsymbol{r} \in \mathbb{N}_0^d$. Assume that f is a function on $\mathbb{R}^d$ such that for every $\boldsymbol{k} \le \boldsymbol{r}$, $D^{\boldsymbol{k}} f \in L_p(\mathbb{R}^d;\mu)$. Put for $\boldsymbol{k} \le \boldsymbol{r}$ and $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$,*

$$g(\boldsymbol{x}^e) := D^{\bar{\boldsymbol{k}}^e} f(\boldsymbol{x}^e, \bar{\boldsymbol{x}}^e).$$

Then $D^{\boldsymbol{s}} g \in L_p(\mathbb{R}^{|e|};\mu)$ for every $\boldsymbol{s} \le \boldsymbol{k}^e$ and almost every $\bar{\boldsymbol{x}}^e \in \mathbb{R}^{d-|e|}$.

Theorem 3.8 *Let λ be an even integer. Then we have the inequality*

$$\|f\|_{W_2^r(\mathbb{R}^d;\mu)} \ll \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W_2^r(\mathbb{R}^d;\mu). \tag{3.37}$$

Moreover, we have the norm equivalence for $\lambda = 4$,

$$\|f\|_{W_2^r(\mathbb{R}^d;\mu)} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W_2^r(\mathbb{R}^d;\mu). \tag{3.38}$$

Proof. In the case $d = 1$, this theorem combines Lemma 3.5 and Theorem 3.6. Both the relations (3.37) and (3.38) can be proven in the same way. For simplicity we prove (3.38) for the case $d = 2$. The general case can be proven by induction on d.

Since the linear combinations of the polynomials $\phi_{\boldsymbol{k}}$, $\boldsymbol{k} = (k_1, k_2) \in \mathbb{N}_0^2$, are dense in the normed spaces $W_{2,w}^r(\mathbb{R}^2)$ and $\mathcal{H}^{r_\lambda}(\mathbb{R}^2)$, it is sufficient to prove the case $d = 2$ for polynomials f of the form

$$f = \sum_{k_1,k_2=0}^{N} \hat{f}_{\boldsymbol{k}} \phi_{\boldsymbol{k}}.$$

Let $f \in W_{2,w}^r(\mathbb{R}^2)$ be such a polynomial. From Lemma 3.7 it follows that $f(\cdot, x_2) \in W_2^r(\mathbb{R};\mu)$, and consequently, by Theorem 3.6 $f(\cdot, x_2) \in \mathcal{H}^{r_\lambda}(\mathbb{R})$ for almost everywhere $x_2 \in \mathbb{R}$. We make use of the temporary notation:

$$h_{k_2}(x_1) := \int_{\mathbb{R}} f(x_1, x_2)\phi_{k_2}(x_2)\mathrm{d}\mu(x_2).$$

By applying successively the case $d = 1$ of the lemma with respect to variables x_2 and x_1, and using Fubini's theorem, we obtain

$$
\begin{aligned}
\|f\|^2_{W^r_{2,w}(\mathbb{R}^2)} &= \sum_{s_1,\,s_2=0}^{r} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|D^{(0,s_2)}(D^{(s_1,0)} f(x_1,x_2))\right|^2 \mathrm{d}\mu(x_2)\mathrm{d}\mu(x_1) \\
&= \sum_{s_1=0}^{r} \int_{\mathbb{R}} \sum_{s_2=0}^{r} \int_{\mathbb{R}} \left|D^{(0,s_2)}(D^{(s_1,0)} f(x_1,x_2))\right|^2 \mathrm{d}\mu(x_2)\, \mathrm{d}\mu(x_1) \\
&= \sum_{s_1=0}^{r} \int_{\mathbb{R}} \left\|D^{(s_1,0)} f(x_1,\cdot)\right\|^2_{W^r_2(\mathbb{R};\mu)} \,\mathrm{d}\mu(x_1) \\
&\ll \sum_{s_1=0}^{r} \int_{\mathbb{R}} \left\|D^{(s_1,0)} f(x_1,\cdot)\right\|^2_{\mathcal{H}^{r_\lambda}(\mathbb{R})} \,\mathrm{d}\mu(x_1) \\
&\ll \sum_{s_1=0}^{r} \int_{\mathbb{R}} \sum_{k_2=0}^{N} |\rho_{\lambda,r,k_2} D^{(s_1,0)} h_{k_2}(x_1)|^2 \,\mathrm{d}\mu(x_1) \\
&\ll \sum_{k_2=0}^{N} \rho^2_{\lambda,r,k_2} \sum_{s_1=0}^{r} \int_{\mathbb{R}} \left|D^{(s_1,0)} h_{k_2}(x_1)\right|^2 \,\mathrm{d}\mu(x_1) \\
&= \sum_{k_2=0}^{N} \rho^2_{\lambda,r,k_2} \|h_{k_2}\|^2_{W^r_2(\mathbb{R};\mu)} \\
&\ll \sum_{k_2=0}^{N} \rho^2_{\lambda,r,k_2} \sum_{k_1=0}^{N} |\rho_{\lambda,r,k_1} \hat{h}_{k_2}(k_1)|^2 \\
&= \sum_{k_2=0}^{N} \rho^2_{\lambda,r,k_2} \sum_{k_1=0}^{N} |\rho_{\lambda,r,k_1} \hat{f}(k_1,k_2)|^2 = \|f\|^2_{\mathcal{H}^{r_\lambda}(\mathbb{R}^2)}\,.
\end{aligned}
$$

□

Notice that the norm equivalence (3.38) for $\lambda = 2$ has been proven in [8, Lemma 3.4] (see also [10, pages 687–689]) .

Due to the norm equivalence (3.38) in Theorem 3.8, we identify $W^r_2(\mathbb{R}^d;\mu)$ with $\mathcal{H}^{r_\lambda}(\mathbb{R}^d)$ for the case when $\lambda = 4$ and $r \in \mathbb{N}$. From Theorem 3.4 and the norm inequality $\|\cdot\|_{L_p(\mathbb{R}^d;\mu)} \le C\,\|\cdot\|_{L_2(\mathbb{R}^d;\mu)}$ for $1 \le p \le 2$, we derive the following result on right convergence rate of sampling n-widths.

Theorem 3.9 *We have for $1 \le q \le 2$, $r \in \mathbb{N}$ and $\lambda = 4$,*

$$
\varrho_n(\boldsymbol{W}^r_2(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \asymp n^{-\frac{3r}{4}} (\log n)^{\frac{3(d-1)r}{4}}. \tag{3.39}
$$

We finish this section with some conjectures.

Conjecture 3.10 *We have for any $r \in \mathbb{N}$ and even integer $\lambda > 4$,*

$$
\|f\|_{W^r_2(\mathbb{R}^d;\mu)} \gg \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}\,, \quad f \in W^r_2(\mathbb{R}^d;\mu). \tag{3.40}
$$

If Conjecture 3.10 holds true, then from this conjecture and Theorems 3.8 and 3.4 we can deduce the following results.

Conjecture 3.11 *We have for any $r \in \mathbb{N}$ and even integer $\lambda > 4$,*

$$\|f\|_{W_2^r(\mathbb{R}^d;\mu)} \asymp \|f\|_{\mathcal{H}^{r_\lambda}(\mathbb{R}^d)}, \quad f \in W_2^r(\mathbb{R}^d;\mu). \tag{3.41}$$

Conjecture 3.12 *We have for any $d \geq 2$, $r \in \mathbb{N}$ and even integer $\lambda > 4$,*

$$\varrho_n(\boldsymbol{W}_2^r(\mathbb{R}^d;\mu), L_2(\mathbb{R}^d;\mu)) \asymp n^{-r_\lambda}(\log n)^{r_\lambda(d-1)}.$$

Acknowledgments: This work is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) in the frame of the NAFOSTED-SNSF Joint Research Project under Grant IZVSZ2_229568. A part of this work was done when the author was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for providing a fruitful research environment and working condition. The author extends special thanks to Paul Nevai for providing the paper [1].

References

[1] S. Bonan. Applications of G. Freud's theory, I. *Approximation Theory, IV (C. K. Chui et al., Eds.), Acad. Press*, pages pp. 347–351, 1984.

[2] S. Bonan and P. Nevai. Orthogonal polynomials and their derivative. *J. Approx. Theory*, 40:134–147, 1984.

[3] H. J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147–269, 2004.

[4] D. Dũng. Numerical weighted integration of functions having mixed smoothness. *J. Complexity*, 78:101757, 2023.

[5] D. Dũng. Sparse-grid sampling recovery and numerical integration of functions having mixed smoothness. *Acta Mathematica Vietnamica*, 49:377–426, 2024.

[6] D. Dũng. Weighted sampling recovery of functions with mixed smoothness. *arXiv:2405.16400*, 2024.

[7] D. Dũng. Sampling reconstruction and integration of functions on $\mathbb{R}^d$ endowed with a measure. *VIASM preprint*, ViAsM25.15, 2025.

[8] D. Dũng and V. K. Nguyen. Optimal numerical integration and approximation of functions on $\mathbb{R}^d$ equipped with Gaussian measure. *IMA Journal of Numer. Anal.*, 44:1242–1267, 2024.

[9] D. Dũng, V. N. Temlyakov, and T. Ullrich. *Hyperbolic Cross Approximation.* Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser/Springer, 2018.

[10] J. Dick, C. Irrgeher, G. Leobacher, and F. Pillichshammer. On the optimal order of integration in Hermite spaces with finite smoothness. *SIAM J. Numer. Anal.*, 56:684–707, 2018.

[11] M. Dolbeault, D. Krieg, and M. Ullrich. A sharp upper bound for sampling numbers in L_2. *Appl. Comput. Harmon. Anal.*, 63:113–134, 2023.

[12] G. Freud. On the coefficients in the recursion formulae of orthogonal polynomials. *Proc. R. Irish Acad., Sect. A*, 76:1–6, 1976.

[13] P. Junghanns, G. Mastroianni, and I. Notarangelo. *Weighted Polynomial Approximation and Numerical Methods for Integral Equations*. Birkhäuser, 2021.

[14] D. S. Lubinsky. A survey of weighted polynomial approximation with exponential weights. *Surveys in Approximation Theory*, 3:1–105, 2007.

[15] H. N. Mhaskar. *Introduction to the Theory of Weighted Polynomial Approximation*. World Scientific, Singapore, 1996.

[16] V. Nguyen and W. Sickel. Pointwise multipliers for Sobolev and Besov spaces of dominating mixed smoothness. *J. Math. Anal. Appl.*, 452:62–90, 2017.

[17] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume II: Standard Information for Functionals*. EMS Tracts in Mathematics, Vol. 12, Eur. Math. Soc. Publ. House, Zürich, 2010.

[18] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Univ. Press, Princeton, NJ, 1970.

[19] Y. Suzuki and T. Karvonen. Construction of optimal algorithms for function approximation in Gaussian Sobolev spaces . *arXiv Preprint*, arXiv:2402.02917 [math.NA], 2024.

[20] V. N. Temlyakov. *Multivariate Approximation*. Cambridge University Press, 2018.