Sampling reconstruction and integration of functions on $\mathbb{R}^d$ endowed with a measure

Dinh Dũng[a]

[a]Information Technology Institute, Vietnam National University, Hanoi
144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
Email: dinhzung@gmail.com

August 15, 2025

Abstract

This paper examines the performance and optimality of sparse-grid linear sampling algorithms for the approximate reconstruction of functions possessing mixed smoothness on $\mathbb{R}^d$ based a set of n sampled values. The target functions belong to Sobolev spaces with measure $W_p^r(\mathbb{R}^d;\mu)$ of mixed smoothness. The approximation error is measured by the norm of the Lebesgue space with measure $L_q(\mathbb{R}^d;\mu)$ for $1 \le q < p \le \infty$. The underlying measure μ is defined via a density function of tensor-product exponential weight. The optimality of linear sampling algorithms is investigated in terms of sampling n-widths. We introduced a novel method for constructing sparse-grid linear sampling algorithms which achieve upper bounds of the corresponding sampling n-widths and moreover, the right convergence rate in the case $d = 1$. As consequences, we derived from these results on sapling recovery convergence rates of the generated quadratures for numerical integration of functions in $W_p^r(\mathbb{R}^d;\mu)$.

Keywords and Phrases: Sampling widths; Sobolev spaces with measure of mixed smoothness; Sparse-grid Smolyak grids; Numerical integration; Quadrature; Convergence rate.

MSC (2020): 41A25; 41A46; 41A63; 41A81; 65D30; 65D32.

1 Introduction

The present paper investigates the performance and optimality of linear algorithms for the approximate recovery and quadrature of functions with mixed smoothness on $\mathbb{R}^d$, endowed with an exponential positive measure. We examine how effectively these sampling algorithms and quadrature formulas can reconstruct functions and their integrals from a finite set of sampled values, with a focus on theoretical efficiency and provable approximation guarantees.

We first introduce Sobolev spaces with measure of mixed smoothness of functions on $\mathbb{R}^d$. Let

$$w(\boldsymbol{x}) := \bigotimes_{i=1}^{d} w(x_i), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{1.1}$$

be the tensor product of d copies of the generating univariate exponential weight

$$w(x) := \exp\left(-a|x|^\lambda + b\right), \quad \lambda > 0, \quad a > 0, \quad b \in \mathbb{R}. \tag{1.2}$$

In what follows, we fix the weight w and hence the parameters λ, a, b.

Let Ω be a Lebesgue-measurable set on $\mathbb{R}^d$. Let μ be the positive measure on Ω defined by

$$\mu(A) := \int_A w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$$

for any measurable subset A in Ω, i.e., the weight w is the density function of μ. With an abuse, we also write μ in the tensor product from as:

$$\mu(\boldsymbol{x}) := \bigotimes_{i=1}^{d} \mu(x_i), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{1.3}$$

Let $1 \le q \le \infty$. We denote by $L_q(\Omega; \mu)$ the Lebesgue space of all measurable functions f on Ω such that for $1 \le q < \infty$, the norm

$$\|f\|_{L_q(\Omega;\mu)} := \left(\int_\Omega |f(\boldsymbol{x})|^q \mathrm{d}\mu(\boldsymbol{x})\right)^{1/q} = \left(\int_\Omega |f(\boldsymbol{x})|^q w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\right)^{1/q} \tag{1.4}$$

is finite, and assuming f is continuous on Ω for $q = \infty$, the norm

$$\|f\|_{L_\infty(\Omega;\mu)} := \|f\|_{C(\Omega)} := \sup_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})| \tag{1.5}$$

is finite. For $r \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W_p^r(\Omega; \mu)$ of mixed smoothness r is defined as the normed space of all functions $f \in L_p(\Omega; \mu)$ such that the weak partial derivative $D^{\boldsymbol{k}} f$ belongs to $L_p(\Omega; \mu)$ for every $\boldsymbol{k} \in \mathbb{N}_0^d$ satisfying the inequality $|\boldsymbol{k}|_\infty \le r$. The norm of a function f in this space is defined by

$$\|f\|_{W_p^r(\Omega;\mu)} := \left(\sum_{|\boldsymbol{k}|_\infty \le r} \|D^{\boldsymbol{k}} f\|_{L_p(\Omega;\mu)}^p\right)^{1/p}. \tag{1.6}$$

The well-known Gaussian-measured spaces $L_p(\mathbb{R}^d; \gamma)$ and $W_p^r(\mathbb{R}^d; \gamma)$ are used in many applications. Here the standard Gaussian measure γ is defined via the density function $w_{\mathrm{g}}(\boldsymbol{x}) := (2\pi)^{-d/2} \exp(-|\boldsymbol{x}|_2^2/2)$.

Let X be a normed space of functions on Ω. Given sample points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \Omega$, we consider the approximate recovery of a continuous function f on Ω from their values $f(\boldsymbol{x}_1), \ldots, f(\boldsymbol{x}_k)$ by a linear sampling algorithm S_k on Ω of the form

$$S_k(f) := \sum_{i=1}^{k} f(\boldsymbol{x}_i) h_i, \tag{1.7}$$

where $h_1, \ldots, h_k$ are given continuous functions on Ω. For convenience, we assume that some of the sample points $\boldsymbol{x}_i$ may coincide. The approximation error is measured by the norm $\|f - S_k(f)\|_X$. Denote by $\mathcal{S}_n(\Omega)$ the family of all linear sampling algorithms S_k of the form (1.7) with $k \le n$. Let $F \subset X$ be a set of continuous functions on Ω. To study the optimality of linear sampling algorithms from $\mathcal{S}_n(\Omega)$ for F and their convergence rates we use the (linear) sampling n-width

$$\varrho_n(F, X) := \inf_{S_n \in \mathcal{S}_n(\Omega)} \sup_{f \in F} \|f - S_n(f)\|_X. \tag{1.8}$$

A significant body of research has been dedicated to the problem of unweighted linear sampling recovery of functions possessing mixed smoothness on compact domains. In particular, Smolyak sparse-grid sampling algorithms have been widely employed in both approximation theory and numerical analysis, especially for the sampling recovery of functions with mixed smoothness on domains such as cubes and tori. These algorithms have proven to be highly effective, with numerous investigations addressing various aspects including theoretical approximation properties and numerical implementation. For a thorough overview and bibliography, see [2, 11, 21, 25, 7]. Specifically, understanding the asymptotic behavior of the sampling n-widths $\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)\big)$, where $\mathbb{T}^d$ denotes the d-dimensional torus, constitutes a key research focus. Here and in what follows, for a normed space X of functions on Ω, the boldface $\boldsymbol{X}$ denotes the unit ball in X. For a detailed survey and bibliography, see [7, 11]. It is noteworthy that, for the cases $1 < p < q \le 2$, $2 \le p < q < \infty$ and $p = 2$, $q = \infty$, the optimal convergence can be attained by Smolyak sparse-grid algorithms. To date, no alternative algorithms have been demonstrated to achieve such asymptotic optimality. One can also, in a non-constructive manner, deduce the right convergence rate of these sampling n-widths for the case $1 < q \le 2 \le p \le \infty$ (for detail see, e.g., [7]).

Furthermore, the problem of optimal sampling recovery of functions on $\mathbb{R}^d$ equipped with standard Gaussian measure has been investigated in [10, 8, 24]. In that context, we have established in [10], again in a non-constructive manner, the right convergence rate of the sampling n-widths

$$\varrho_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)\big) \asymp n^{-r} (\log n)^{r(d-1)} \tag{1.9}$$

for $2 < p \le \infty$, and

$$\varrho_n\big(\boldsymbol{W}^r_2(\mathbb{R}^d;\gamma), L_2(\mathbb{R}^d;\gamma)\big) \asymp n^{-r/2} (\log n)^{r(d-1)/2} \tag{1.10}$$

which is obtained by using inequalities between sampling widths and Kolmogorov widths and the right convergence rate of Kolmogorov widths proven in the same paper [10]. Let

$1 < p < \infty$, $\lambda > 1$ and $r_\lambda := r(1 - 1/\lambda)$. Then we have recently proven in [8] that

$$n^{-r_\lambda}(\log n)^{r_\lambda(d-1)} \ll \varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_p(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)}. \tag{1.11}$$

In the present work, we focus on the construction of sparse-grid linear sampling algorithms for the approximate recovery of functions with mixed smoothness on $\mathbb{R}^d$. Specifically, we consider functions belonging to Sobolev spaces equipped with a measure $W^r_p(\mathbb{R}^d;\mu)$, and we measure the approximation error in the norm of the Lebesgue space $L_q(\mathbb{R}^d;\mu)$. Here, the parameters p and q may differ and vary satisfying the condition $1 \le q < p \le \infty$. The case $1 \le p < q \le \infty$ is excluded from consideration since in this case we do not have a continuous embedding of $W^r_p(\mathbb{R}^d;\mu)$ into $L_q(\mathbb{R}^d;\mu)$. The case $1 \le p = q \le \infty$ has been investigated in [8] for $\lambda > 1$. Our investigation centers on the asymptotic optimality of these linear sampling algorithms, evaluated in terms of the relevant sampling n-widths. It is worth emphasizing that this framework of sampling recovery is highly relevant to various theoretical and applied fields, particularly those involving standard Gaussian measure γ and other probability measures.

We briefly describe the main results of the present paper.

Let $1 \le q < p \le \infty$, $r > 1/p$. Then, we prove that

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \ll \begin{cases} n^{-r}(\log n)^{(r+1/2)(d-1)} & \text{if } \ 1 < q < p < \infty, \\ n^{-r}(\log n)^{(r+1)(d-1)} & \text{if either } q = 1 \text{ or } p = \infty, \end{cases} \tag{1.12}$$

and

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \gg n^{-r}(\log n)^{r(d-1)} \ \text{ if } \ 1 \le q < p \le \infty. \tag{1.13}$$

In the one-dimensional case, we prove the right convergence rate

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R};\mu), L_q(\mathbb{R};\mu)) \asymp n^{-r} \ \text{ if } \ 1 \le q < p \le \infty \text{ and } r > 1/p. \tag{1.14}$$

The sparse-grid linear sampling algorithms that achieve the upper bounds (1.12) for the case $1 < q < p < \infty$ are constructed through a process of *assembling* B-spline quasi-interpolation Smolyak sampling algorithms which are designed for the related Sobolev spaces on the integer-shifted d-cubes which cover $\mathbb{R}^d$. This is a novel method for constructing sparse-grid linear sampling algorithms for approximate reconstruction of functions on $\mathbb{R}^d$ endowed with measure. It crucially differs from classical methods of weighted sampling reconstruction of functions based on polynomial interpolation, see, e.g., [17] for a survey and bibliography on weighted polynomial interpolation of function on $\mathbb{R}$.

The lower bounds in (1.13) are derived from known lower bounds of Kolmogorov n-widths $d_n\big(\boldsymbol{W}^r_p(\mathbb{T}^d), L_q(\mathbb{T}^d)\big)$. Differing from the case $1 < p = q < \infty$, the convergence rates of the sampling n-widths in (1.12) and (1.14) in the case $1 \le q < p \le \infty$ do not depend on the main parameter λ in the weight w, the density function of the measure μ, and coincide with those of the related unweighted sampling n-widths (see, e.g., [11]).

The gap between the upper bounds (1.12) and the lower bounds in (1.13) for the sampling n-widths, is a logarithmic factor if $d \ge 2$. The main parameter r in the convergence rates is the same as that in the convergence rates of the related unweighted sampling n-widths (for detail, see, e.g., [7, Theorem 2.9]).

We are interested in approximation of integrals

$$\int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) \tag{1.15}$$

for functions f lying in the space $W^r_p(\mathbb{R}^d;\mu)$. To approximate them we use weighted quadratures of the form

$$Q_n f := \sum_{i=1}^n \omega_i f(\boldsymbol{x}_i), \tag{1.16}$$

where $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^d$ are the integration nodes and $\omega_1, \dots, \omega_n$ the integration weights.

Let F be a set of continuous functions on $\mathbb{R}^d$. Denote by $\mathcal{Q}_n(\mathbb{R}^d)$ the family of all quadratures Q_k of the form (1.16) with $k \le n$. The optimality of quadratures from $\mathcal{Q}_n(\mathbb{R}^d)$ for $f \in F$ is measured by

$$\mathrm{Int}_n(F) := \inf_{Q_n \in \mathcal{Q}_n(\mathbb{R}^d)} \sup_{f \in F} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) - Q_n f \right|. \tag{1.17}$$

Observe that every sampling algorithm $S_n \in \mathcal{S}_n(\mathbb{R}^d)$ generates the weighted quadrature $Q_n \in \mathcal{Q}_n(\mathbb{R}^d)$ by integrating $S_n(f)$ with measure μ over $\mathbb{R}^d$. Moreover, the integration error can be estimated by the error of sampling recovery by $S_n(f)$ in the norm of $L_1(\mathbb{R}^d;\mu)$ (for detail, see Section 3). Due to this observation, from the main results on sampling reconstruction we obtain the following.

For $1 < p < \infty$, we prove the upper and lower bounds

$$n^{-r}(\log n)^{r(d-1)/2} \ll \mathrm{Int}_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)) \ll n^{-r}(\log n)^{(r+1)(d-1)}. \tag{1.18}$$

(The upper bound still holds true for $p = \infty$.) In the one-dimensional case, for $1 < p \le \infty$, $r > 1/p$, we prove the right convergence rate

$$\mathrm{Int}_n(\boldsymbol{W}^r_p(\mathbb{R};\mu)) \asymp n^{-r}. \tag{1.19}$$

We shortly give comments on some works related to the results (1.21) and (1.19). There is a large number of works on high-dimensional unweighted integration over the unit d-cube $\mathbb{I}^d := [0,1]^d$ for functions having a mixed smoothness (see [11, 13, 25] for results and bibliography). However, there are only a few works on high-dimensional weighted integration for functions having a mixed smoothness. The problem of numerical integration (1.15)–(1.17) has been studied in [15, 16, 12, 6, 10, 14] for functions in the Gaussian-measured space $W^r_p(\mathbb{R}^d;\gamma)$. Recently, in [10, Theorem 2.3] for the space $W^r_p(\mathbb{R}^d,\gamma)$ with $r \in \mathbb{N}$ and $1 < p < \infty$, we have constructed asymptotically optimal quadratures Q^γ_n of the form (1.16) which achieve the right convergence rate:

$$\sup_{f \in \boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \mathrm{d}\gamma(\boldsymbol{x}) - Q^\gamma_n f \right| \asymp \mathrm{Int}_n\big(\boldsymbol{W}^r_p(\mathbb{R}^d;\gamma)\big) \asymp n^{-r}(\log n)^{(d-1)/2}. \tag{1.20}$$

In constructing the asymptotically optimal quadrature Q_n^γ in (1.20), we used a technique assembling a quadrature for the Sobolev spaces on the unit d-cube to the integer-shifted d-cubes. This technique is extended and developed in the present paper to constructing sampling algorithms. For the set $\boldsymbol{W}_1^r(\mathbb{R}^d;\mu)$ with $\lambda > 1$, the upper and lower bounds

$$n^{-r_\lambda}(\log n)^{r_\lambda(d-1)} \ll \mathrm{Int}_n(\boldsymbol{W}_1^r(\mathbb{R}^d;\mu)) \ll n^{-r_\lambda}(\log n)^{(r_\lambda+1)(d-1)} \tag{1.21}$$

have been proven in [6].

Finally, it is worth emphasizing that in all the results on sampling recovery and numerical integration with the measure μ of the present paper, the primary parameter λ which is the most significant in shaping the properties of the associated weight w and measure μ, is assumed simply *a positive number*. This significantly distinguishes our setting from the classical theory of weighted approximation (see, e.g., [19], [18], [17]), where it is typically assumed that $\lambda > 1$ in the weight w (Freud-type weight).

The paper is organized as follows. In Section 2, we prove upper and lower bounds of $\varrho_n(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu))$ for $d \in \mathbb{N}$, $1 \le q < p \le \infty$, and construct linear sampling algorithms which achieve the upper bound for $d \ge 2$ and the right convergence rate for $d = 1$. In Section 3, as consequences, we derived from the results on sapling recovery convergence rates of the generated quadratures for numerical integration of functions in $W_p^r(\mathbb{R}^d;\mu)$.

Notation. Denote $\boldsymbol{x} =: (x_1, ..., x_d)$ for $\boldsymbol{x} \in \mathbb{R}^d$; $\boldsymbol{1} := (1, ..., 1) \in \mathbb{R}^d$; for $0 < \lambda < \infty$ and $\boldsymbol{x} \in \mathbb{R}^d$, $|\boldsymbol{x}|_\lambda := \left(\sum_{j=1}^d |x_j|^\lambda\right)^{1/\lambda}$ and $|\boldsymbol{x}|_\infty := \max_{1\le j\le d} |x_j|$. We use letter C to denote general positive constants which may take different values. For the quantities $A_n(f,\boldsymbol{k})$ and $B_n(f,\boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f,\boldsymbol{k}) \ll B_n(f,\boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ independent of $n, f, \boldsymbol{k}$ such that $A_n(f,\boldsymbol{k}) \le CB_n(f,\boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f,\boldsymbol{k}) \gg B_n(f,\boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f,\boldsymbol{k}) \asymp B_n(f,\boldsymbol{k})$ if $A_n(f,\boldsymbol{k}) \ll B_n(f,\boldsymbol{k})$ and $B_n(f,\boldsymbol{k}) \ll A_n(f,\boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G. For a Banach space X, denote by the boldface $\boldsymbol{X}$ the unit ball in X.

2 Sparse-grid sampling algorithms

In this section, we establish upper and lower bounds of $\varrho_n\big(\boldsymbol{W}_p^r(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)\big)$ for $1 \le q < p \le \infty$, and construct linear sampling algorithms which achieve the upper bounds. In the one dimensional case ($d = 1$), we obtain the right convergence rate of $\varrho_n\big(\boldsymbol{W}_p^r(\mathbb{R};\mu), L_q(\mathbb{R};\mu)\big)$.

We construct linear sampling algorithms that achieve the upper bounds (1.12) through a process of assembling sampling algorithms which are tailored for the related Sobolev spaces on the integer-shifted d-cubes which cover $\mathbb{R}^d$. Some important linear sampling algorithms such as Smolyak sampling algorithms based on trigonometric interpolation or periodic B-spline quasi-interpolations, have been constructively designed for periodic functions with Sobolev mixed smoothness (for detail, see [11, Section 5]). To employ these constructions for for constructing linear sampling algorithms for functions on $\mathbb{R}^d$ with Sobolev smoothness, it is necessary to adapt and modify the underlying algorithms accordingly.

Denote by $\tilde{C}(\mathbb{I}^d)$, $\tilde{L}_q(\mathbb{I}^d)$ and $\tilde{W}^r_p(\mathbb{I}^d)$ the subspaces of $C(\mathbb{I}^d)$, $L_q(\mathbb{I}^d)$ and $W^r_p(\mathbb{I}^d)$, respectively, of all functions f on the d-unit cube $\mathbb{I}^d := [0,1]^d$, which can be extended to the whole $\mathbb{R}^d$ as 1-periodic functions in each variable (denoted again by f). Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$. Let $S_n \in \mathcal{S}_n(\mathbb{I}^d)$ be a sampling algorithm. Assume it holds that

$$\|f - S_n(f)\|_{\tilde{L}_q(\mathbb{I}^d)} \le C n^{-\alpha} (\log n)^\beta \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{2.1}$$

Then based on S_n, we will construct a sampling algorithm belonging to $\mathcal{S}_n(\mathbb{R}^d)$, which approximates $f \in W^r_p(\mathbb{R}^d;\mu)$ with the same error bound as in (2.1) for the approximation error measured in the norm of $L_q(\mathbb{R}^d;\mu)$. Such a sampling algorithm will be constructed by assembling sampling algorithms which are designed for the related Sobolev spaces on the integer-shifted d-cubes which cover $\mathbb{R}^d$. Let us process this construction.

Fix a number $\theta > 0$ and put $\mathbb{I}^d_\theta := [-\theta, 1+\theta]^d$. Denote by $\tilde{C}(\mathbb{I}^d_\theta)$, $\tilde{L}_q(\mathbb{I}^d_\theta)$ and $\tilde{W}^r_p(\mathbb{I}^d_\theta)$ the subspaces of $C(\mathbb{I}^d_\theta)$, $L_q(\mathbb{I}^d_\theta)$ and $W^r_p(\mathbb{I}^d_\theta)$, respectively, of all functions f which can be extended to the whole $\mathbb{R}^d$ as $(1+2\theta)$-periodic functions in each variable (denoted again by f). A sampling algorithm $S_n \in \mathcal{S}_n(\mathbb{I}^d)$ induces the sampling algorithm $S_{\theta,n} \in \mathcal{S}_n(\mathbb{I}^d_\theta)$ defined for a function $f \in \tilde{C}(\mathbb{I}^d_\theta)$ by

$$S_{\theta,n}(f)(\boldsymbol{x}) := S_n(f(\boldsymbol{x}/(1+2\theta) + \theta\mathbf{1})), \quad \boldsymbol{x} \in \mathbb{I}^d_\theta.$$

From (2.1) it follows that

$$\|f - S_{\theta,n}(f)\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \le C n^{-\alpha} (\log n)^\beta \|f\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d_\theta).$$

We define for $n \in \mathbb{N}$,

$$m_n := \left(\delta^{-1}\alpha \log n\right)^{1/\lambda}, \tag{2.2}$$

and for $\boldsymbol{k} \in \mathbb{Z}^d$,

$$n_{\boldsymbol{k}} := \begin{cases} \lfloor \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|_\lambda^\lambda} + 1 \rfloor & \text{if } |\boldsymbol{k}|_\lambda < m_n, \\ 0 & \text{if } |\boldsymbol{k}|_\lambda \ge m_n, \end{cases} \tag{2.3}$$

where an appropriate fixed value of parameter $\delta > 0$ will be chosen below,

$$\varrho^{-1} := V^d_\lambda \sum_{s=0}^\infty s^d e^{-\frac{a\delta}{\alpha} s^\lambda} < \infty,$$

and V^d_λ denote the volume of the set

$$B^d_\lambda := \left\{ \boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}|_\lambda \le 1 \right\}.$$

We write $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \boldsymbol{k} + \mathbb{I}^d_\theta$ for $\boldsymbol{k} \in \mathbb{Z}^d$, and denote by $f_{\theta,\boldsymbol{k}}$ the restriction of f on $\mathbb{I}^d_{\theta,\boldsymbol{k}}$ for a function f on $\mathbb{R}^d$.

It is well-known that one can constructively define a unit partition $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ such that

(i) $\varphi_{\boldsymbol{k}} \in C_0^\infty(\mathbb{R}^d)$ and $0 \le \varphi_{\boldsymbol{k}}(\boldsymbol{x}) \le 1, \quad \boldsymbol{x} \in \mathbb{R}^d, \quad \boldsymbol{k} \in \mathbb{Z}^d$;

(ii) $\operatorname{supp} \varphi_{\boldsymbol{k}}$ are contained in the interior of $\mathbb{I}^d_{\theta,\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$;

(iii) $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_{\boldsymbol{k}}(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \mathbb{R}^d$;

(iv) $\|\varphi_{\boldsymbol{k}}\|_{W^r_p(\mathbb{I}^d_{\theta,\boldsymbol{k}})} \le C_{r,d,\theta}, \quad \boldsymbol{k} \in \mathbb{Z}^d$,

(see, e.g., [23, Chapter VI, 1.3]).

Due to the item (ii), we can define the $(1+2\theta)$-periodic functions $\tilde{f}_{\theta,\boldsymbol{k}}$ on $\mathbb{R}^d$ for $\boldsymbol{k} \in \mathbb{Z}^d$ by putting

$$\tilde{f}_{\theta,\boldsymbol{k}}(\boldsymbol{x}) := f_{\theta,\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k})\varphi_{\boldsymbol{k}}(\boldsymbol{x}+\boldsymbol{k}), \quad \boldsymbol{x} \in \mathbb{I}^d_\theta.$$

For $n \in \mathbb{N}$, taking the sequence $(n_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d}$ given as in (2.3) and satisfying the condition

$$\sum_{|\boldsymbol{k}|_\lambda < m_n} n_{\boldsymbol{k}} \le n,$$

we define the linear sampling algorithm $S^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ generated from S_n by

$$\left(S^\mu_{\theta,n} f\right)(\boldsymbol{x}) := \sum_{|\boldsymbol{k}|<m_n} \left(S_{\theta,n_{\boldsymbol{k}}} \tilde{f}_{\theta,\boldsymbol{k}}\right)(\boldsymbol{x}-\boldsymbol{k}). \tag{2.4}$$

Theorem 2.1 *Let $1 \le q < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$, $\theta > 0$. Assume that for any $n \in \mathbb{N}$, there is a linear sampling algorithm $S_n \in \mathcal{S}_n(\mathbb{I}^d)$ such that the convergence rate (2.1) holds. Then for any $n \in \mathbb{N}$, based on this sampling algorithm, one can construct a sampling algorithm $S^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ of the form (2.4) so that*

$$\|f - S^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \le C n^{-\alpha} (\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu). \tag{2.5}$$

Proof. We preliminarily decompose a function in $W^r_p(\mathbb{R}^d;\mu)$ into a sum of functions on $\mathbb{R}^d$ having support contained in integer translations of the d-cube $\mathbb{I}^d_\theta$. Then a desired sampling algorithm for $W^r_p(\mathbb{R}^d;\mu)$ will be the sum of integer-translated dilations of S_n. Notice that

$$\mathbb{R}^d = \bigcup_{\boldsymbol{k}\in\mathbb{Z}^d} \mathbb{I}^d_{\theta,\boldsymbol{k}},$$

where $\mathbb{I}^d_{\theta,\boldsymbol{k}} := \mathbb{I}^d_\theta + \boldsymbol{k}$. From the items (ii) and (iii) in the definition of unit partition it is implied that

$$f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f_{\theta,\boldsymbol{k}} \varphi_{\boldsymbol{k}}, \tag{2.6}$$

where $f_{\theta,\boldsymbol{k}}$ denotes the restriction of f to $\mathbb{I}^d_{\theta,\boldsymbol{k}}$. Hence we have

$$\begin{aligned} \|f - S^\mu_{\theta,n}(f)\|_{L_q(\mathbb{R}^d;\mu)} \le & \sum_{|\boldsymbol{k}|_\lambda < m_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \\ & + \sum_{|\boldsymbol{k}|_\lambda \ge m_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)}. \end{aligned} \tag{2.7}$$

For the fixed θ, there exists a constant C depending p, λ, a, θ only such that $w^{-1/p}(\boldsymbol{x}) \le w^{-1/p}(\boldsymbol{k})$ for every $\boldsymbol{x} \in \mathbb{I}^d_{\theta,\boldsymbol{k}}$. Therefore,

$$\|f_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \le C w^{-1/p}(\boldsymbol{k}) \|f\|_{W^r_p(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \le C w^{-1/p}(\boldsymbol{k}) \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \tag{2.8}$$

Since $W^r_p(\mathbb{I}^d)$ is a multiplication algebra (see [20, Theorem 3.16]), from (2.8) and property (iv) of the unit partition $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, we have that $\tilde{f}_{\theta,\boldsymbol{k}} := f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k}) \in \tilde{W}^r_p(\mathbb{I}^d_\theta)$, and

$$\begin{aligned} \|\tilde{f}_{\theta,\boldsymbol{k}}\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} &\le C \|f_{\theta,\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \cdot \|\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \\ &\le C w^{-1/p}(\boldsymbol{k}) \|f\|_{W^r_p(\mathbb{R}^d;\mu)}. \end{aligned} \tag{2.9}$$

Similarly,

$$\left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \le C w^{1/q}(\boldsymbol{k}) \left\| \tilde{f}_{\theta,\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right) \right\|_{\tilde{L}_q(\mathbb{I}^d_\theta)}. \tag{2.10}$$

Because $q < p$, from the definition of the univariate weight w in (1.2) it follows that there are numbers C and $0 < \delta' < a\,(1/q - 1/p)$ such that

$$w^{1/q-1/p}(\boldsymbol{k}) \le C e^{-\delta' |\boldsymbol{k}|^\lambda_\lambda}, \quad \boldsymbol{k} \in \mathbb{Z}^d. \tag{2.11}$$

We choose a number δ in (2.2), satisfying the condition

$$\delta \max(1, a/\alpha) < \delta'. \tag{2.12}$$

Firstly, with this choice of δ, let us check that $S^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$, i.e., $m \le n$ where m denotes the number of sample points in $S^\mu_{\theta,n}$. Indeed, denoting

$$B^d_\lambda(s) := \left\{ \boldsymbol{x} \in \mathbb{R}^d : \, |\boldsymbol{x}|_\lambda \le s \right\},$$

and $V^d_\lambda(s)$ the volume of $B^d_\lambda(s)$, we get

$$\begin{aligned} m &\le \sum_{|\boldsymbol{k}|_\lambda < m_n} n_{\boldsymbol{k}} \le \sum_{|\boldsymbol{k}|=1}^{\lfloor m_n \rfloor} \varrho n e^{-\frac{a\delta}{\alpha}|\boldsymbol{k}|^\lambda_\lambda} \le n\varrho \sum_{s=0}^{\lfloor m_n \rfloor} \sum_{\boldsymbol{k}\in B^d_\lambda(s)} e^{-\frac{a\delta}{\alpha}s^\lambda} \\ &\ll n\varrho \sum_{s=0}^{\lfloor m_n \rfloor} V^d_\lambda(s) e^{-\frac{a\delta}{\alpha}s^\lambda} \ll n\varrho\, V^d_\lambda \sum_{s=0}^{\infty} s^d e^{-\frac{a\delta}{\alpha}s^\lambda} \le n. \end{aligned} \tag{2.13}$$

Secondly, we prove the bound (3.3). By (2.3), (2.1) (2.9) and (2.10) we derive the estimates

$$\begin{aligned} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k}) \right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll w^{1/q}(\boldsymbol{k}) \left\| \tilde{f}_{\theta,\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right) \right\|_{\tilde{L}_q(\mathbb{I}^d_\theta)} \\ &\ll w^{1/q}(\boldsymbol{k}) n_{\boldsymbol{k}}^{-\alpha} (\log n_{\boldsymbol{k}})^\beta \|f(\cdot + \boldsymbol{k})\varphi_{\boldsymbol{k}}(\cdot + \boldsymbol{k})\|_{\tilde{W}^r_p(\mathbb{I}^d_\theta)} \\ &\ll w^{1/q}(\boldsymbol{k}) w^{-1/p}(\boldsymbol{k}) n^{-\alpha} (\log n)^\beta e^{\delta \frac{a}{\alpha}|\boldsymbol{k}|^\lambda_\lambda} \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \end{aligned}$$

where the numbers $n_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{Z}^d$, are defined as in (2.3). Hence, by using the inequality (2.11) we get

$$\left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} \ll e^{-\varepsilon|\boldsymbol{k}|_\lambda^\lambda} n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)},$$

where $\varepsilon := \delta' - \delta a/\alpha > 0$ due to (2.12). This in a similar manner as (2.13) implies that

$$\begin{aligned}
\sum_{|\boldsymbol{k}|_\lambda < m_n} \left\| f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}} - S_{\theta,n_{\boldsymbol{k}}}\left(\tilde{f}_{\theta,\boldsymbol{k}}\right)(\cdot - \boldsymbol{k})\right\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll \sum_{|\boldsymbol{k}|_\lambda < m_n} e^{-\varepsilon|\boldsymbol{k}|_\lambda^\lambda} n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\
&\le n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}| < m_n} e^{-\varepsilon|\boldsymbol{k}|_\lambda^\lambda} \\
&\ll n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}.
\end{aligned}$$

Again in a similar manner as (2.13) we have by (2.2), (2.11) and the inequality $\delta'/\delta > 1$ provided with (2.12),

$$\begin{aligned}
\sum_{|\boldsymbol{k}|_\lambda \ge m_n} \|f_{\theta,\boldsymbol{k}}\varphi_{\boldsymbol{k}}\|_{L_q(\mathbb{I}^d_{\theta,\boldsymbol{k}};\mu)} &\ll \sum_{|\boldsymbol{k}| \ge m_n} w^{1/q}(\boldsymbol{k}) w^{-1/p}(\boldsymbol{k}) \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \\
&\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{|\boldsymbol{k}|_\lambda \ge m_n} e^{-\delta'|\boldsymbol{k}|_\lambda^\lambda} \\
&\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} \sum_{s \ge \lfloor m_n \rfloor} V^d_\lambda(s) e^{-\delta' s^\lambda} \\
&\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} V^d_\lambda \sum_{s \ge \lfloor m_n \rfloor} s^d e^{-\delta' s^\lambda} \\
&\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} m_n^d e^{-\delta' m_n^\lambda} \sum_{s=0}^\infty s^d e^{-\frac{a\delta}{\alpha} s^\lambda} \\
&\ll \|f\|_{W^r_p(\mathbb{R}^d;\mu)} (\log n)^{d/\lambda} e^{-\delta' \alpha \log n/\delta} \sum_{s=0}^\infty s^d e^{-\frac{a\delta}{\alpha} s^\lambda} \\
&\ll n^{-\alpha}(\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}.
\end{aligned}$$

From the last two estimates and (2.7) we obtain (3.3). □

We introduce Smolyak sampling algorithms for 1-periodic functions on $\mathbb{R}^d$ based B-spline quasi-interpolation, which satisfy (2.1). For a given number $\ell \in \mathbb{N}$, denote by M_ℓ the cardinal B-spline of order ℓ with support $[0, \ell]$ and knots at the points $0, 1, ..., \ell$. We fixed an even number $\ell \in \mathbb{N}$ and take the cardinal B-spline $M := M_\ell$ of order ℓ. Let $\Lambda = \{\lambda(j)\}_{|j| \le \mu}$ be a given finite even sequence, i.e., $\lambda(-j) = \lambda(j)$ for some $\mu \ge \frac{\ell}{2} - 1$. We define the linear operator Q for functions f on $\mathbb{R}$ by

$$Q(f)(x) := \sum_{s \in \mathbb{Z}} \Lambda(f, s) M(x - s + \ell/2), \tag{2.14}$$

where

$$\Lambda(f, s) := \sum_{|j| \le \mu} \lambda(j) f(s - j). \tag{2.15}$$

The operator Q is local and bounded in $C(\mathbb{R})$ (see [4, p. 100–109]). An operator Q of the form (2.14)–(2.15) is called a quasi-interpolation operator in $C(\mathbb{R})$ if it reproduces $\mathcal{P}_{\ell-1}$, i.e., $Q(f) = f$ for every $f \in \mathcal{P}_{\ell-1}$, where $\mathcal{P}_{\ell-1}$ denotes the set of d-variate polynomials of degree at most $\ell - 1$ in each variable.

We present two well-known examples of quasi-interpolation operator. A piecewise linear quasi-interpolation operator is defined as

$$Q(f)(x) := \sum_{s \in \mathbb{Z}} f(s) M(x - s + 1),$$

where M is the piecewise linear B-spline with support $[0, 2]$ and knots at the integer points $0, 1, 2$. It is related to the classical Faber-Schauder basis of the hat functions Another example is the cubic quasi-interpolation operator

$$Q(f)(x) := \sum_{s \in \mathbb{Z}} \frac{1}{6}\{-f(s-1) + 8f(s) - f(s+1)\} M(x - s + 2),$$

where M is the symmetric cubic B-spline with support $[0, 4]$ and knots at the integer points $0, 1, 2, 4$. For more examples of B-spline quasi-interpolation operators, see, e.g., [1, 4].

Since $M(\ell\, 2^k x) = 0$ for every $k \in \mathbb{N}_0$ and $x \notin (0, 1)$, we can extend the restriction to the interval $[0, 1]$ of the B-spline $M(\ell\, 2^k \cdot)$ to an 1-periodic function on the whole $\mathbb{R}$. Denote this periodic extension by N_k and define

$$N_{k,s}(x) := N_k(x - h^{(k)} s), \ k \in \mathbb{Z}_+, \ s \in I(k),$$

where

$$I(k) := \{0, 1, ..., \ell 2^k - 1\}.$$

Then we have for 1-periodic functions f on $\mathbb{R}$,

$$Q_k(f)(x) \ = \sum_{\boldsymbol{s} \in I(\boldsymbol{k})} a_{k,s}(f) N_{k,s}(x), \quad \forall x \in \mathbb{R}. \tag{2.16}$$

For convenience we define the univariate operator Q_{-1} by putting $Q_{-1}(f) := 0$.

We define the univariate B-spline $N_{k,s}$ by

$$N_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}) := \bigotimes_{i=1}^{d} N_{k_i,s_i}(x_i), \ \boldsymbol{k} \in \mathbb{Z}^d_+, \ \boldsymbol{s} \in I(\boldsymbol{k}),$$

where

$$I(\boldsymbol{k}) := \prod_{i=1}^{d} I(k_i).$$

Let the operators $q_{\boldsymbol{k}}$ be defined by

$$q_{\boldsymbol{k}} := \prod_{i=1}^{d} \Big(Q_{k_i} - Q_{k_i - 1} \Big), \ \boldsymbol{k} \in \mathbb{Z}^d_+.$$

where the univariate operator $Q_{k_i} - Q_{k_i-1}$ is applied to the univariate function f by considering f as a function of variable x_i with the other variables held fixed.

From the refinement equation for the B-spline M (see, e.g., [4, (4.3.4)]), in the univariate case, we can represent the component functions $q_{\boldsymbol{k}}(f)$ as

$$q_{\boldsymbol{k}}(f) = \sum_{\boldsymbol{s} \in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}, \tag{2.17}$$

where where the coefficient functionals $c_{\boldsymbol{k},\boldsymbol{s}}(f)$ are explicitly constructed as linear combinations of at most m_0 of function values of f for some $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}$ and f.

For $m \in \mathbb{N}$, the well known periodic Smolyak grid of points $G^d(m)$ is defined as

$$G^d(m) := \{\boldsymbol{x} = 2^{-\boldsymbol{k}}\boldsymbol{s} : \boldsymbol{k} \in \mathbb{N}^d, \ |\boldsymbol{k}|_1 = m, \ \boldsymbol{s} \in I(\boldsymbol{k})\}.$$

Here and in what follows, we use the notations: $\boldsymbol{xy} := (x_1y_1, ..., x_dy_d)$ and $2^{\boldsymbol{x}} := (2^{x_1}, ..., 2^{x_d})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.

For $m \in \mathbb{N}_0$, we define the operator R_m by

$$R_m(f) := \sum_{|\boldsymbol{k}|_1 \le m} q_{\boldsymbol{k}}(f) \ = \sum_{|\boldsymbol{k}|_1 \le m} \ \sum_{\boldsymbol{s} \in I(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) \, N_{\boldsymbol{k},\boldsymbol{s}}.$$

From (2.17) one can see that for a functions f on $\mathbb{I}^d$, R_m defines a sampling algorithm on $\mathbb{I}^d$ of the form (1.7):

$$R_m(f) \ = \sum_{2^{-\boldsymbol{k}}\boldsymbol{s} \in G^d(m)} f(2^{-\boldsymbol{k}}\boldsymbol{s}) \varphi_{\boldsymbol{k},\boldsymbol{s}},$$

where $n := \ |G^d(m)|$, and $\varphi_{\boldsymbol{k},\boldsymbol{s}}$ are explicitly constructed as linear combinations of at most at most m_0 B-splines $N_{\boldsymbol{k},\boldsymbol{j}}$ for some $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}, m$ and f. The operator R_m is also called Smolyak (sparse-gird) sampling algorithm initiated and used by him [22] for quadrature and interpolation for functions with mixed smoothness. It plays an important role in sampling recovery of multivariate functions and its applications (see [2, 11] for comments and bibliography).

Lemma 2.2 *Let $1 \le q \le p \le \infty$ and $1/p < r < \ell$. For $n \in \mathbb{N}$, let m_n be the largest integer number such that $|G^d(m_n)| \le n$. Then we have $R_{m_n} \in \mathcal{S}_n(\mathbb{I}^d)$ and*

$$\sup_{f \in \tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d)} \big\| f - R_{m_n}(f) \big\|_{\tilde{L}_q(\mathbb{I}^d)} \ll \begin{cases} n^{-r} (\log n)^{(r+1/2)(d-1)}, & 1 < q \le p < \infty; \\ n^{-r} (\log n)^{(r+1)(d-1)}, & \textit{either } q = 1 \textit{ or } p = \infty. \end{cases} \tag{2.18}$$

Proof. The case $1 < q \le p < \infty$ in (2.18) was proven in [5, Corollary 4.1]. We consider the case when either $q = 1$ or $p = \infty$ in (2.18). Let $\tilde{H}^r_p(\mathbb{I}^d)$ be the Hölder-Nikol'skii space of 1-periodic functions on $\mathbb{R}^d$ of mixed smoothness r bounded in the space $\tilde{L}_p(\mathbb{I}^d)$ (see, e.g., [11] for the definition). Then the bound (2.18) follows from the embedding $\tilde{W}^r_p(\mathbb{I}^d)$ into $\tilde{H}^r_p(\mathbb{I}^d)$ and the bound

$$\sup_{f \in \tilde{\boldsymbol{H}}^r_p(\mathbb{I}^d)} \big\| f - R_{m_n}(f) \big\|_{\tilde{L}_q(\mathbb{I}^d)} \ll n^{-r} (\log n)^{(r+1)(d-1)},$$

which can be proven in the same way as the non-periodic version [9, Theorem 3.3] . □

If $S_n = R_{m_n}$ where the operator R_{m_n} is defined as in Lemma 2.2, we write $S^\mu_{\theta,n} := R^\mu_{\theta,n}$.

Let $n \in \mathbb{N}$ and let X be a Banach space and F a central symmetric compact set in X. Then the Kolmogorov n-width of F is defined by

$$d_n(F,X) := \inf_{L_n} \sup_{f\in F} \inf_{g\in L_n} \|f-g\|_X,$$

where the left-most infimum is taken over all subspaces L_n of dimension $\le n$ in X. If X is a normed space of functions on Ω and $F \subset X$ is a set of continuous functions on Ω, then from the definitions we have

$$\varrho_n(F,X) \ge d_n(F,X). \tag{2.19}$$

This inequality will be employed to establish lower bounds for sampling n-widths,

Lemma 2.3 *Let $1 \le q < p \le \infty$ Then we have*

$$d_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \asymp n^{-r}(\log n)^{(d-1)r}. \tag{2.20}$$

For detail on the proof of this lemma see, e.g., in [11, Theorems 4.2.5, 4.3.1, 4.3.6 & 4.3.7] and related comments on the asymptotic optimality of the hyperbolic cross approximation.

Theorem 2.4 *Let $1 \le q < p \le \infty$, $1/p < r < \ell$ and $\theta > 0$, and denote*

$$\varrho_n := \varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)).$$

Then for any $n \in \mathbb{N}$, based on the sampling algorithm $S_n := R_{m_n} \in \mathcal{S}_n(\mathbb{I}^d)$ in Lemma 2.2, then one can construct a sampling algorithm $R^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ of the form (2.4) so that

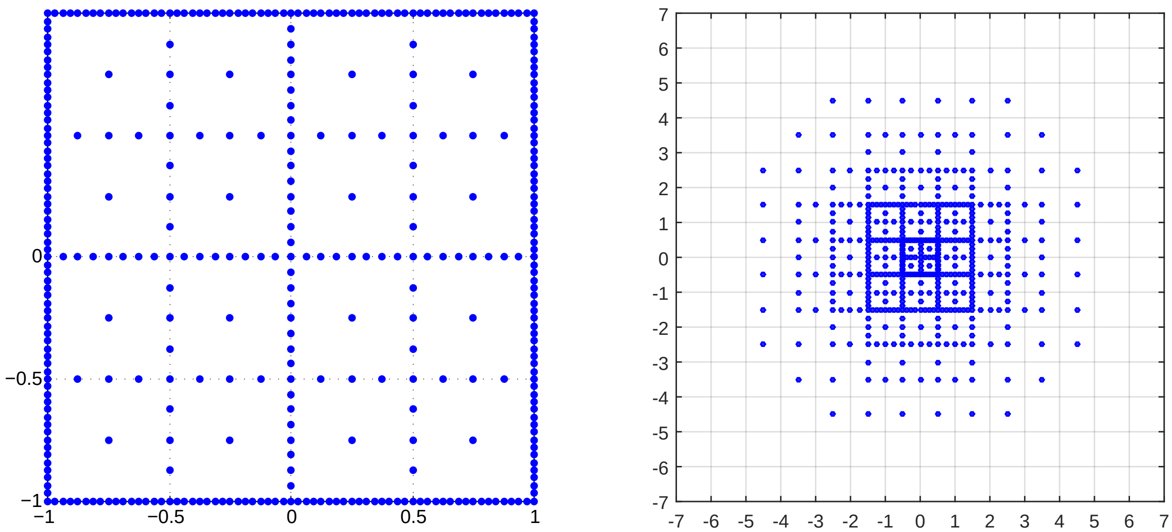
$$\varrho_n \le \sup_{f\in \boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \left\|f - R^\mu_{\theta,n}(f)\right\|_{L_q(\mathbb{R}^d;\mu)} \ll \begin{cases} n^{-r}(\log n)^{(r+1/2)(d-1)}, & 1<q<p<\infty; \\ n^{-r}(\log n)^{(r+1)(d-1)}, & \text{either } q=1 \text{ or } p=\infty. \end{cases} \tag{2.21}$$

Moreover,

$$\varrho_n \gg n^{-r}(\log n)^{r(d-1)}. \tag{2.22}$$

Proof. For a fixed $\theta > 0$, we define $R^\mu_{\theta,n} := S^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ as the sampling algorithm described in Theorem 2.1 with $S_n = R_{m_n} \in \mathcal{S}_n(\mathbb{I}^d)$. The upper bounds (2.21) follow from Lemma 2.2 and Theorem 2.1 with $\alpha = r$ and $\beta = (d-1)(r+1/2)$ for $1<q<p<\infty$, and $\beta = (d-1)(r+1)$ for the other cases.

We next prove the lower bound (2.22). If f is a 1-periodic function on $\mathbb{R}^d$ and $f \in$



A classical Smolyak grid   An assembling of Smolyak grids

Figure 1: Different sparse grids on function domains ($d = 2$)

$\tilde{W}^r_p(\mathbb{I}^d)$, then for $1 < p < \infty$,

$$
\begin{aligned}
\|f\|^p_{W^r_p(\mathbb{R}^d;\mu)} &= \sum_{|\boldsymbol{r}|_\infty \le r} \int_{\mathbb{R}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\
&= \sum_{|\boldsymbol{r}|_\infty \le r} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x}+\boldsymbol{k})|^p w(\boldsymbol{x}+\boldsymbol{k}) \mathrm{d}\boldsymbol{x} \\
&\ll \sum_{|\boldsymbol{r}|_\infty \le r} \int_{\mathbb{I}^d} |D^{\boldsymbol{r}} f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \sum_{\boldsymbol{k} \in \mathbb{Z}^d} w(|\boldsymbol{k} - (\operatorname{sign} \boldsymbol{k})|) \\
&\ll \|f\|^p_{\tilde{W}^r_p(\mathbb{I}^d)}.
\end{aligned}
$$

The bound $\|f\|_{W^r_\infty(\mathbb{R}^d;\mu)} \ll \|f\|_{\tilde{W}^r_\infty(\mathbb{I}^d)}$ can be proven similarly with a slight modification. On the other hand,

$$
\|f\|_{\tilde{L}_q(\mathbb{I}^d)} = \|f(\cdot + \mathbf{1})\|_{\tilde{L}_q(\mathbb{I}^d)} \ll \|f\|_{L_q(\mathbb{R}^d;\mu)},
$$

where recall, $\mathbf{1} := (1, 1, ..., 1) \in \mathbb{R}^d$. Hence we get by the inequality (2.19) and Lemma 2.3 the lower bound (2.22):

$$
\begin{aligned}
\varrho_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) &\ge d_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu), L_q(\mathbb{R}^d;\mu)) \\
&\gg d_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d), \tilde{L}_q(\mathbb{I}^d)) \gg n^{-r} (\log n)^{r(d-1)}.
\end{aligned}
$$

□

The sampling algorithms $R^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ of the form (2.4) in Theorem 2.4, are based on very sparse sample points contained in the set

$$
\left\{ \boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}|_\lambda \le C (\log n)^{1/\lambda} \right\}
$$

(see Figure 1), and completely different from the sample points in step hyperbolic cross for the case $1 < p = q < \infty$ considered in [8]. In Figure 1, a classical Smolyak grid on the domain $[-1,1]^2$ is in the left picture, and an assembled grid in the two-dimensional function domain $\mathbb{R}^2$ for the weight $w(x_1,x_2) := \exp(-x_1^2 - x_2^2)$ in the right picture, is designed on classical Smolyak grids.

In the one-dimensional case, we have the following refined result.

Corollary 2.5 *Let $1 \le q < p \le \infty$, $1/p < r < \ell$ and $\theta > 0$. For any $n \in \mathbb{N}$, based on the sampling algorithm $R_{m_n} \in \mathcal{S}_n(\mathbb{I})$ in Lemma 2.2, one can construct the sampling algorithm $R^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R})$ as in (2.4) so that*

$$\varrho_n(\boldsymbol{W}^r_p(\mathbb{R};\mu), L_q(\mathbb{R};\mu)) \asymp \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R};\mu)} \left\| f - R^\mu_{\theta,n}(f) \right\|_{L_q(\mathbb{R}^d;\mu)} \asymp n^{-r}. \tag{2.23}$$

Here we point out that, in constructing the sampling algorithm $S^\mu_{\theta,n} = R^\mu_{\theta,n}$ as in (2.4), one could replace the B-spline sampling algorithms $S_n = R_{m_n}$ and Lemma 2.2 with their trigonometric modifications. In particular, a related version of Lemma 2.2, proven in [3], can be employed to obtain the same bounds for convergence rate of the sampling recovery as in Theorem 2.4 and Corollary 2.5.

3 Numerical integration

Notice that every sampling algorithm $S_n \in \mathcal{S}_n(\mathbb{R}^d)$ of the form (1.7) generates in a natural way a weighted quadrature $Q_n \in \mathcal{Q}_n(\mathbb{R}^d)$ by the formula

$$Q_n f := \int_{\mathbb{R}^d} S_n f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \sum_{i=1}^n \omega_i f(\boldsymbol{x}_i) \tag{3.1}$$

with the integration weights

$$\omega_i := \int_{\mathbb{R}^d} h_i(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}). \tag{3.2}$$

Moreover, the error of numerical integration by the quadrature Q_n can be estimated via the error in the norm $L_1(\mathbb{R}^d,\mu)$ of the sampling recovery by S_n. More precisely, it holds the inequality

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \, \mathrm{d}\mu(\boldsymbol{x}) - Q_n f \right| \le \|f - S_n(f)\|_{L_1(\mathbb{R}^d;\mu)}.$$

These observations together with the inequality (2.19) and the results on Kolmogorov n-widths in Lemma 2.3 allow to derive from the results on sampling recovery in Section 2 the following results.

Corollary 3.1 *Let $1 < p \le \infty$ and $\alpha > 0$, $\beta \ge 0$, $\theta > 0$. Assume that for any $n \in \mathbb{N}$, there is a linear sampling algorithm $S_n \in \mathcal{S}_n(\mathbb{I}^d)$ such that the convergence rate (2.1) holds. Let the sampling algorithm $S^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ be constructed by the formula (2.4) as* *in Theorem 2.1. For any $n \in \mathbb{N}$, let $Q^\mu_{\theta,n} \in \mathcal{Q}_n(\mathbb{R}^d)$ be the quadrature constructed by the formula (3.1)–(3.2). Then we have*

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \,\mathrm{d}\mu(\boldsymbol{x}) - Q^\mu_{\theta,n} f \right| \le C n^{-\alpha} (\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}, \quad f \in W^r_p(\mathbb{R}^d;\mu). \tag{3.3}$$

Proof. Let $f \in W^r_p(\mathbb{R}^d;\mu)$. We have by Theorem 2.1 for $q = 1$,

$$\left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \,\mathrm{d}\mu(\boldsymbol{x}) - Q^\mu_{\theta,n} f \right| \le \|f - S^\mu_{\theta,n}(f)\|_{L_1(\mathbb{R}^d;\mu)} \le C n^{-\alpha} (\log n)^\beta \|f\|_{W^r_p(\mathbb{R}^d;\mu)}$$

□

Corollary 3.2 *Let $1 < p \le \infty$, $1/p < r < \ell$ and $\theta > 0$. Let the sampling algorithm $R^\mu_{\theta,n} \in \mathcal{S}_n(\mathbb{R}^d)$ be constructed by the formula (2.4) as in Theorem 2.4 for $S_n := R_{m_n} \in \mathcal{S}_n(\mathbb{I}^d)$ as in Lemma 2.2. For any $n \in \mathbb{N}$, let $Q^\mu_{\theta,n} \in \mathcal{Q}_n(\mathbb{R}^d)$ be the quadrature constructed by the formula (3.1)–(3.2). Then we have*

$$\mathrm{Int}_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)) \le \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R}^d;\mu)} \left| \int_{\mathbb{R}^d} f(\boldsymbol{x}) \,\mathrm{d}\mu(\boldsymbol{x}) - Q^\mu_{\theta,n} f \right| \ll n^{-r} (\log n)^{(r+1)(d-1)}. \tag{3.4}$$

Moreover, if in addition, $p < \infty$,

$$\mathrm{Int}_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)) \gg n^{-r} (\log n)^{r(d-1)/2}. \tag{3.5}$$

Proof. The upper bounds (3.4) follow from Lemma 2.2 and Corollary 3.2 with $\alpha = r$ and $\beta = (r+1)(d-1)$. We next prove the lower bound (3.5). In the proof of Theorem 2.4 it has been proven that

$$\|f\|_{W^r_p(\mathbb{R}^d;\mu)} \ll \|f\|_{\tilde{W}^r_p(\mathbb{I}^d)}, \quad f \in \tilde{W}^r_p(\mathbb{I}^d). \tag{3.6}$$

Hence we get by the lower bound (3.5) for $1 < p < \infty$:

$$\mathrm{Int}_n(\boldsymbol{W}^r_p(\mathbb{R}^d;\mu)) \gg \mathrm{Int}_n(\tilde{\boldsymbol{W}}^r_p(\mathbb{I}^d)) \gg n^{-r} (\log n)^{r(d-1)/2}.$$

Here, for the last inequality, see, e.g., [11, Theorem 8.2.1]. □

Corollary 3.3 *Under the assumption and notation of Corollary 3.2 we have for $d = 1$,*

$$\mathrm{Int}_n(\boldsymbol{W}^r_p(\mathbb{R};\mu)) \asymp \sup_{f \in \boldsymbol{W}^r_p(\mathbb{R};\mu)} \left| \int_{\mathbb{R}} f(x) \,\mathrm{d}\mu(x) - Q^\mu_{\theta,n} f \right| \asymp n^{-r}. \tag{3.7}$$

Proof. This corollary in the case $1 < p < \infty$ as well as the upper bound in (3.7) in the case $p = \infty$ follow from Corollary 3.2. The lower bound bound in (3.7) in the case $p = \infty$ is derived from the inequalities (3.6) and

$$\mathrm{Int}_n(\boldsymbol{W}^r_\infty(\mathbb{R};\mu)) \gg \mathrm{Int}_n(\tilde{\boldsymbol{W}}^r_\infty(\mathbb{I})) \gg n^{-r} (\log n)^{r(d-1)/2}.$$

Here, for the last inequality, see, e.g., [25, Theorem 2.25]. ∎

Acknowledgments: This work is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) in the frame of the NAFOSTED-SNSF Joint Research Project under Grant IZVSZ2_229568. A part of this work was done when the author was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for providing a fruitful research environment and working condition. The author specially thanks Dr. Nguyen Van Kien for drawing Figure 1.

References

[1] C. D. Boor, K. Höllig, and S. Riemenschneider. *Box Splines*. Springer, 2013.

[2] H. J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147–269, 2004.

[3] G. Byrenheid and T. Ullrich. Optimal sampling recovery of mixed order Sobolev embeddings via discrete Littlewood-Paley type characterizations. *Anal. Math.*, 43(2):133–191, 2017.

[4] C. K. Chui. *An Introduction to Wavelets*. Academic Press, 1992.

[5] D. Dũng. B-spline quasi-interpolation sampling representation and sampling recovery in Sobolev spaces of mixed smoothness. *Acta Math. Vietnamica*, 43:83–110, 2018.

[6] D. Dũng. Numerical weighted integration of functions having mixed smoothness. *J. Complexity*, 78:101757, 2023.

[7] D. Dũng. Sparse-grid sampling recovery and numerical integration of functions having mixed smoothness. *Acta Mathematica Vietnamica*, 49:377–426, 2024.

[8] D. Dũng. Weighted sampling recovery of functions with mixed smoothness. *arXiv:2405.16400*, 2024.

[9] D. Dũng. B-spline quasi-interpolant representations and sampling recovery of functions with mixed smoothness. *J. Complexity*, 27:541–567, 2011.

[10] D. Dũng and V. K. Nguyen. Optimal numerical integration and approximation of functions on $\mathbb{R}^d$ equipped with Gaussian measure. *IMA Journal of Numer. Anal.*, 44:1242–1267, 2024.

[11] D. Dũng, V. N. Temlyakov, and T. Ullrich. *Hyperbolic Cross Approximation*. Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser/Springer, 2018.

[12] J. Dick, C. Irrgeher, G. Leobacher, and F. Pillichshammer. On the optimal order of integration in Hermite spaces with finite smoothness. *SIAM J. Numer. Anal.*, 56:684–707, 2018.

[13] J. Dick, F. Y. Kuo, and I. H. Sloan. High-dimensional integration: the quasi-Monte Carlo way. *Acta Numer.*, 22:133–288, 2013.

[14] T. Goda, K. Suzuki, and T. Yoshiki. Optimal order quadrature error bounds for infinitedimensional higher order digital sequences. *Found. Comput. Math.*, 18:433–458, 2018.

[15] C. Irrgeher, P. Kritzer, G. Leobacher, and F. Pillichshammer. Integration in Hermite spaces of analytic functions. *J. Complexity*, 31:308–404, 2015.

[16] C. Irrgeher and G. Leobacher. High-dimensional integration on the $\mathbb{R}^d$, weighted Hermite spaces, and orthogonal transforms. *J. Complexity*, 31:174–205, 2015.

[17] P. Junghanns, G. Mastroianni, and I. Notarangelo. *Weighted Polynomial Approximation and Numerical Methods for Integral Equations.* Birkhäuser, 2021.

[18] D. S. Lubinsky. A survey of weighted polynomial approximation with exponential weights. *Surveys in Approximation Theory*, 3:1–105, 2007.

[19] H. N. Mhaskar. *Introduction to the Theory of Weighted Polynomial Approximation.* World Scientific, Singapore, 1996.

[20] V. Nguyen and W. Sickel. Pointwise multipliers for Sobolev and Besov spaces of dominating mixed smoothness. *J. Math. Anal. Appl.*, 452:62–90, 2017.

[21] E. Novak and H. Woźniakowski. *Tractability of Multivariate Problems, Volume II: Standard Information for Functionals.* EMS Tracts in Mathematics, Vol. 12, Eur. Math. Soc. Publ. House, Zürich, 2010.

[22] S. Smolyak. Quadrature and interpolation formulas for tensor products of certain classes of functions. *Dokl. Akad. Nauk*, 148:1042–1045, 1963.

[23] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, Princeton, NJ, 1970.

[24] Y. Suzuki and T. Karvonen. Construction of optimal algorithms for function approximation in Gaussian Sobolev spaces . *arXiv Preprint*, arXiv:2402.02917 [math.NA], 2024.

[25] V. N. Temlyakov. *Multivariate Approximation.* Cambridge University Press, 2018.