Simultaneous spatial-parametric collocation approximation for parametric PDEs with log-normal random inputs

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Abstract

We proved convergence rates of fully discrete multi-level simultaneous linear collocation approximation of solutions to parametric elliptic PDEs on bounded polygonal domain with log-normal random inputs based on a finite number of their values at points in the spatial-parametric domain. These convergence rates significantly improve the best-known convergence rates of fully discrete collocation approximation and with some logarithm factors coincide with the convergence rates of best n-term approximation. These results are obtained as consequences of general results on multi-level linear sampling recovery in abstract Bochner spaces.

Keywords and Phrases: High dimensional approximation; Sampling recovery; Bochner spaces; Linear collocation approximation; Least squares approximation; Parametric PDEs with random inputs; Infinite dimensional holomorphic function; Convergence rate.

Mathematics Subject Classifications (2010): 65C30, 65N15, 65N35, 41A25.

1 Introduction and main results

Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain (recall that a polygonal domain in $\mathbb{R}^2$ is a polygon (which may have precludes cusps and slits) with a finite number of straight sides). Consider the divergence-form diffusion elliptic equation

$$-\operatorname{div}(a\nabla u) \;=\; f \quad \text{in} \quad D, \quad u|_{\partial D} \;=\; 0, \tag{1.1}$$

for a given fixed right-hand side f and a spatially variable scalar diffusion coefficient a. Denote by $V := H^1_0(D)$ the energy space and $V' = H^{-1}(D)$ the dual space of V. Assume that $f \in H^{-1}(D)$ and $a \in L_\infty(D)$ (in what follows this preliminary assumption always holds without mention). If a satisfies the ellipticity assumption

$$0 < a_{\min} \le a \le a_{\max} < \infty,$$

by the well-known Lax-Milgram lemma, there exists a unique weak solution $u \in V$ to the equation (1.1) satisfying the equation

$$\int_D a \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} \;=\; \langle f, v \rangle, \quad \forall v \in V.$$

For the equation (1.1), we consider the diffusion coefficients having a parametric form $a = a(\boldsymbol{y})$, where $\boldsymbol{y} = (y_j)_{j \in \mathbb{N}}$ is a sequence of real-valued parameters ranging in the set $\mathbb{R}^\infty$. Denote by $u(\boldsymbol{y})$ the weak solution to the parametric diffusion divergence-form elliptic equation

$$-\operatorname{div}(a(\boldsymbol{y}) \nabla u(\boldsymbol{y})) \;=\; f \quad \text{in} \quad D, \quad u(\boldsymbol{y})|_{\partial D} \;=\; 0. \tag{1.2}$$

In the present paper, we consider the case when the diffusion coefficient a is of the lognormal form

$$a(\boldsymbol{y}) = \exp(b(\boldsymbol{y})), \quad \text{with} \;\; b(\boldsymbol{y}) = \sum_{j=1}^{\infty} y_j \psi_j, \tag{1.3}$$

and y_j are i.i.d. standard Gaussian random variables, where $\psi_j \in L_\infty(D)$.

For $r \in \mathbb{N}_0$ and $\varkappa \in \mathbb{R}$, the Kondrat'ev spaces $\mathcal{K}^r_\varkappa(D)$ and $\mathcal{W}^r_\infty(D)$ are defined as the weighted normed spaces of functions on D equipped with norms

$$\|u\|_{\mathcal{K}^r_\varkappa(D)} := \sum_{|\boldsymbol{\alpha}| \le r} \|\tau_D^{|\boldsymbol{\alpha}| - \varkappa} D^{\boldsymbol{\alpha}} u\|_{L^2(D)} \qquad \text{and} \qquad \|u\|_{\mathcal{W}^r_\infty(D)} := \sum_{|\boldsymbol{\alpha}| \le r} \|\tau_D^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} u\|_{L^\infty(D)},$$

respectively. Here, $\tau_D : D \to [0,1]$ denotes a fixed smooth function that coincides with the distance to the nearest corner, in a neighborhood of each corner, $D^{\boldsymbol{\alpha}}$ denotes the weak partial derivative of order $\boldsymbol{\alpha} \in \mathbb{N}_0^2$, and $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2$. The function spaces $\mathcal{K}^r_\varkappa(D)$ and $\mathcal{W}^r_\infty(D)$ endowed with these norms are Banach spaces, and $\mathcal{K}^r_\varkappa(D)$ are separable Hilbert spaces. An embedding of these spaces is $\mathcal{K}^1_1(D) \hookrightarrow H^1_0(D)$ [11, Lemma 1.5], i.e., there exits a positive constant C such that

$$\|v\|_{H^1_0(D)} \;\le\; C \|v\|_{\mathcal{K}^1_1(D)}, \quad v \in \mathcal{K}^1_1(D). \tag{1.4}$$

By [24, Theorem 3.29], for $r \ge 2$, $f \in \mathcal{K}^{r-2}_{\varkappa-1}(D)$ and $a \in \mathcal{W}^{r-1}_\infty(D)$, the solution to (1.1) belongs to $\mathcal{K}^r_{\varkappa+1}(D)$ provided that with

$$\rho(a) := \operatorname*{ess\,inf}_{\boldsymbol{x} \in D} \Re(a(\boldsymbol{x})) > 0, \quad \text{and} \quad |\varkappa| < \frac{\rho(a)}{\nu \, \|a\|_{L_\infty(D)}},$$

where ν is a constant depending on D and r.

In computational uncertainty quantification, the problem of efficient approximation for parametric and stochastic PDEs has been of great interest and achieved significant progress in recent years. Depending on a particular setting, as usual, this problem leads to an approximation problem in a Bochner space $L_2(U, X; \mu)$ with an appropriate separable Hilbert space X, an infinite-dimensional domain U and a probability measure μ on U, where parametric solutions $u(\boldsymbol{y})$, $\boldsymbol{y} \in U$, to parametric and stochastic PDEs, are treated as elements of $L_2(U, X; \mu)$ and U the parametric domain. There is a vast number of works on this topic to not mention all of them. We point out just some works [1, 2, 3, 4, 5, 6, 7, 9, 14, 16, 13, 12, 17, 18, 20, 21, 24, 25, 26, 30, 31, 32, 33, 34] which are directly related to the problem setting in our paper. In semi-discrete (intrusive and non-intrusive) approximations, [2, 3, 5, 6, 9, 14, 16, 13, 12, 21, 22, 24, 25, 30, 31, 32, 34] discretizations are processed only with respect to the parametric variables, but not spatial variables, the approximants still belong to the infinite dimensional spatial space and hence are not useful for practical applications. They should themselves be approximated by spatial discretizations by means of finite elements, wavelets or spectral Galerkin methods, etc.. The problem of fully discrete (and multi-level) approximation of parametric PDEs with random inputs and of relevant infinite-dimensional holomorphic functions has been investigated in the works [4, 17, 18, 20, 21, 22, 24, 26, 33] based on some spatial approximation properties and parametric summabilities of the GPC expansion coefficients of solutions. Some bounds for convergence rates of spectral and collocation fully discrete approximation have been proven in these papers. Observe that by the problem setting a convergence rate of any fully discrete approximation is not better than the convergence rate given by the spatial approximation properties. Moreover, it was proven that they coincide in the case of sufficiently small spatial regularity. In the case of higher spatial regularity, the known bounds of convergence rates of fully discrete approximation are less than this regularity and depend essentially on parametric summability properties of the generalized polynomial chaos (GPC) expansion coefficients of solutions (see the above cited papers for detail).

A natural question arising is whether the known convergence rates are optimal and whether one can improve them. The aim of the present paper is to improve these known bounds in the case of higher spatial regularity of the solution of the parametric equation (1.2) with log-normal inputs (1.3). More precisely, we are interested in the problem of fully discrete simultaneous spatial-parametric linear collocation approximation of solutions u to parametric PDEs (1.2) on bounded polygonal domain D with log-normal random inputs (1.3) and its convergence rate, based on a finite number of its particular spatial-parametric values $u(\boldsymbol{x}_1, \boldsymbol{y}_1), ..., u(\boldsymbol{x}_n, \boldsymbol{y}_n)$. The approximation error is measured by the norm of the Bochner space $L_2(\mathbb{R}^\infty, V; \gamma) = V \otimes L_2(\mathbb{R}^\infty; \gamma)$ associated with the energy space V and the infinite standard Gaussian measure γ on $\mathbb{R}^\infty$ (see Subsection 2.2 for definition of Bochner space). The present paper can be considered as a continuation of the works [7, 21, 24]. In [21, 24], we have investigated the problem of fully discrete multi-level collocation approximation of stochastic and parametric elliptic PDEs with log-normal inputs by using sparse-grid GPC Lagrange interpolation at the zeros of Hermite polynomials. In the recent paper [7], we have investigated semi-discrete collocation approximation of these equations by employing extended least squares sampling algorithms in Bochner spaces. In the present paper, we develop and combine the different techniques used in these papers for solving the formulated problem.

We give a short description of main contribution of the present paper with some comments.

Let $r \in \mathbb{N}$, $r \geq 2$, $f \in \mathcal{K}^{r-2}_{\varkappa-1}(D)$ and $\psi_k \in W^1_\infty(D) \cap \mathcal{W}^{r-1}_\infty(D)$, $k \in \mathbb{N}$. For $i = 1, 2$, let the sequences $\boldsymbol{b}_i := (b_{i,j})_{j \in \mathbb{N}}$ be defined by

$$b_{1,j} := \|\psi_j\|_{L^\infty}, \quad \text{and} \quad b_{2,j} := \max\left\{ \|\psi_j\|_{W^1_\infty(D)}, \|\psi_j\|_{\mathcal{W}^{r-1}_\infty(D)} \right\},$$

and satisfy the condition $\boldsymbol{b}_i \in \ell^{p_i}(\mathbb{N})$, respectively, with $0 < p_1 \leq p_2 < 1$ and $p_1 < 2/3$. Let the numbers α and β be defined by

$$\alpha := \frac{r-1}{2}, \quad \beta := \left(\frac{1}{p_1} - 1\right)\frac{\alpha}{\alpha+\delta}, \quad \delta := \frac{1}{p_1} - \frac{1}{p_2}.$$

Then for all $n \geq 2$ there exist points $(\boldsymbol{x}_1, \boldsymbol{y}_1), ..., (\boldsymbol{x}_n, \boldsymbol{y}_n) \in D \times \mathbb{R}^\infty$ and functions $\varphi_1, ..., \varphi_n \in V$ and $h_1, ..., h_n \in L_2(\mathbb{R}^\infty, \mathbb{R}; \gamma)$ such that for the linear sampling algorithm S_n on the spatial-parametric domain $D \times \mathbb{R}^\infty$ defined by

$$S_n(v)(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^{n} v(\boldsymbol{x}_i, \boldsymbol{y}_i)\varphi_i(\boldsymbol{x})h_i(\boldsymbol{y}), \quad \boldsymbol{x} \in D, \quad \boldsymbol{y} \in \mathbb{R}^\infty,$$

and for the parametric solution u to the equation (1.2) with log-normal random inputs (1.3), it holds true the error bounds

$$\left\|u - S_n u\right\|_{L_2(\mathbb{R}^\infty, V;\gamma)} \leq \begin{cases} Cn^{-\alpha} & \text{if } \alpha \leq 1/p_2 - 3/2, \\ C_\varepsilon n^{-\alpha}(\log n)^{\alpha+1+\varepsilon} & \text{if } 1/p_2 - 3/2 < \alpha < 1/p_2 - 1, \\ C_\varepsilon n^{-\alpha}(\log n)^{\alpha+2+\varepsilon} & \text{if } \alpha = 1/p_2 - 1, \\ Cn^{-\beta}(\log n)^{\beta+\alpha/(\alpha-\beta)} & \text{if } \alpha > 1/p_2 - 1, \end{cases} \tag{1.5}$$

for an arbitrarily small number $\varepsilon > 0$, where the constants C and C_ε are independent of u and n.

The convergence rate $n^{-\alpha}$ in (1.5) in the case of small spatial regularity $\alpha \leq 1/p_2 - 3/2$ is as expected. The convergence rates in (1.5) in the cases of higher spatial regularity $\alpha > 1/p_2 - 3/2$ significantly improve the best-known convergence rates of fully discrete collocation approximation of u which is $n^{-\beta'}$ where $\beta' := \beta - \frac{1}{2}\frac{\alpha}{\alpha+\delta}$ (cf. [21, 22, 24]). Moreover, in the particular intermediate case $1/p_2 - 3/2 < \alpha \leq 1/p_2 - 1$, the convergence rate $n^{-\alpha}$ still holds with a logarithm factor. Notice also that with some logarithm factors, the convergence rates in (1.5) coincide with the convergence rates $n^{-\min(\alpha,\beta)}$ of fully discrete best n-term approximation of the parametric solution u in the space $L_2(\mathbb{R}^\infty, V; \gamma)$ (cf. [4, 17, 18, 20, 21, 22, 24, 26, 33]).

Notice that S_n is a fully discrete multi-level collocation approximation method. At each level in S_n, the spatial component is based on finite element Lagrange interpolation associated with triangulations of the spatial domain D, while the parametric component is based on Hermite-Lagrange GPC interpolation in the case $\alpha \leq 1/p_2 - 3/2$, and on extended least squares sampling algorithms in the cases $\alpha > 1/p_2 - 3/2$.

Following the approach in [7, 21, 24], the convergence rate results in (1.5) and constructions of the linear sampling algorithms S_n realizing them, are obtained as consequences of general results on multi-level linear sampling recovery in abstract Bochner spaces. In the case of small spatial regularity $\alpha \leq 1/p_2 - 3/2$, we use a modification of methods of sparse-grid Hermite-Lagrange GPC interpolation in Bochner spaces in [21]. In the cases of higher spatial regularity $\alpha > 1/p_2 - 3/2$, we used extended least squares sampling algorithms in Bochner spaces and some recent results on linear sampling recovery of functions in reproducing kernel Hilbert spaces [8, 23, 27, 28]. We would like to emphasize that the general theory of sampling recovery in abstract Bochner spaces presented in this paper as well as [7, 21, 24] is applicable to uncertainty quantification for a wide class of parametric and stochastic PDEs.

This paper is organized as follows.

In Section 2, we investigate multi-level linear sampling recovery of functions in an abstract Bochner space $L_2(U, X; \mu)$ for a Hilbert space X and a probability measure space (U, Σ, μ), based on some weighted ℓ_2-summability of their coefficients of an orthonormal expansion, and some approximation properties in the space X. The received results are then applied to $L_2(\mathbb{R}^\infty, X; \gamma)$, the Bochner space associated with a Hilbert space X and the infinite standard Gaussian measure γ on $\mathbb{R}^\infty$. In Section 3, we apply the results on multi-level linear sampling recovery in the space $L_2(\mathbb{R}^\infty, X; \gamma)$ in the previous section to $(\boldsymbol{b}, \xi, \delta, X)$-holomorphic functions on $\mathbb{R}^\infty$. In Section 4, we construct fully discrete multi-level sparse-grid sampling algorithms for $(\boldsymbol{b}, \xi, \delta, X)$-holomorphic functions, based on GPC Lagrange-Hermite interpolation, and prove convergence rates of the approximation by them. In Section 5, we prove that the parametric solution u to the parametric elliptic PDEs (1.1) on bounded polygonal domain with log-normal inputs (1.3) is a $(\boldsymbol{b}_j, \xi, \delta, V)$-holomorphic function. This allows prove convergence rates of a fully discrete multi-level collocation algorithm by applying the results in Sections 3 and 4. In Section 6, we discuss various least squares sampling algorithms for functions in the reproducing kernel Hilbert space, and inequalities between sampling n-widths and Kolmogorov n-widths of the unit ball of this space, and how to apply these inequalities to obtain corresponding convergence rates of multi-level linear sampling recovery in abstract Bochner spaces and of fully discrete multi-level collocation approximation of the parametric solution u to the parametric elliptic PDEs (1.1) on bounded polygonal domain with log-normal inputs (1.3).

Notation As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, and $\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}$. We denote $\mathbb{R}^\infty$ the set of all sequences $\boldsymbol{y} = (y_j)_{j \in \mathbb{N}}$ with $y_j \in \mathbb{R}$. Denote by $\mathbb{F}$ the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j \in \mathbb{N}}$ such that their support $\operatorname{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. If $\boldsymbol{a} = (a_j)_{j \in \mathcal{J}}$ is a set of positive numbers with any index set $\mathcal{J}$, then we use the notation $\boldsymbol{a}^{-1} := (a_j^{-1})_{j \in \mathcal{J}}$. We use letters C and K to denote general positive constants which may take different values, and $C_{a,b,\ldots,d}$ and $K_{a,b,\ldots,d}$ constants depending on $a, b, \ldots, d$. For the quantities $A_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ such that $A_n(f, \boldsymbol{k}) \leq C B_n(f, \boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f, \boldsymbol{k}) \gg B_n(f, \boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f, \boldsymbol{k}) \asymp B_n(f, \boldsymbol{k})$ if $A_n(f, \boldsymbol{k}) \ll B_n(f, \boldsymbol{k})$ and $B_n(f, \boldsymbol{k}) \ll A_n(f, \boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G.

2 Sampling recovery in Bochner spaces

In this section, we investigate multi-level linear sampling recovery of functions in abstract Bochner spaces $L_2(U, X; \mu)$ for a Hilbert space X and a probability measure space (U, Σ, μ), based on some weighted ℓ_2-summability of their coefficients of an orthonormal expansion, and some approximation properties in the space X. We develop the methods used in [7, 21, 24] to construct fully discrete (multi-level) sampling algorithms by using the approximation properties in the space X combined with extended least squares sampling algorithms at each level, and prove convergence rates of approximation by them. The received results are then applied to $L_2(\mathbb{R}^\infty, X; \gamma)$, the Bochner space associated with a Hilbert space X and the infinite standard Gaussian measure γ on $\mathbb{R}^\infty$.

2.1 Function spaces

Let (U, Σ, μ) be a probability measure space with Σ being countably generated and let X be a complex separable Hilbert space. Denote by $L_2(U, X; \mu)$ the Bochner space of strongly μ-measurable mappings v from U to X, equipped with the norm

$$\|v\|_{L_2(U,X;\mu)} := \left(\int_U \|v(\boldsymbol{y})\|_X^2 \, \mathrm{d}\mu(\boldsymbol{y}) \right)^{1/2}. \tag{2.1}$$

Notice that because Σ is countably generated, $L_2(U, \mathbb{C}; \mu)$ is separable by [19, Proposition 3.4.5]. Hence $L_2(U, X; \mu)$ is a separable complex Hilbert space and, moreover,

$$L_2(U, X; \mu) = L_2(U, \mathbb{C}; \mu) \otimes X.$$

Let $(\varphi_s)_{s\in\mathbb{N}}$ be a fixed orthonormal basis of $L_2(U, \mathbb{C}; \mu)$. Then a function $v \in L_2(U, X; \mu)$ can be represented by the expansion

$$v(\boldsymbol{y}) = \sum_{s\in\mathbb{N}} v_s \, \varphi_s(\boldsymbol{y}), \quad v_s \in X, \tag{2.2}$$

with the series convergence in $L_2(U, X; \mu)$, where

$$v_s := \int_U v(\boldsymbol{y}) \, \overline{\varphi_s(\boldsymbol{y})} \, \mathrm{d}\mu(\boldsymbol{y}), \quad s \in \mathbb{N}.$$

Moreover, for every $v \in L_2(U, X; \mu)$ represented by the series (2.2), Parseval's identity holds

$$\|v\|_{L_2(U,X;\mu)}^2 \ = \ \sum_{s\in\mathbb{N}} \|v_s\|_X^2.$$

Let $\boldsymbol{\sigma} = (\sigma_s)_{s\in\mathbb{N}}$ be a non-decreasing sequence of positive numbers strictly larger than 1 such that $\boldsymbol{\sigma}^{-1} := \left(\sigma_s^{-1}\right)_{s\in\mathbb{N}} \in \ell_2(\mathbb{N})$. For given U and μ, denote by $H_{X,\boldsymbol{\sigma}}$ the linear subspace of all functions $v \in L_2(U, X; \mu)$ such that the norm

$$\|v\|_{H_{X,\boldsymbol{\sigma}}} := \left(\sum_{s\in\mathbb{N}} (\sigma_s \|v_s\|_X)^2 \right)^{1/2} < \infty.$$

In particular, the space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is the linear subspace in $L_2(U, \mathbb{C}; \mu)$ equipped with its own inner product

$$\langle f, g \rangle_{H_{\mathbb{C},\boldsymbol{\sigma}}} := \sum_{s\in\mathbb{N}} \sigma_s^2 \langle f, \varphi_s \rangle_{L_2(U,\mathbb{C};\mu)} \overline{\langle g, \varphi_s \rangle_{L_2(U,\mathbb{C};\mu)}}.$$

The space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is a reproducing kernel Hilbert space with the reproducing kernel

$$K(\cdot,\boldsymbol{y}) := \sum_{s\in\mathbb{N}} \sigma_s^{-2}\varphi_s(\cdot)\overline{\varphi_s(\boldsymbol{y})}$$

having the eigenfunctions $(\varphi_s)_{s\in\mathbb{N}}$ and the eigenvalues $\left(\sigma_{1,s}^{-1}\right)_{s\in\mathbb{N}}$. Moreover, $K(\boldsymbol{x},\boldsymbol{y})$ satisfies the finite trace assumption

$$\int_U K(\boldsymbol{y},\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) \ < \ \infty.$$

Let $(V_k)_{k\in\mathbb{N}_0}$ be a given sequence of subspaces $V_k \subset X$ of dimension 2^k, and $(P_k)_{k\in\mathbb{N}_0}$ a given sequence of uniformly bounded linear projectors P_k from X onto V_k. We extend the operators P_k (which with an abuse is denoted again by P_k) to $L_2(U,X;\mu)$ by the formula

$$(P_k v)(\boldsymbol{y}) := P_k(v(\boldsymbol{y})).$$

Let $(\sigma_{i;s})_{s\in\mathbb{N}}$, $i = 1,2$, be given non-decreasing sequences of numbers strictly larger than 1, such that $\boldsymbol{\sigma}_i^{-1} := \left(\sigma_{i,s}^{-1}\right)_{s\in\mathbb{N}} \in \ell_2(\mathbb{N})$. For a given number $\alpha > 0$, denote by H_X^α the linear subspace in $L_2(U,X;\mu)$ of all v such that the norm

$$\|v\|_{H_X^\alpha} := \left(\|v\|^2_{H_{X,\boldsymbol{\sigma}_1}} + \|v\|^2_{H^\alpha_{X,\boldsymbol{\sigma}_2}}\right)^{1/2} < \infty, \tag{2.3}$$

where the semi-norm $\|v\|_{H^\alpha_{X,\boldsymbol{\sigma}_2}}$ is defined by

$$\|v\|_{H^\alpha_{X,\boldsymbol{\sigma}_2}} := \sup_{k\in\mathbb{N}_0} 2^{\alpha k}\,\|v - P_k v\|_{H_{X,\boldsymbol{\sigma}_2}}. \tag{2.4}$$

Similarly, for given numbers $\alpha > 0$ and $\tau > 0$, denote by $H_X^{\alpha,\tau}$ the linear subspace in $L_2(U,X;\mu)$ of all v such that the norm

$$\|v\|_{H_X^{\alpha,\tau}} := \left(\|v\|^2_{H_{X,\boldsymbol{\sigma}_1}} + \|v\|^2_{H^{\alpha,\tau}_{X,\boldsymbol{\sigma}_2}}\right)^{1/2} < \infty, \tag{2.5}$$

where the semi-norm $\|v\|_{H^{\alpha,\tau}_{X,\boldsymbol{\sigma}_2}}$ is defined by

$$\|v\|_{H^{\alpha,\tau}_{X,\boldsymbol{\sigma}_2}} := \left(\sum_{k\in\mathbb{N}_0}\left(2^{\alpha k}k^{-\tau}\,\|v - P_k v\|_{H_{X,\boldsymbol{\sigma}_2}}\right)^2\right)^{1/2} < \infty. \tag{2.6}$$

Later on we will see that under certain assumptions the parametric solution u to the parametric elliptic PDE (1.2) with log-normal inputs (1.3) belongs to a certain space H_X^α. However, such a space has not Hilbertian structure and hence is not appropriate for constructing least squares sampling algorithms. Due to the continuous embedding $H_X^\alpha \hookrightarrow H_X^{\alpha,\tau}$ for any $\tau > 1$, the parametric solution u can be treated as an element of the Hilbert space $H_X^{\alpha,\tau}$ for which we are able to construct least squares sampling algorithms.

We will need the following lemma.

Lemma 2.1. *We have*

$$\|v - P_k v\|_{H_{X,\boldsymbol{\sigma}_2}} = \left(\sum_{s\in\mathbb{N}} (\sigma_{2;s}\|v_s - P_k v_s\|_X)^2\right)^{1/2}, \quad v \in H_{X,\boldsymbol{\sigma}_2}.$$

Proof. By the definition,

$$\|v - P_k v\|_{H_{X,\boldsymbol{\sigma}_2}} := \left(\sum_{s\in\mathbb{N}} (\sigma_s\|(v - P_k v)_s\|_X)^2\right)^{1/2} < \infty.$$

Due to the equality $(v - P_k v)_s = v_s - (P_k v)_s$, to prove the lemma it is sufficient to show $(P_k v)_s = P_k v_s$. Since $\boldsymbol{\sigma}_2^{-1} \in \ell_2(\mathbb{N})$, then the series (2.2) converges unconditionally in $L_2(U, X; \mu)$ to v for $v \in H_{X,\boldsymbol{\sigma}_2}$. Indeed, by the Hölder inequality we have for $v \in H_{X,\boldsymbol{\sigma}_2}$,

$$\begin{aligned}\sum_{s\in\mathbb{N}} \|v_s\varphi_s\|_{L_2(U,X;\mu)} &= \sum_{s\in\mathbb{N}} \|v_s\|_X\|\varphi_s\|_{L_2(U,\mathbb{C};\mu)} = \sum_{s\in\mathbb{N}} \|v_s\|_X \\ &\le \left(\sum_{s\in\mathbb{N}}(\sigma_s\|v_s\|_X)^2\right)^{1/2}\left(\sum_{s\in\mathbb{N}}\sigma_s^{-2}\right)^{1/2} < \infty.\end{aligned}$$

This yields that the series (2.2) absolutely and hence unconditionally converges in $L_2(U, \mathbb{C}; \mu)$ to v, since in a Banach space the absolute convergence implies the unconditional convergence. From this unconditional convergence and the uniform boundedness of the linear projectors $(P_k)_{k\in\mathbb{N}_0}$ we derive

$$P_k v = \sum_{s\in\mathbb{N}} P_k v_s \varphi_s,$$

and therefore, $(P_k v)_s = P_k v_s$. ∎

2.2 Extended least squares sampling algorithms in Bochner spaces

Let A^X be a linear operator in $L_2(U, X; \mu)$ defined for $v \in L_2(U, X; \mu)$ by

$$v \mapsto \sum_{k\in\mathbb{N}}\left(\sum_{s\in\mathbb{N}} a_{k,s}v_s\right)\varphi_k, \tag{2.7}$$

where $(a_{k,s})_{(k,s)\in\mathbb{N}^2}$ is an infinite dimensional matrix. Then A^X uniquely defines the linear operator $A^{\mathbb{C}}$ in $L_2(U, \mathbb{C}; \mu)$ given for $f \in L_2(U, \mathbb{C}; \mu)$ by

$$f \mapsto \sum_{k\in\mathbb{N}}\left(\sum_{s\in\mathbb{N}} a_{k,s}f_k\right)\varphi_k. \tag{2.8}$$

Conversely, a linear operator $A^{\mathbb{C}}$ in $L_2(U, \mathbb{C}; \mu)$ of the form (2.8) uniquely defines the A^X linear operator of the form (2.7) in $L_2(U, X; \mu)$, i.e., this is an one-onto-one correspondence between the linear operators in $L_2(U,X;\mu)$ and $L_2(U,\mathbb{C};\mu)$. Similarly, there is an one-onto-one correspondence between the sets of linear subspaces in $L_2(U,X;\mu)$ and $L_2(U,\mathbb{C};\mu)$. Based on these correspondences, we can see that there are one-onto-one correspondences between the spaces $H^\alpha_{\mathbb{C}}$, $H^{\alpha,\tau}_{\mathbb{C}}$, $H_{\mathbb{C}}$ and H^α_X, $H^{\alpha,\tau}_X$, H_X, respectively.

Let E be a normed space of functions on Ω. Given sample points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in \Omega$, we consider the approximate recovery of a function f on Ω from their values $f(\boldsymbol{x}_1),\dots,f(\boldsymbol{x}_n)$ by a linear sampling algorithm S_n on Ω of the form

$$S_n(f) := \sum_{i=1}^n f(\boldsymbol{x}_i) h_i, \tag{2.9}$$

where $h_1,\dots,h_n$ are given functions on Ω. For convenience, we assume that some of the sample points $\boldsymbol{x}_i$ may coincide. The approximation error is measured by the norm $\|f - S_n(f)\|_E$. Denote by $\mathcal{S}_n$ the family of all linear sampling algorithms S_k of the form (2.9) with $k \le n$. Let $F \subset E$ be a set of functions on Ω. To study the optimality of linear sampling algorithms from $\mathcal{S}_n$ for F and their convergence rates we use the (linear) sampling n-width

$$\varrho_n(F,E) := \inf_{S_n \in \mathcal{S}_n} \sup_{f \in F} \|f - S_n(f)\|_E.$$

Denote by $B_{X,\boldsymbol{\sigma}}$, B^α_X and $B^{\alpha,\tau}_X$ the unit balls in the spaces $H_{X,\boldsymbol{\sigma}}$, H^α_X and $H^{\alpha,\tau}_X$, respectively.

The following result has been proven in [7].

Lemma 2.2. *Given arbitrary sample points $\boldsymbol{y}_1,\dots,\boldsymbol{y}_n \in U$ and functions $h_1,\dots,h_n \in L_2(\mathbb{R}^\infty,\mathbb{C};\mu)$, for the sampling algorithm S^X_n in $L_2(U,X;\mu)$ defined by (2.9), we have*

$$\sup_{v \in B_{X,\boldsymbol{\sigma}}} \left\|v - S^X_n v\right\|_{L_2(U,X;\mu)} = \sup_{f \in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\|f - S^{\mathbb{C}}_n f\right\|_{L_2(U,\mathbb{C};\mu)}.$$

In the next step we extend this result to the space $H^{\alpha,\tau}_X$. For $k \in \mathbb{N}_0$ and $v \in H^{\alpha,\tau}_X$, we define

$$\delta_k v := \ P_k v - P_{k-1} v, \ k \in \mathbb{N}, \quad \delta_0 v = P_0 v. \tag{2.10}$$

We then can represent every $v \in H^{\alpha,\tau}_X$ by the series

$$v \ = \ \sum_{k \in \mathbb{N}_0} \delta_k v$$

converging in $L_2(U,X;\mu)$ absolutely and hence, unconditionally, and satisfying

$$\sup_{k \in \mathbb{N}_0} 2^{\alpha k} \, \|\delta_k v\|_{H^{\alpha,\tau}_{X,\boldsymbol{\sigma}_2}} \ \le \ 2\|v\|_{H^{\alpha,\tau}_{X,\boldsymbol{\sigma}_2}} \ < \infty.$$

For $m \in \mathbb{N}_0$, we introduce the sets

$$\delta_m B^{\alpha,\tau}_X := \ \left\{\delta_m v : \, v \in B^{\alpha,\tau}_X\right\}, \quad \delta_m B^{\alpha,\tau}_{\mathbb{C}} := \ \left\{\delta_m f : \, f \in B^{\alpha,\tau}_{\mathbb{C}}\right\}.$$

Lemma 2.3. *Let $\alpha > 1/q_2$. Assume that there is a constant K such that for every $v \in H_X^{\alpha,\tau}$ and every $m \in \mathbb{N}_0$, $\|P_m v\|_{H_X^{\alpha,\tau}} \le K \|v\|_{H_X^{\alpha,\tau}}$. Then for every $m \in \mathbb{N}_0$ there holds the equality*

$$\sup_{0 \neq \delta_m v \in \delta_m B_X^{\alpha,\tau}} \frac{\left\|A^X \delta_m v\right\|_{L_2(U,X;\mu)}}{\|\delta_m v\|_{H_X^{\alpha,\tau}}} = \sup_{0 \neq \delta_m f \in \delta_m B_{\mathbb{C}}^{\alpha,\tau}} \frac{\left\|A^X \delta_m f\right\|_{L_2(U,\mathbb{C};\mu)}}{\|\delta_m f\|_{H_{\mathbb{C}}^{\alpha,\tau}}}. \tag{2.11}$$

Proof. From the assumptions we can see that for every $v \in H_X^{\alpha,\tau}$ and every $m \in \mathbb{N}_0$, $\|\delta_m v\|_{H_X^{\alpha,\tau}} \le 2K \|v\|_{H_X^{\alpha,\tau}}$. Denote by N_X and $N_{\mathbb{C}}$ the left-hand side and right-hand side of (2.11), respectively. For $\delta_m f \in \delta_m B_{\mathbb{C}}^{\alpha,\tau}$ represented as

$$\delta_m f = \sum_{s \in \mathbb{N}} (\delta_m f)_s \varphi_s,$$

we have

$$\|A^{\mathbb{C}} \delta_m f\|^2_{L_2(U,\mathbb{C};\mu)} \le N_{\mathbb{C}}^2 \left(\|\delta_m f\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}_1}} + \|\delta_m f\|^2_{H^{\alpha,\tau}_{\mathbb{C},\boldsymbol{\sigma}_2}} \right).$$

The last inequality is equivalent to inequality

$$\sum_{i \in \mathbb{N}} \left| \sum_{s \in \mathbb{N}} a_{i,s} (\delta_m f)_s \right|^2 \le N_{\mathbb{C}}^2 \left(\sum_{s \in \mathbb{N}} (\sigma_{1;s} |(\delta_m f)_s|)^2 + \sum_{k \in \mathbb{N}_0} \sum_{s \in \mathbb{N}} \left(2^{\alpha k} k^{-\tau} \sigma_{2;s} |(\delta_m f - P_k(\delta_m f))_s| \right)^2 \right).$$

For $\delta_m v \in \delta_m H_X^{\alpha,\tau}$, we have

$$\delta_m v = \sum_{s \in \mathbb{N}} (\delta_m v)_s \varphi_s$$

with

$$\sum_{s \in \mathbb{N}} (\sigma_{1;s} \|(\delta_m v)_s\|_X)^2 + \sum_{k \in \mathbb{N}_0} \sum_{s \in \mathbb{N}} \left(2^{\alpha k} k^{-\tau} \sigma_{2;s} \|(\delta_m v - P_k(\delta_m v))_s\|_X \right)^2 < \infty,$$

and

$$\|A^X \delta_m v\|^2_{L_2(U,X;\mu)} = \sum_{i \in \mathbb{N}} \left\| \sum_{s \in \mathbb{N}} a_{i,s} (\delta_m v)_s \right\|_X^2. \tag{2.12}$$

Let $(\psi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of X. Then $(\varphi_k \psi_j)_{k,j \in \mathbb{N}}$ is an orthonormal basis of $L_2(U, X; \mu)$. We have

$$\delta_m v = \sum_{j \in \mathbb{N}} \sum_{s \in \mathbb{N}} (\delta_m v)_s^j \psi_j \varphi_s = \sum_{j \in \mathbb{N}} (\delta_m v^j) \psi_j.$$

Then,

$$A^X \delta_m v = \sum_{s \in \mathbb{N}} \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} a_{i,s} (\delta_m v)_s^j \psi_j \varphi_m.$$

By applying (2.12) to $\delta_m f = \delta_m v^j$, we obtain for every $k \in \mathbb{N}_0$,

$$\begin{aligned}
&\|A^X \delta_m v\|^2_{L_2(U,X;\mu)} \\
&= \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \left| \sum_{s\in\mathbb{N}} a_{i,s} (\delta_m v)^j_s \right|^2 = \sum_{j\in\mathbb{N}} \|A^{\mathbb{C}} (\delta_m v)^j\|^2_{L_2(U,\mathbb{C};\mu)} \\
&\leq N_X^2 \left(\sum_{j\in\mathbb{N}} \sum_{s\in\mathbb{N}} \left(\sigma_{1;s} |((\delta_m v)^j)_s|\right)^2 + \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}_0} \sum_{s\in\mathbb{N}} \left(2^{\alpha k} k^{-\tau} \sigma_{2;s} |((\delta_m v)^j - P_k((\delta_m v)^j))_s|\right)^2 \right) \\
&=: N_X^2 (B_1^2 + B_2^2).
\end{aligned}$$

For the term B_2^2, we have

$$\begin{aligned}
B_2^2 &= \sum_{k\in\mathbb{N}_0} \sum_{s\in\mathbb{N}} \left(2^{\alpha k} k^{-\tau} \sigma_{2;s}\right)^2 \sum_{j\in\mathbb{N}} |((\delta_m v)^j - P_k((\delta_m v))^j)_s|^2 \\
&= \sum_{k\in\mathbb{N}_0} \sum_{s\in\mathbb{N}} \left(2^{\alpha k} k^{-\tau} \sigma_{2;s} |((\delta_m v)^j - P_k((\delta_m v))^j)_s|\right)^2 = \|\delta_m v\|^2_{H^{\alpha,\tau}_{X,\boldsymbol{\sigma}_2}}.
\end{aligned}$$

By the same way we can show

$$B_1 = \|\delta_m v\|_{H_{X,\boldsymbol{\sigma}_1}}.$$

Summing up we prove the inequality

$$N_X \leq N_{\mathbb{C}}.$$

In order to prove the inverse inequality, let $(f^n)_{n\in\mathbb{N}} \subset H^{\alpha,\tau}_{\mathbb{C}}$ be a sequence such that $\|(\delta_m f)^n\|_{H^{\alpha,\tau}_{\mathbb{C}}} = 1$, and

$$\lim_{n\to\infty} \left\|A^{\mathbb{C}} (\delta_m f)^n\right\|_{L_2(U,\mathbb{C};\mu)} = N_{\mathbb{C}}.$$

Define $(\delta_m v)^n := (\delta_m f)^n \psi_1$. Then $\|(\delta_m v)^n\|_{H^{\alpha,\tau}_X} = 1$ and

$$\begin{aligned}
\left\|A^X (\delta_m v)^n\right\|^2_{L_2(U,X;\mu)} &= \sum_{k\in\mathbb{N}} \left\| \sum_{s\in\mathbb{N}} a_{k,s} \langle (\delta_m f)^n, \varphi_s \rangle_{L_2(U,\mathbb{C};\mu)} \psi_1 \right\|^2_X \\
&= \sum_{k\in\mathbb{N}} \left| \sum_{s\in\mathbb{N}} a_{k,s} \langle (\delta_m f)^n, \varphi_s \rangle_{L_2(U,\mathbb{C};\mu)} \right|^2 \\
&= \left\|A^{\mathbb{C}} (\delta_m f)^n\right\|^2_{L_2(U,\mathbb{C};\mu)} \to N_{\mathbb{C}} \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This proves the inequality

$$N_X \geq N_{\mathbb{C}}.$$

□

From this lemma we can extend the result of Lemma 2.2 to the sets $\delta_m B^{\alpha,\tau}_X$ and $\delta_m B^{\alpha,\tau}_{\mathbb{C}}$.

Corollary 2.1. *Let $\alpha > 1/q_2$. Assume that there is a constant K such that for every $v \in H_X^{\alpha,\tau}$ and every $m \in \mathbb{N}_0$, $\|P_m v\|_{H_X^{\alpha,\tau}} \leq K \|v\|_{H_X^{\alpha,\tau}}$. Given arbitrary sample points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k \in U$ and functions $h_1, \ldots, h_k \in L_2(U, \mathbb{C}; \mu)$, for the sampling algorithm S_n^X in $L_2(U, X; \mu)$ defined by (2.9), we have for every $m \in \mathbb{N}_0$,*

$$\sup_{0 \neq \delta_m v \in \delta_m B_X^{\alpha,\tau}} \frac{\left\|\delta_m v - S_n^X(\delta_m v)\right\|_{L_2(U,X;\mu)}}{\|\delta_m v\|_{H_X^{\alpha,\tau}}} = \sup_{0 \neq \delta_m f \in \delta_m B_{\mathbb{C}}^{\alpha,\tau}} \frac{\left\|\delta_m f - S_n^{\mathbb{C}}(\delta_m f)\right\|_{L_2(U,\mathbb{C};\mu)}}{\|\delta_m f\|_{H_{\mathbb{C}}^{\alpha,\tau}}}.$$

Proof. This corollary is Lemma 2.3 for $A^X = I^X - S_n^X$, where I^X denotes the identity operator in $L_2(U, X; \mu)$. ∎

Let $n \in \mathbb{N}$ and E be a normed space and F a central symmetric compact set in E. Then the Kolmogorov n-width of F is defined by

$$d_n(F, E) := \inf_{L_n} \sup_{f \in F} \inf_{g \in L_n} \|f - g\|_E,$$

where the left-most infimum is taken over all subspaces L_n of dimension at most n in E. The Kolmogorov n-width $d_n(F, E)$ is a characterization of the best approximation of elements in F by elements in linear subspaces of dimension at most n.

We recall a concept of weighted least squares sampling algorithm in the space $L_2(U, \mathbb{C}; \mu)$ (cf. also [7]). Recall that $(\varphi_s)_{s \in \mathbb{N}}$ is the fixed orthonormal basis of $L_2(U, \mathbb{C}; \mu)$ considered in Subsection 2.2. For $c, n, m \in \mathbb{N}$ with $cn \geq m$, let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{cn} \in U$ be points, $\omega_1, \ldots, \omega_{cn} \geq 0$ be weights, and $\Phi_m = \operatorname{span}\{\varphi_s\}_{s=1}^m$ the subspace spanned by the functions φ_s, $s = 1, ..., m$. The weighted least squares sampling algorithm $S_{cn}^{\mathbb{C}} f = S_{cn}^{\mathbb{C}}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{cn}, \omega_1, \ldots, \omega_{cn}, \Phi_m) f$ of a function $f \colon U \to \mathbb{C}$ is given by

$$S_{cn}^{\mathbb{C}} f = \arg\min_{g \in \Phi_m} \sum_{i=1}^{cn} \omega_i |f(\boldsymbol{y}_i) - g(\boldsymbol{y}_i)|^2. \tag{2.13}$$

This weighted least squares sampling algorithm can be computed using the Moore-Penrose inverse, which gives the solution of smallest error for over-determined systems where no exact solution can be expected. In particular, for $\boldsymbol{L} = [\varphi_s(\boldsymbol{y}_i)]_{i=1,\ldots,cn; s=1,\ldots,m}$ and $\boldsymbol{W} = \operatorname{diag}(\omega_1, \ldots, \omega_n)$ we have

$$S_{cn}^{\mathbb{C}} f = \sum_{s=1}^{m} \hat{g}_s \varphi_s \quad \text{with} \quad (\hat{g}_1, \ldots, \hat{g}_m)^\top = (\boldsymbol{L}^* \boldsymbol{W} \boldsymbol{L})^{-1} \boldsymbol{L}^* \boldsymbol{W} (f(\boldsymbol{y}_1), \ldots, f(\boldsymbol{y}_{cn}))^\top. \tag{2.14}$$

Notice that $S_{cn}^{\mathbb{C}}$ is a linear sampling algorithm of the form

$$S_{cn}^{\mathbb{C}} f = \sum_{i=1}^{cn} f(\boldsymbol{y}_i) h_i(\boldsymbol{y}) \tag{2.15}$$

for some $h_1, ..., h_{cn}$ in $L_2(U, \mathbb{C}; \mu)$.

The extended least squares algorithm to the Bochner space $L_2(U, X; \mu)$ can be defined by replacing $f \in L_2(U, \mathbb{C}; \mu)$ with $v \in L_2(U, X; \mu)$:

$$S_{cn}^X v = \sum_{i=1}^{cn} v(\boldsymbol{y}_i) h_i(\boldsymbol{y}). \tag{2.16}$$

For $m \in \mathbb{N}$ let the probability measure $\nu = \nu(m)$ be defined by

$$\mathrm{d}\nu(\boldsymbol{y}) := \varrho(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) := \frac{1}{2}\left(\frac{1}{m}\sum_{s=1}^{m}|\varphi_s(\boldsymbol{y})|^2 + \frac{\sum_{s=m+1}^{\infty}|\sigma_s^{-1}\varphi_s(\boldsymbol{y})|^2}{\sum_{s=m+1}^{\infty}\sigma_s^{-2}}\right)\mathrm{d}\mu(\boldsymbol{y}). \tag{2.17}$$

Let $m := n$ and $\lceil 20n\log n\rceil$ sample points be drawn i.i.d. with respect to ν. Let further $\boldsymbol{y}_1,\dots,\boldsymbol{y}_{c_pn} \in U$ be the subset of the sample points fulfilling [23, Theorem 3] with $c_pn \asymp m$ and $\omega_i := \frac{c_pn}{\lceil 20n\log n\rceil}(\varrho(\boldsymbol{y}_i))^{-1}$. For every $n \in \mathbb{N}$, let

$$S_{c_pn}^{\mathbb{C}}f := \sum_{i=1}^{c_pn} f(\boldsymbol{y}_i)h_i, \tag{2.18}$$

be the least squares sampling algorithm constructed as in (2.13)–(2.15) for these sample points and weights, where $h_1,...,h_{c_pn} \in L_2(U,\mathbb{C};\mu)$.

The following result [23, Theorem 3] gives a error bound of the approximation by $S_{c_pn}^{\mathbb{C}}$ via Kolmogorov n-widths.

Lemma 2.4. *Let $0 < p < 2$ and $d_s := d_s(F, L_2(U,\mathbb{C};\mu))$. Assume that $F \subset L_2(U,\mathbb{C};\mu)$ is such that there is a metric on F satisfying the condition that F is continuously embedded into $L_2(U,\mathbb{C};\mu)$, and the function evaluation $f \mapsto f(\boldsymbol{y})$ is continuous on F for every $\boldsymbol{y} \in F$. Then there is a constant $c_p \in \mathbb{N}$ such that the least squares sampling algorithms $S_{c_pn}^{\mathbb{C}}$ defined as in* (2.18) *satisfy the inequalities*

$$\varrho_n(F, L_2(U,\mathbb{C};\mu)) \ \le\ \sup_{f\in F}\left\|v - S_{c_pn}^{\mathbb{C}}f\right\|_{L_2(U,\mathbb{C};\mu)} \ \le \left(\frac{1}{n}\sum_{s\ge n} d_s^p\right)^{1/p}.$$

The sampling algorithms $S_{c_pn}^{\mathbb{C}}$ in Lemma 2.4 is a weighted least squares algorithm using samples from a set of c_pn points that is subsampled from a set of $c_pn\log n$ i.i.d. random points with respect to the probability measure ν. Depending on the function class F, the algorithm using the full set of random points may be constructive but the subsampling is not constructive. We will explain and discuss it and other least squares sampling algorithms in detail in Section 6.

Lemma 2.5. *Let $0 < p < 2$ and $d_j := d_j(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(U,\mathbb{C};\mu))$. Let the least squares linear sampling algorithm $S_{c_pn}^{\mathbb{C}}$ be as in* (2.18)*. Then the extended least squares linear sampling algorithm*

$$S_{c_pn}^{X}v := \sum_{i=1}^{c_pn} v(\boldsymbol{y}_i)h_i$$

defined by the formula (2.16)*, satisfies the inequality*

$$\varrho_n(B_{X,\boldsymbol{\sigma}}, L_2(U,X;\mu)) \ \le\ \sup_{v\in B_{X,\boldsymbol{\sigma}}}\left\|v - S_{c_pn}^{X}v\right\|_{L_2(U,X;\mu)} \ \le \left(\frac{1}{n}\sum_{j\ge n} d_j^p\right)^{1/p}.$$

Proof. We first consider the particular case when $X = \mathbb{C}$. In this case $B_{\mathbb{C},\sigma}$ is a subset of the unit ball of the reproducing kernel Hilbert space $H_{\mathbb{C},\boldsymbol{\sigma}}$ equipped with a metric as the norm of this space, which is continuously embedded into $L_2(U, X; \mu)$ and for which the function evaluation $f \mapsto f(\boldsymbol{y})$ is continuous on $B^\alpha_{\mathbb{C}}$ for every $\boldsymbol{y} \in B^\alpha_{\mathbb{C}}$. This means that the assumption of Lemma 2.4 is satisfied for $F = B_{\mathbb{C},\sigma}$. The lemma has been proven for the particular case when $X = \mathbb{C}$. Hence, by using Lemma 2.2 we prove the lemma in the general case. ∎

In a similar way from Lemma 2.4 and Corollary 2.1 we derive

Lemma 2.6. *Let $0 < p < 2$, $m \in \mathbb{N}_0$, $\alpha > 1/q_2$ and $d_j := d_j(\delta_m B^{\alpha,\tau}_{\mathbb{C}}, L_2(U, \mathbb{C}; \mu))$. Assume that there is a constant K such that for every $v \in H^{\alpha,\tau}_X$ and every $m \in \mathbb{N}_0$, $\|P_m v\|_{H^{\alpha,\tau}_X} \le K \|f\|_{H^{\alpha,\tau}_X}$. Then there is a constant $c_p \in \mathbb{N}$ such that we have the following. For every $m \in \mathbb{N}_0$ and every $n \in \mathbb{N}$, there are $\boldsymbol{y}_1, ..., \boldsymbol{y}_{c_p n}$ and $h_1, ..., h_{c_p n} \in L_2(U, \mathbb{C}; \mu)$ such that the linear (least squares) sampling algorithm*

$$S^X_{c_p n} v := \sum_{i=1}^{c_p n} v(\boldsymbol{y}_i) h_i$$

satisfies the inequalities

$$\varrho_n(\delta_m B^{\alpha,\tau}_X, L_2(U, X; \mu)) \ \le \ \sup_{\delta_m v \in \delta_m B^{\alpha,\tau}_X} \left\| \delta_m v - S^X_{c_p n}(\delta_m v) \right\|_{L_2(U,X;\mu)} \ \le \left(\frac{1}{n} \sum_{j \ge n} d_j^p \right)^{1/p}.$$

Let $\Lambda \subset \mathbb{N}$ be a finite set. For $v \in L_2(U, X; \mu)$, we define the truncation of the GPC expansion of v

$$S_\Lambda v := \sum_{s \in \Lambda} (\delta_m v)_s \varphi_s.$$

The following lemma has been proven in [7, Lemma 2.2]

Lemma 2.7. *Let $0 < q \le 2$ and $\boldsymbol{\sigma} = (\sigma_s)_{s \in \mathbb{N}}$ a given non-decreasing sequence of numbers strictly larger than 1, $\boldsymbol{\sigma}^{-1} \in \ell_q(\mathbb{N})$. For $\xi > 0$ and $M > 0$, we introduce the set*

$$\Lambda(\xi) := \quad \left\{ s \in \mathbb{N} : \ \sigma_s \le \xi^{1/q} \right\}.$$

We have

$$\sup_{v \in B_{X,\boldsymbol{\sigma}}} \|v - S_{\Lambda(\xi)} v\|_{L_2(U,X;\mu)} \ \le \ \xi^{-1/q}. \tag{2.19}$$

Moreover, if in addition $\left\|\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{N})} \le 1$, then for all $n \ge 2$ there exists a number ξ_n such that $\dim(\mathcal{V}(\Lambda(\xi_n)) \le n$, *and*

$$d_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(U, \mathbb{C}; \mu)) \ \le \ \sup_{f \in B_{\mathbb{C},\boldsymbol{\sigma}}} \|f - S_{\Lambda(\xi_n)} f\|_{L_2(U,\mathbb{C};\mu)} \ \le 2^{1/q} n^{-1/q} \quad \forall n \in \mathbb{N}. \tag{2.20}$$

Lemma 2.8. *Let $\alpha > 1/q_2$. Assume that there is a constant K such that for every $v \in H^{\alpha,\tau}_X$ and every $m \in \mathbb{N}_0$, $\|P_m v\|_{H^{\alpha,\tau}_X} \le K \|v\|_{H^{\alpha,\tau}_X}$. For $\xi > 0$ and $m \in \mathbb{N}_0$, we introduce the set*

$$\Lambda_m(\xi) := \ \left\{ \sigma_{2;s} \le \xi^{1/q_1} 2^{-\alpha m} m^\tau, \ \sigma_{1;s} \le \xi^{1/q_1} \right\}.$$

We have

$$\|\delta_m v - S_{\Lambda(\xi)}(\delta_m v)\|_{L_2(U,X;\mu)} \ \le\ 2K\xi^{-2/q_1}\|\delta_m v\|_{H_X^{\alpha,\tau}}, \quad v \in H_X^{\alpha,\tau}. \tag{2.21}$$

Moreover, for all $n \ge 2$ *there exists a number* ξ_n *such that* $\dim(\mathcal{V}(\Lambda_m(\xi_n)) \le n$, *and*

$$\begin{aligned} d_n(\delta_m B_X^{\alpha,\tau}, L_2(U,X;\mu)) \ &\le \sup_{\delta_m v \in \delta_m B_X^{\alpha,\tau}} \|\delta_m v - S_{\Lambda_m(\xi_n)}(\delta_m v)\|_{L_2(U,X;\mu)} \\ &\le\ C\,(2^m n)^{-\beta}(m+\log n)^{\beta\tau/\alpha} \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.22}$$

Proof. From the assumptions we can see that for every $v \in H_X^{\alpha,\tau}$ and every $m \in \mathbb{N}_0$, $\|\delta_m v\|_{H_X^{\alpha,\tau}} \le 2K\,\|v\|_{H_X^{\alpha,\tau}}$. For a function $v \in B_{\mathbb{C}}^{\alpha,\tau}$, putting

$$(\delta_m v)_\xi := \sum_{\sigma_{1;s}^{q_1} \le \xi} (\delta_m v)_s \varphi_s,$$

we get

$$\|(\delta_m v) - S_{\Lambda_m(\xi)}(\delta_m v)\|_{L_2(U,X;\mu)} \ \le\ \|(\delta_m v) - (\delta_m v)_\xi\|_{L_2(U,X;\mu)} + \|(\delta_m v)_\xi - S_{\Lambda_m(\xi)}(\delta_m v)\|_{L_2(U,X;\mu)}.$$

For the norm $\|(\delta_m v) - (\delta_m v)_\xi\|_{L_2(U,X;\mu)}$ we have that

$$\begin{aligned} \|(\delta_m v) - (\delta_m v)_\xi\|^2_{L_2(U,X;\mu)} \ &=\ \sum_{\sigma_{1;s} > \xi^{1/q_1}} \|(\delta_m v)_s\|_X^2 \ =\ \sum_{\sigma_{1;s} > \xi^{1/q_1}} (\sigma_{1;s}\|(\delta_m v)_s\|_X)^2 \sigma_{1;s}^{-2} \\ &\le\ \xi^{-2/q_1} \sum_{\sigma_{1;s} > \xi^{1/q_1}} (\sigma_{1;s}\|(\delta_m v)_s\|_X)^2 \\ &\le\ (2K)^2 \xi^{-2/q_1} \|\delta_m v\|^2_{H_X^{\alpha,\tau}}. \end{aligned}$$

For the norm $\|(\delta_m v)_\xi - S_{\Lambda_m(\xi)}(\delta_m v)\|_{L_2(U,X;\mu)}$, we obtain

$$\begin{aligned} \|(\delta_m v)_\xi - S_{\Lambda_m(\xi)}(\delta_m v)\|^2_{L_2(U,X;\mu)} \ &=\ \sum_{\sigma_{1;s} \le \xi^{1/q_1},\ \sigma_{2;s} > \xi^{1/q_1} 2^{-\alpha m} m^\tau} \|(\delta_m v)_s\|_X^2 \\ &\le\ \sum_{\sigma_{1;s} \le \xi^{1/q_1},\ \sigma_{2;s} > \xi^{1/q_1} 2^{-\alpha m} m^\tau} \Big(2^{\alpha m} m^{-\tau} \xi^{-1/q_1} \sigma_{2;s}\,\|(\delta_m v)_s\|_X\Big)^2 \\ &\le\ \xi^{-2/q_1} \sum_{s \in \mathbb{N}} \big(2^{\alpha m} m^{-\tau} \sigma_{2;s}\,\|(\delta_m v)_s\|_X\big)^2 \\ &\le\ (2K)^2 \xi^{-2/q_1} \|\delta_m v\|^2_{H_X^{\alpha,\tau}}. \end{aligned}$$

These estimates yield (2.21).

For the dimension of the space $\mathcal{V}(\Lambda_m(\xi))$, with $q := q_2\alpha > 1$ and $1/q' + 1/q = 1$, we have that

$$
\begin{aligned}
\dim \mathcal{V}(\Lambda_m(\xi)) &= |\Lambda_m(\xi)| = \sum_{s\in\Lambda_m(\xi)} 1 = 2^{-m} \sum_{\sigma_{1;s}\le\xi^{1/q_1},\ \sigma_{2;s}>\xi^{1/q_1}2^{-\alpha m}m^\tau} 2^m \\
&\le 2^{-m} \sum_{s\in\mathbb{N}:\ \sigma_{1;s}\le\xi^{1/q_1}} \sum_{(k,s)\in\mathbb{N}_0\times\mathbb{N}:\ \sigma_{2;s}>\xi^{1/q_1}2^{-\alpha k}k^\tau} 2^k \\
&\le 2^{-m} \sum_{\sigma_{1;\boldsymbol{s}}\le\xi^{1/q_1}} \xi^{1/(q_1\alpha)} (\log\xi)^{\tau/\alpha} \sigma_{2;\boldsymbol{s}}^{-1/\alpha} \\
&\le 2^{-m}\xi^{1/(q_1\alpha)} (\log\xi)^{\tau/\alpha} \left(\sum_{\sigma_{1;\boldsymbol{s}}\le\xi^{1/q_1}} \sigma_{2;\boldsymbol{s}}^{-q_2}\right)^{1/q} \left(\sum_{\sigma_{1;\boldsymbol{s}}\le\xi^{1/q_1}} 1\right)^{1/q'} \\
&\le 2^{-m}\xi^{1/(q_1\alpha)} (\log\xi)^{\tau/\alpha} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \sigma_{2;\boldsymbol{s}}^{-q_2}\right)^{1/q} \left(\sum_{\boldsymbol{s}\in\mathbb{F}} \xi\sigma_{1;\boldsymbol{s}}^{-q_1}\right)^{1/q'} \\
&\le 2^{-m}\xi^{1+\delta/\alpha} (\log\xi)^{\tau/\alpha}.
\end{aligned}
$$

For any $n \in \mathbb{N}$, letting ξ_n be a number satisfying the inequalities

$$
\xi^{1+\delta/\alpha} (\log\xi)^{\tau/\alpha} 2^{-m} \le n < 2\xi^{1+\delta/\alpha} (\log\xi)^{\tau/\alpha} 2^{-m}, \tag{2.23}
$$

we derive that $\dim \mathcal{V}(\Lambda_m(\xi_n)) \le n$. On the other hand, by (2.23),

$$
\begin{aligned}
\xi_n^{-1/q_1} &\le C \left(2^m n(\log n + m)^{-\tau/\alpha}\right)^{-\frac{1}{q_1}\frac{\alpha}{\alpha+\delta}} \\
&\le C\,(2^m n)^{-\beta}(m + \log n)^{\beta\tau/\alpha}.
\end{aligned}
$$

This together with (2.21) proves (2.22). ∎

From Corollary 2.1 and Lemma 2.8 we obtain

Corollary 2.2. *Under the assumption and notation of Lemma 2.8, we have*

$$
\|\delta_m f - S_{\Lambda(\xi)}(\delta_m f)\|_{L_2(U,\mathbb{C};\mu)} \le 2K\xi^{-2/q_1}\|\delta_m f\|_{H^{\alpha,\tau}_{\mathbb{C}}}, \quad \forall f \in H^{\alpha,\tau}_{\mathbb{C}}.
$$

Moreover, for all $n \ge 2$ there exists a number ξ_n such that $\dim(\mathcal{V}(\Lambda_m(\xi_n)) \le n$, *and*

$$
\begin{aligned}
d_n(\delta_m B^{\alpha,\tau}_{\mathbb{C}}, L_2(U,\mathbb{C};\mu)) &\le \sup_{\delta_m f\in\delta_m B^{\alpha,\tau}_{\mathbb{C}}} \|\delta_m f - S_{\Lambda_m(\xi_n)}(\delta_m f)\|_{L_2(U,\mathbb{C};\mu)} \\
&\le C\,(2^m n)^{-\beta}(m + \log n)^{\beta\tau/\alpha} \quad \forall n \in \mathbb{N}.
\end{aligned}
$$

2.3 Multi-level least squares sampling algorithms

Let $0 < p < 2$ and $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of increasing positive numbers whose values will be selected later, such that $\xi_n \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$, we consider the multi-level least squares operator $\mathcal{S}_n^X$ defined by

$$\mathcal{S}_n^X v \ = \ \sum_{k=0}^{k_n} S_{c_p n_k}^X(\delta_k v), \tag{2.24}$$

where

$$n_k := \lfloor c_p^{-1} n 2^{-k} \rfloor, \quad k_n := \lfloor \log \xi_n \rfloor, \tag{2.25}$$

and

$$S_{c_p n_k}^X v := \sum_{i=1}^{c_p n_k} v(\boldsymbol{y}_{k,i}) h_{k,i} \tag{2.26}$$

are the least squares sampling algorithms and $c_p > 0$ the constant as in Lemma 2.4.

Since $\delta_k v$ belongs to the subspace $V_k + V_{k-1}$ of dimension at most $2^k + 2^{k-1}$ and the number of sampling points in the sampling algorithms $S_{c_p n_k}^X$ is n_k. The computational complexity $\mathrm{Comp}\left(\mathcal{S}_n^X\right)$ of the operator $\mathcal{S}_n^X$ can be defined as

$$\mathrm{Comp}\left(\mathcal{S}_n^X\right) := \sum_{k=0}^{k_n} \left(2^k + 2^{k-1}\right) n_k.$$

Theorem 2.1. *Let $0 < q_1 \le q_2 < 2$ and p be a fixed number such that $1/q_2 < p < 2$. Let the numbers β and δ be defined as*

$$\beta := \frac{1}{q_1}\frac{\alpha}{\alpha+\delta}, \quad \delta := \frac{1}{q_1} - \frac{1}{q_2}. \tag{2.27}$$

Let τ be any number such that $\tau > 1$ if $\alpha \le 1/q_2$, and $\tau = \alpha/(\alpha - \beta)$ if $\alpha > 1/q_2$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of increasing positive numbers ξ_n satisfying the condition

$$\begin{cases} \xi_n \ \le \ n \ < \ 2\xi_n & \text{if } \alpha \le 1/q_2, \\ \xi_n^{q_1(\alpha+\delta)} \ \le \ n \ < \ 2\,\xi_n^{q_1(\alpha+\delta)} & \text{if } \alpha > 1/q_2. \end{cases} \tag{2.28}$$

Assume that there is a constant $K := K_\tau$ such that for every $v \in H_X^{\alpha,\tau}$ and every $m \in \mathbb{N}_0$, $\|P_m v\|_{H_X^{\alpha,\tau}} \le K\,\|v\|_{H_X^{\alpha,\tau}}$. Then for all $n \ge 2$ there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in U$ and $h_{k,1}, ..., h_{k,n_k} \in L_2(U, \mathbb{C}; \mu)$, $k = 0, ..., k_n$, such that for the operator $\mathcal{S}_n^X$ defined as in (2.24)–(2.26),

$$\mathrm{Comp}\left(\mathcal{S}_n^X\right) \ \le \ Cn\log n.$$

$$\sup_{v\in B_X^{\alpha,\tau}} \|v - \mathcal{S}_n^X v\|_{L_2(U,X;\mu)} \ \le \ C_\tau \begin{cases} n^{-\alpha}(\log n)^\tau & \text{if } \alpha < 1/q_2, \\ n^{-\alpha}(\log n)^{1+\tau} & \text{if } \alpha = 1/q_2, \\ n^{-\beta}(\log n)^\tau & \text{if } \alpha > 1/q_2. \end{cases}$$

Proof. Let $v \in B_X^{\alpha,\tau}$ be given. By the assumptions we have

$$\|\delta_k v\|_{H_X^{\alpha,\tau}} \le 2K \quad \forall k \in \mathbb{N}_0. \tag{2.29}$$

By the triangle inequality,

$$\|v - \mathcal{S}_n^X v\|_{L_2(U,X;\mu)} \ \le\ \|v - P_{k_n} v\|_{L_2(U,X;\mu)} \ +\ \sum_{k=0}^{k_n} \|\delta_k v - S_{n_k}^X(\delta_k v)\|_{L_2(U,X;\mu)}.$$

By Parseval's identity and the equality $k_n := \lfloor \log \xi_n \rfloor$ we deduce that

$$\begin{aligned}\|v - P_{k_n} v\|_{L_2(U,X;\mu)}^2 \ &= \ \sum_{s\in\mathbb{N}} \|(v - P_{k_n} v)_s\|_X^2 \\ &\le\ 2^{-2\alpha k_n}\,(k_n)^{2\tau}\, 2^{2\alpha k_n}\,(k_n)^{-2\tau} \sum_{s\in\mathbb{N}} \left(\sigma_{2;s}\|(v - P_{k_n} v)_s\|_X\right)^2 \\ &\le\ 2^{-2\alpha\lfloor \log \xi_n\rfloor}\,(k_n)^{2\tau}\,\|v\|_{H^\alpha_{X,\boldsymbol{\sigma}_2}} \ \le\ 2^{-2\alpha\lfloor \log \xi_n\rfloor}\,(\log \xi_n)^{2\tau}.\end{aligned}$$

This means that

$$\|v - P_{k_n} v\|_{L_2(U,X;\mu)} \ \le\ 2^{-\alpha\lfloor \log \xi_n\rfloor}\,(\log \xi_n)^{\tau}. \tag{2.30}$$

Assume $\alpha \le 1/q_2$. By (2.29),

$$\|\delta_k v\|_{H_{X,\boldsymbol{\sigma}_2}} \ \le\ 2K 2^{-\alpha k} k^\tau, \quad k \in \mathbb{N}_0. \tag{2.31}$$

By Lemma 2.5 for $\boldsymbol{\sigma} = \boldsymbol{\sigma}_2$ and $q = q_2$, there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in U$, $h_{k,1}, ..., h_{k,n_k} \in L_2(U,\mathbb{C};\mu)$ such that

$$\|\delta_k v - S_{n_k}^X(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ \|\delta_k v\|_{H_{X,\boldsymbol{\sigma}_2}} \left(\frac{1}{n_k}\sum_{j\ge n_k} d_j^p\right)^{1/p},$$

where $d_j := d_j(B_{\mathbb{C},\boldsymbol{\sigma}_2}, L_2(U,\mathbb{C};\mu))$. Hence, by Lemma 2.7 for $\boldsymbol{\sigma} = \boldsymbol{\sigma}_2$ and $q = q_2$, (2.31) and the inequality $p/q_2 > 1$, we derive for $k \le k_n$,

$$\|\delta_k v - S_{n_k}^X(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ C\,2^{-\alpha k} k^\tau \left(\frac{1}{n_k}\sum_{j\ge n_k} j^{-p/q_2}\right)^{1/p} \ \le\ C\,2^{(1/q_2-\alpha)k} k^\tau n^{-1/q_2}. \tag{2.32}$$

For any $n \in \mathbb{N}$, choose ξ_n as a number satisfying the inequalities

$$\xi_n \ \le\ n \ <\ 2\xi_n.$$

By the equality $k_n := \lfloor \log \xi_n \rfloor$ and the equivalence $\xi_n \asymp n$, from (2.30) and (2.32) we deduce that

$$\|v - P_{k_n} v\|_{L_2(U,X;\mu)} \ \le\ 2^{-\alpha\lfloor \log \xi_n\rfloor}(\log \xi_n)^\tau \ \le\ 2^\alpha n^{-\alpha}(\log n)^\tau,$$

and for $\alpha < 1/q_2$,

$$\sum_{k=0}^{k_n} \|\delta_k v - S_{c_p n}^X(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ C\,n^{-1/q_2} \sum_{k=0}^{k_n} 2^{(1/q_2-\alpha)k} k^\tau \ \le\ C\,n^{-\alpha}(\log n)^\tau,$$

and for $\alpha = 1/q_2$,

$$\sum_{k=0}^{k_n} \|\delta_k v - S^X_{c_p n}(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ C\, n^{-1/q_2} \sum_{k=0}^{k_n} k^\tau \ \le\ C\, n^{-\alpha} (\log n)^{1+\tau}.$$

Summing up, we arrive at the inequalities

$$\|v - \mathcal{S}^X_n v\|_{L_2(U,X;\mu)} \ \le\ C\, n^{-\alpha} (\log n)^\tau, \quad \text{if} \ \ \alpha < 1/q_2,$$

and

$$\|v - \mathcal{S}^X_n v\|_{L_2(U,X;\mu)} \ \le\ C\, n^{-\alpha} (\log n)^{1+\tau}, \quad \text{if} \ \ \alpha = 1/q_2.$$

For the computational complexity $\text{Comp}\left(\mathcal{S}^X_n v\right)$ we have that

$$\text{Comp}\left(\mathcal{S}^X_n v\right) \ \le 2 \sum_{k=0}^{k_n} 2^k n_k \ \le\ 2 \sum_{k=0}^{k_n} 2^k (\xi_n 2^{-k}) \ =\ 2\xi_n \sum_{k=0}^{k_n} 1 \ \le\ C n \log n.$$

We now consider the case $\alpha > 1/q_2$. In this case we have $\alpha > \beta$. For any $n \in \mathbb{N}$, choose ξ_n as a number satisfying the inequalities

$$\xi_n^{q_1(\alpha+\delta)} \ \le\ n \ <\ 2\, \xi_n^{q_1(\alpha+\delta)}.$$

By the equivalence $\xi_n \asymp n^{\frac{1}{q_1}\frac{1}{\alpha+\delta}}$, from (2.30) we deduce that

$$\|v - P_{k_n} v\|_{L_2(U,X;\mu)} \ \le\ 2^{-\alpha \lfloor \log \xi_n \rfloor} (\log \xi_n)^\tau \ \le\ C\, n^{-\beta} (\log n)^\tau .$$

By Lemma 2.8, there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in U$, $h_{k,1}, ..., h_{k,n_k} \in L_2(U, \mathbb{C}; \mu)$ such that

$$\|\delta_k v - S^X_{n_k}(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ \|\delta_k v\|_{H^{\alpha,\tau}_X} \left(\frac{1}{n_k} \sum_{j \ge n_k} d_j^p \right)^{1/p},$$

where $d_j := d_j(\delta_k B^{\alpha,\tau}_{\mathbb{C}}, L_2(U, \mathbb{C}; \mu))$. Hence, by Corollary 2.2 and the inequality $p\beta > 1$, we derive for $k \le k_n$,

$$\|\delta_k v - S^X_{n_k}(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ C\, (2^k n_k)^{-\beta} (k + \log n_k)^{\beta\tau/\alpha} \ \le\ C\, n^{-\beta} (\log n)^{\beta\tau/\alpha}.$$

Hence, we have

$$\sum_{k=0}^{k_n} \|\delta_k v - S^X_{c_p n}(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ \sum_{k=0}^{k_n} C\, n^{-\beta} (\log n)^{\beta\tau/\alpha} \ \le\ C\, n^{-\beta} (\log n)^{1+\beta\tau/\alpha},$$

Since $\tau = \frac{\alpha}{\alpha - \beta}$, we have $1 + \beta\tau/\alpha = \tau$. Hence, we have

$$\sum_{k=0}^{k_n} \|\delta_k v - S^X_{c_p n}(\delta_k v)\|_{L_2(U,X;\mu)} \ \le\ C\, n^{-\beta} (\log n)^\tau,$$

Summing up, we arrive at the inequality

$$\|v - \mathcal{S}_n^X v\|_{L_2(U,X;\mu)} \;\le\; C\, n^{-\beta} (\log n)^{\tau}, \quad \text{if} \;\; \alpha > 1/q_2.$$

For the computational complexity $\operatorname{Comp}\left(\mathcal{S}_n^X v\right)$ we have that

$$\operatorname{Comp}\left(\mathcal{S}_n^X v\right) \;\le 2 \sum_{k=0}^{k_n} 2^k n_k \;\le\; 2 \sum_{k=0}^{k_n} 2^k \left(n 2^{-k}\right) \;=\; 2n \sum_{k=0}^{k_n} 1 \;\le\; C n \log n.$$

□

By the continuous embedding $H_X^\alpha \hookrightarrow H_X^{\alpha,1+\varepsilon}$ for any $\varepsilon > 0$, Theorem 2.1 can be reformulated more conveniently for the space H_X^α as follows.

Theorem 2.2. *Let $0 < q_1 \le q_2 < 2$ and ε be any positive number. Let the number β be defined as in (2.27). Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of increasing positive numbers ξ_n satisfying the condition (2.28). Assume that there is a constant K_ε such that for every $v \in H_X^{\alpha,1+\varepsilon}$ and every $m \in \mathbb{N}_0$, $\|P_m v\|_{H_X^{\alpha,1+\varepsilon}} \le K_\varepsilon \|f\|_{H_X^{\alpha,1+\varepsilon}}$. Then for all $n \ge 2$ there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in U$ and $h_{k,1}, ..., h_{k,n_k} \in L_2(U,\mathbb{C};\mu)$, $k = 0, ..., k_{\lceil n/\log n \rceil}$, such that for the operator $\bar{\mathcal{S}}_n^X := \mathcal{S}_{\lceil n/\log n\rceil}^X$ defined as in (2.24)–(2.26),*

$$\operatorname{Comp}\left(\bar{\mathcal{S}}_n^X\right) \;\le\; n,$$

and

$$\sup_{v\in B_X^\alpha} \|v - \bar{\mathcal{S}}_n^X v\|_{L_2(U,X;\mu)} \;\le\; \begin{cases} C_\varepsilon n^{-\alpha} (\log n)^{\alpha+1+\varepsilon} & \text{if } \alpha < 1/q_2, \\ C_\varepsilon n^{-\alpha} (\log n)^{\alpha+2+\varepsilon} & \text{if } \alpha = 1/q_2, \\ C n^{-\beta} (\log n)^{\beta+\alpha/(\alpha-\beta)} & \text{if } \alpha > 1/q_2. \end{cases}$$

Next, we apply Theorem 2.2 to Bochner spaces with infinite tensor-product standard Gaussian measure relevant to the applications to holomorphic functions and solutions to parametric PDEs with random inputs in Sections 3 and 5, respectively.

We recall a concept of standard Gaussian measure $\gamma(\boldsymbol{y})$ on $\mathbb{R}^\infty$ as the infinite tensor product of copies of the one-dimensional standard Gaussian measure $\gamma(y_i)$:

$$\gamma(\boldsymbol{y}) := \bigotimes_{j\in\mathbb{N}} \gamma(y_j), \quad \boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in \mathbb{R}^\infty.$$

(The sigma algebra for $\gamma(\boldsymbol{y})$ is generated by the set of cylinders $A := \prod_{j\in\mathbb{N}} A_j$, where $A_j \subset \mathbb{R}$ are univariate γ-measurable sets and only a finite number of A_i are different from $\mathbb{R}$. For such a set A, we have $\gamma(A) = \prod_{j\in\mathbb{N}} \gamma(A_j)$).

Let X be a separable Hilbert space. Then a function $v \in L_2(\mathbb{R}^\infty, X;\gamma)$ can be represented by the Hermite GPC expansion

$$v = \sum_{\boldsymbol{s}\in\mathbb{F}} v_{\boldsymbol{s}}\, H_{\boldsymbol{s}}, \quad v_{\boldsymbol{s}} \in X, \tag{2.33}$$

with

$$H_{\boldsymbol{s}}(\boldsymbol{y}) = \bigotimes_{j \in \mathbb{N}} H_{s_j}(y_j), \quad v_{\boldsymbol{s}} := \int_{\mathbb{R}^\infty} v(\boldsymbol{y})\, H_{\boldsymbol{s}}(\boldsymbol{y})\, \mathrm{d}\gamma(\boldsymbol{y}), \quad \boldsymbol{s} \in \mathbb{F}.$$

Here $(H_s)_{s \in \mathbb{N}_0}$ are the univariate orthonormal Hermite polynomials, $\mathbb{F}$ is the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j \in \mathbb{N}}$ such that their support $\mathrm{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. Notice that the complex-valued $(H_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ are an orthonormal basis of $L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)$. Moreover, for every $v \in L_2(\mathbb{R}^\infty, X; \gamma)$ represented by the series (2.33), Parseval's identity holds

$$\|v\|^2_{L_2(\mathbb{R}^\infty, X; \gamma)} \ = \ \sum_{\boldsymbol{s} \in \mathbb{F}} \|v_{\boldsymbol{s}}\|_X^2.$$

Let $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ be a non-decreasing sequence of positive numbers strictly larger than 1 such that $\boldsymbol{\sigma}^{-1} := \left(\sigma_{\boldsymbol{s}}^{-1}\right)_{\boldsymbol{s} \in \mathbb{F}} \in \ell_2(\mathbb{F})$. Denote by $H_{X,\boldsymbol{\sigma}}$ the linear subspace in $L_2(\mathbb{R}^\infty, X; \gamma)$ of all v such that the norm

$$\|v\|_{H_{X,\boldsymbol{\sigma}}} := \ \left(\sum_{\boldsymbol{s} \in \mathbb{F}} \left(\sigma_{\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_X\right)^2 \right)^{1/2} \ < \infty.$$

In particular, the space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is the linear subspace in $L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)$ equipped with its own inner product

$$\langle f, g \rangle_{H_{\mathbb{C},\boldsymbol{\sigma}}} := \ \sum_{\boldsymbol{s} \in \mathbb{F}} \sigma_{\boldsymbol{s}}^2 \langle f, H_{\boldsymbol{s}} \rangle_{L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)} \overline{\langle g, H_{\boldsymbol{s}} \rangle_{L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)}}.$$

The space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is a reproducing kernel Hilbert space with the reproducing kernel

$$K(\cdot, \boldsymbol{y}) := \ \sum_{\boldsymbol{s} \in \mathbb{F}} \sigma_{\boldsymbol{s}}^{-2} H_{\boldsymbol{s}}(\cdot) \overline{H_{\boldsymbol{s}}(\boldsymbol{y})}$$

with the eigenfunctions $(H_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$ and the eigenvalues $\left(\sigma_{\boldsymbol{s}}^{-1}\right)_{\boldsymbol{s} \in \mathbb{F}}$. Moreover, $K(\boldsymbol{x}, \boldsymbol{y})$ satisfies the finite trace assumption

$$\int_{\mathbb{R}^\infty} K(\boldsymbol{x}, \boldsymbol{x}) \mathrm{d}\gamma(\boldsymbol{x}) \ < \ \infty.$$

Let $(V_k)_{k \in \mathbb{N}_0}$ be a given sequence of subspaces $V_k \subset X$ of dimension 2^k, and $(P_k)_{k \in \mathbb{N}_0}$ a given sequence of uniformly bounded linear projectors P_k from X onto V_k. Let $(\sigma_{i;\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$, $i = 1, 2$, be given non-decreasing sets of numbers strictly larger than 1, such that $\boldsymbol{\sigma}_i^{-1} := \left(\sigma_{i,\boldsymbol{s}}^{-1}\right)_{\boldsymbol{s} \in \mathbb{F}} \in \ell_2(\mathbb{F})$. For given numbers $\alpha > 0$ and $\tau > 0$, the linear subspaces H_X^α and $H_X^{\alpha,\tau}$ in $L_2(\mathbb{R}^\infty, X; \gamma)$ are defined in the same manner as (2.3) and (2.5), respectively.

Theorem 2.2 for the space $L_2(\mathbb{R}^\infty, X; \gamma)$ is read as

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold for the space $L_2(\mathbb{R}^\infty, X; \gamma)$. Then for all $n \geq 2$ there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in \mathbb{R}^\infty$ and $h_{k,1}, ..., h_{k,n_k} \in L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)$, $k = 0, ..., k_{\lceil n/\log n \rceil}$, such that for the operator $\bar{\mathcal{S}}_n^X := \mathcal{S}^X_{\lceil n/\log n \rceil}$ defined as in* (2.24)–(2.26)*,*

$$\mathrm{Comp}\left(\bar{\mathcal{S}}_n^X\right) \ \leq \ n,$$

and

$$\sup_{v \in B^\alpha_X} \|v - \bar{\mathcal{S}}^X_n v\|_{L_2(\mathbb{R}^\infty, X;\gamma)} \le \begin{cases} C_\varepsilon n^{-\alpha} (\log n)^{\alpha+1+\varepsilon} & \text{if } \alpha < 1/q_2, \\ C_\varepsilon n^{-\alpha} (\log n)^{\alpha+2+\varepsilon} & \text{if } \alpha = 1/q_2, \\ C n^{-\beta} (\log n)^{\beta+\alpha/(\alpha-\beta)} & \text{if } \alpha > 1/q_2. \end{cases}$$

3 Applications to holomorphic functions

In this section, we apply the results on multi-level linear sampling recovery in the space $L_2(\mathbb{R}^\infty, X;\gamma)$ to holomorphic functions on $\mathbb{R}^\infty$.

We recall a concept of $(\boldsymbol{b}, \xi, \delta, X)$-holomorphic functions which has been introduced in [24, Definition 4.1]. For $m \in \mathbb{N}$ and a positive sequence $\boldsymbol{\varrho} = (\varrho_j)_{j=1}^m$, we put

$$\mathcal{S}(\boldsymbol{\varrho}) := \{\boldsymbol{z} \in \mathbb{C}^m \,:\, |\mathfrak{Im} z_j| < \varrho_j \ \forall j\} \qquad \text{and} \qquad \mathcal{B}(\boldsymbol{\varrho}) := \{\boldsymbol{z} \in \mathbb{C}^m \,:\, |z_j| < \varrho_j \ \forall j\}.$$

Let X be a complex separable Hilbert space, $\boldsymbol{b} = (b_j)_{j \in \mathbb{N}}$ a positive sequence, and $\xi > 0$, $\delta > 0$. For $m \in \mathbb{N}$ we say that a positive sequence $\boldsymbol{\varrho} = (\varrho_j)_{j=1}^m$ is $(\boldsymbol{b}, \xi)$-*admissible* if

$$\sum_{j=1}^m b_j \varrho_j \le \xi\,.$$

A function $v \in L_2(\mathbb{R}^\infty, X;\gamma)$ is called $(\boldsymbol{b}, \xi, \delta, X)$-holomorphic if

(i) for every $m \in \mathbb{N}$ there exists $v_m : \mathbb{R}^m \to X$, which, for every $(\boldsymbol{b}, \xi)$-admissible $\boldsymbol{\varrho}$, admits a holomorphic extension (denoted again by v_m) from $\mathcal{S}(\boldsymbol{\varrho}) \to X$; furthermore, for all $m < m'$

$$v_m(y_1, \dots, y_m) = v_{m'}(y_1, \dots, y_m, 0, \dots, 0) \qquad \forall (y_j)_{j=1}^m \in \mathbb{R}^m,$$

(ii) for every $m \in \mathbb{N}$ there exists $\varphi_m : \mathbb{R}^m \to \mathbb{R}_+$ such that $\|\varphi_m\|_{L^2(\mathbb{R}^m;\gamma)} \le \delta$ and

$$\sup_{\boldsymbol{\rho} \text{ is } (\boldsymbol{b},\xi)\text{-adm.}} \ \sup_{\boldsymbol{z} \in \mathcal{B}(\boldsymbol{\varrho})} \|v_m(\boldsymbol{y} + \boldsymbol{z})\|_X \le \varphi_m(\boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \mathbb{R}^m,$$

(iii) with $\tilde{v}_m : \mathbb{R}^\infty \to X$ defined by $\tilde{v}_m(\boldsymbol{y}) := v_m(y_1, \dots, y_m)$ for $\boldsymbol{y} \in \mathbb{R}^\infty$ it holds

$$\lim_{m \to \infty} \|v - \tilde{v}_m\|_{L_2(\mathbb{R}^\infty, X;\gamma)} = 0.$$

The following key result on weighted ℓ_2-summability of $(\boldsymbol{b}, \xi, \delta, X)$-holomorphic functions has been proven in [24, Corollary 4.9].

Lemma 3.1. *Let v be $(\boldsymbol{b},\xi,\delta,X)$-holomorphic for some $\boldsymbol{b}\in\ell_p(\mathbb{N})$ with $0<p<1$. Let $\eta\in\mathbb{N}_0$ and let the sequence $\boldsymbol{\rho}=(\rho_j)_{j\in\mathbb{N}}$ be defined by*

$$\rho_j := b_j^{p-1}\frac{\xi}{4\sqrt{\eta!}}\|\boldsymbol{b}\|_{\ell_p(\mathbb{N})}.$$

Assume that $\boldsymbol{b}$ is a non-increasing sequence and that $b_j^{p-1}\frac{\xi}{4\sqrt{\eta!}}\|\boldsymbol{b}\|_{\ell_p(\mathbb{N})}>1$ for all $j\in\mathbb{N}$. Then we have

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}}(\sigma_{\boldsymbol{s}}\|v_{\boldsymbol{s}}\|_X)^2\right)^{1/2}\le M\ <\infty,\quad \text{with}\quad \left\|\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{N})}\le N<\infty,$$

where $q:=p/(1-p)$, with $|\boldsymbol{s}'|_\infty:=\sup_{j\in\mathbb{N}}s_j'$ the set $\boldsymbol{\sigma}:=(\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ is defined by

$$\sigma_{\boldsymbol{s}}^2 := \sum_{|\boldsymbol{s}'|_\infty\le\eta}\binom{\boldsymbol{s}}{\boldsymbol{s}'}\prod_{j\in\mathbb{N}}\rho_j^{2s_j'}, \tag{3.1}$$

$M=\delta C_{\boldsymbol{b},\xi,\eta}$ with some positive constant $C_{\boldsymbol{b},\xi,\eta}$, and $N=K_{\boldsymbol{b},\xi,\eta}$.

This theorem allows us to apply weighted ℓ_2-summability for collocation approximation of solutions $u(\boldsymbol{y})$ as $(\boldsymbol{b},\xi,\delta,X)$-holomorphic functions on various function spaces X, to a wide range of parametric and stochastic PDEs with log-normal inputs, specially, Kondrat'ev spaces $X=K_\varkappa^r(D)$ for the parametric elliptic PDEs (1.1) on bounded polygonal domain with log-normal inputs (1.3) a detailed analysis of which will be presented in Section 5. For more information see [24].

For a function v defined on $\mathbb{R}^\infty$ taking values in a separable Hilbert space X, we say that v satisfies Assumption 3.1 if

Assumption 3.1. *$\alpha>0$, $\eta\in\mathbb{N}$, $0<p_1\le p_2<1$, $0<p_1<2/3$, $\boldsymbol{b}_i\in\ell_{p_1}(\mathbb{N})$, $i=1,2$, are given non-decreasing sequences, $\xi>0$, $\delta>0$ such that*

$$\rho_{i,j} := b_{i,j}^{p_i-1}\frac{\xi}{4\sqrt{\eta!}}\|\boldsymbol{b}_i\|_{\ell_{p_j}(\mathbb{N})}>1,\ j\in\mathbb{N}. \tag{3.2}$$

$\mathcal{V}=(V_k)_{k\in\mathbb{N}_0}$ is a given sequence of subspaces $V_k\subset L_2(\mathbb{R}^\infty,X;\gamma)$ of dimension 2^k, and $\mathcal{P}=(P_k)_{k\in\mathbb{N}_0}$ a given sequence of linear uniformly bounded projectors from $L_2(\mathbb{R}^\infty,X;\gamma)$ onto V_k such that

(i) $v\in L_2(\mathbb{R}^\infty,X;\gamma)$ *and is* $(\boldsymbol{b}_1,\xi,\delta,X)$*-holomorphic,*

(ii) $(v-P_kv)\in L_2(\mathbb{R}^\infty,X;\gamma)$ *and is* $(\boldsymbol{b}_1,\xi,\delta,X)$*-holomorphic for every* $k\in\mathbb{N}_0$,

(iii) $(v-P_kv)$ *is* $(\boldsymbol{b}_2,\xi,\delta 2^{-\alpha k},X)$*-holomorphic for every* $k\in\mathbb{N}_0$,

(iv) For every $m\in\mathbb{N}_0$ and every linear bounded operator Q in $L_2(\mathbb{R}^\infty,X;\gamma)$ with $\|Q\|\le K$, the function $w:=Qv$ possesses the properties (i)–(iii) with the parameter δ replaced by $K\delta$, the same parameters α,η,p_1,p_2,ξ and sequences $\boldsymbol{b}_1,\boldsymbol{b}_2,\mathcal{V},\mathcal{P}$.

For a space H^α_X with $\left\|\boldsymbol{\sigma}_i^{-1}\right\|_{\ell_q(\mathbb{N})} \le N$, $i = 1,2$, denote

$$B^\alpha_X(M,N) := \left\{ v \in H^{\alpha,\tau}_X : \|v\|_{H^{\alpha,\tau}_X} \le M \right\}.$$

Lemma 3.2. *Let H^α_X and $H^{\alpha,\tau}_X$ with $\tau > 1$ be defined so that for $i = 1,2$, $q_i := p_i/(1-p_i)$ and $\boldsymbol{\sigma}_i := (\sigma_{i,\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ be given by (3.1) for $\boldsymbol{\rho}_i$ as in (3.2), $\left\|\boldsymbol{\sigma}_i^{-1}\right\|_{\ell_{q_i}(\mathbb{F})} \le N$, respectively. Then every function v satisfying Assumption 3.1 belongs to $B^\alpha_X(M,N)$ for $M = \delta C_{\boldsymbol{b},\xi,\eta}$ and $N = K_{\boldsymbol{b},\xi,\eta,\tau}$ with some positive constants $C_{\boldsymbol{b},\xi,\eta}$, and $K_{\boldsymbol{b},\xi,\eta}$. Moreover,*

$$\|P_m v\|_{H^{\alpha,\tau}_X} \le K_{\boldsymbol{b},\xi,\eta,\tau} M \, \|v\|_{H^{\alpha,\tau}_X}, \quad m \in \mathbb{N}_0.$$

Proof. Assume that v satisfies Assumption 3.1. Then we have by Lemma 3.1,

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{1,\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_X)^2 \right)^{1/2} \le M, \quad \text{with} \quad \left\|\boldsymbol{\sigma}_1^{-1}\right\|_{\ell_q(\mathbb{F})} \le N < \infty,$$

and for every $k \in \mathbb{N}_0$,

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{2,\boldsymbol{s}} \|(v - P_k v)_{\boldsymbol{s}}\|_X)^2 \right)^{1/2} \le M 2^{-\alpha k} < \infty, \quad \text{with} \quad \left\|\boldsymbol{\sigma}_2^{-1}\right\|_{\ell_q(\mathbb{F})} \le N < \infty,$$

and in addition, for a every $m \in \mathbb{N}_0$,

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{1,\boldsymbol{s}} \|(P_m)_{\boldsymbol{s}}\|_X)^2 \right)^{1/2} \le K_{\boldsymbol{b},\xi,\eta} M, \quad \left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{2,\boldsymbol{s}} \|(P_m - P_k(P_m))_{\boldsymbol{s}}\|_X)^2 \right)^{1/2} \le K_{\boldsymbol{b},\xi,\eta} M 2^{-\alpha k}.$$

Hence, $v \in B^\alpha_X(M,N)$. Moreover, $\|P_m v\|_{H^{\alpha,\tau}_X} \le K_{\boldsymbol{b},\xi,\eta,\tau} M \, \|v\|_{H^{\alpha,\tau}_X}$, $m \in \mathbb{N}_0$, since $\tau > 1$. ∎

By applying Theorem 2.3, from Lemma 3.2 we obtain

Theorem 3.2. *Let v satisfy Assumption 3.1. Let $q_i := p_i/(1-p_i)$ for $i = 1,2$, and ε be any positive number. Let the number β be defined as in (2.27). Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of increasing positive numbers ξ_n satisfying the condition (2.28). Then for all $n \ge 2$ there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in \mathbb{R}^\infty$ and $h_{k,1}, ..., h_{k,n_k} \in L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)$, $k = 0, ..., k_{\lceil n/\log n \rceil}$, such that for the operator $\bar{\mathcal{S}}^X_n := \mathcal{S}^X_{\lceil n/\log n \rceil}$ defined as in (2.24)–(2.26),*

$$\operatorname{Comp}\left(\bar{\mathcal{S}}^X_n\right) \le n,$$

and

$$\|v - \bar{\mathcal{S}}^X_n v\|_{L_2(\mathbb{R}^\infty, X; \gamma)} \le \begin{cases} C_\varepsilon n^{-\alpha} (\log n)^{\alpha+1+\varepsilon} & \text{if } \alpha < 1/q_2, \\ C_\varepsilon n^{-\alpha} (\log n)^{\alpha+2+\varepsilon} & \text{if } \alpha = 1/q_2, \\ C n^{-\beta} (\log n)^{\beta+\alpha/(\alpha-\beta)} & \text{if } \alpha > 1/q_2. \end{cases}$$

4 Multi-level sparse-grid interpolation algorithms

In this section, we construct fully discrete multi-level sparse-grid sampling algorithms for the $(\boldsymbol{b}, \xi, \delta, X)$-holomorphic functions satisfying Assumption 3.1 except the item 4, based on GPC Lagrange-Hermite interpolation, and prove convergence rates of the approximation by them. It terns out that in the case $\alpha \le 1/q_2 - 1/2$, these sampling algorithms give a convergence rate better than the extended least squares sampling algorithm in Theorem 3.2. The construction and techniques used in this section are a modification of those in [21, 22].

For $m \in \mathbb{N}_0$, let $Y_m = (y_{m;k})_{k \in \pi_m}$ be the increasing sequence of the $m+1$ roots of the Hermite polynomial H_{m+1}, ordered as

$$y_{m,-j} < \cdots < y_{m,-1} < y_{m,0} = 0 < y_{m,1} < \cdots < y_{m,j} \quad \text{if } m = 2j,$$

$$y_{m,-j} < \cdots < y_{m,-1} < y_{m,1} < \cdots < y_{m,j} \quad \text{if } m = 2j-1,$$

where

$$\pi_m := \begin{cases} \{-j, -j+1, ..., -1, 0, 1, ..., j-1, j\} & \text{if } m = 2j; \\ \{-j, -j+1, ..., -1, 1, ..., j-1, j\} & \text{if } m = 2j-1. \end{cases}$$

(in particular, $Y_0 = (y_{0;0})$ with $y_{0;0} = 0$).

For a function v on $\mathbb{R}$, taking values in a Hilbert space X and $m \in \mathbb{N}_0$, we define the Lagrange interpolation operator I_m by

$$I_m(v) := \sum_{k \in \pi_m} v(y_{m;k}) L_{m;k}, \quad L_{m;k}(y) := \prod_{j \in \pi_m \ j \neq k} \frac{y - y_{m;j}}{y_{n;k} - y_{m;j}},$$

(in particular, $I_0(v) = v(y_{0,0}) L_{0,0}(y) = v(0)$ and $L_{0,0}(y) = 1$). Notice that $I_m(v)$ is a function on $\mathbb{R}$ taking values in X and interpolating v at $y_{m;k}$, i.e., $I_m(v)(y_{m;k}) = v(y_{m;k})$.

We define the univariate operator Δ_m for $m \in \mathbb{N}_0$ by

$$\Delta_m := I_m - I_{m-1},$$

with the convention $I_{-1} = 0$.

For a function v on $\mathbb{R}^\infty$, taking values in a Hilbert space X, we introduce the tensor product operator $\Delta_{\boldsymbol{s}}$, $\boldsymbol{s} \in \mathbb{F}$, by

$$\Delta_{\boldsymbol{s}}(v) := \bigotimes_{j \in \mathbb{N}} \Delta_{s_j}(v),$$

where the univariate operator Δ_{s_j} is successively applied to the univariate function $\bigotimes_{i<j} \Delta_{s_i}(v)$ by considering it as a function of variable y_j with the other variables held fixed. We define for $\boldsymbol{s} \in \mathbb{F}$,

$$I_{\boldsymbol{s}}(v) := \bigotimes_{j \in \mathbb{N}} I_{s_j}(v), \quad L_{\boldsymbol{s};\boldsymbol{k}} := \bigotimes_{j \in \mathbb{N}} L_{s_j;k_j}, \quad \pi_{\boldsymbol{s}} := \prod_{j \in \mathbb{N}} \pi_{s_j},$$

(the operator $I_{\boldsymbol{s}}$ is defined in the same manner as $\Delta_{\boldsymbol{s}}$).

For $\boldsymbol{s} \in \mathbb{F}$ and $\boldsymbol{k} \in \pi_{\boldsymbol{s}}$, let $E_{\boldsymbol{s}}$ be the subset in $\mathbb{F}$ of all $\boldsymbol{e}$ such that e_j is either 1 or 0 if $s_j > 0$, and e_j is 0 if $s_j = 0$, and let $\boldsymbol{y}_{\boldsymbol{s};\boldsymbol{k}} := (y_{s_j;k_j})_{j\in\mathbb{N}} \in \mathbb{R}^\infty$. Put $|\boldsymbol{s}|_1 := \sum_{j\in\mathbb{N}} s_j$ for $\boldsymbol{s} \in \mathbb{F}$. It is easy to check that the interpolation operator $\Delta_{\boldsymbol{s}}$ can be represented in the form

$$\Delta_{\boldsymbol{s}}(v) \;=\; \sum_{\boldsymbol{e}\in E_{\boldsymbol{s}}} (-1)^{|\boldsymbol{e}|_1} I_{\boldsymbol{s}-\boldsymbol{e}}(v) \;=\; \sum_{\boldsymbol{e}\in E_{\boldsymbol{s}}} (-1)^{|\boldsymbol{e}|_1} \sum_{\boldsymbol{k}\in\pi_{\boldsymbol{s}-\boldsymbol{e}}} v(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}) L_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}. \tag{4.1}$$

For a given finite set $\Lambda \subset \mathbb{F}$, we introduce the GPC interpolation operator I_Λ by

$$I_\Lambda := \sum_{\boldsymbol{s}\in\Lambda} \Delta_{\boldsymbol{s}}.$$

From (4.1) we obtain the sparse-grid interpolation sampling algorithm

$$I_\Lambda(v) \;=\; \sum_{\boldsymbol{s}\in\Lambda} \sum_{\boldsymbol{e}\in E_{\boldsymbol{s}}} (-1)^{|\boldsymbol{e}|_1} \sum_{\boldsymbol{k}\in\pi_{\boldsymbol{s}-\boldsymbol{e}}} v(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}) L_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}. \tag{4.2}$$

Next, we introduce the multi-level sparse-grid interpolation sampling algorithm $\mathcal{I}_\xi$ for $\xi > 1$ by

$$\mathcal{I}_\xi v \;=\; \sum_{k=0}^{\lfloor \log_2 \xi \rfloor} I_{\Lambda_k(\xi)}(\delta_k v). \tag{4.3}$$

where

$$\Lambda_k(\xi) := \begin{cases} \left\{\boldsymbol{s} \in \mathbb{F} : \, \sigma_{2;\boldsymbol{s}} \le \left(2^{-k}\xi\right)^{1/q_2} \right\} & \text{if } \alpha \le 1/q_2 - 1/2; \\ \left\{\boldsymbol{s} \in \mathbb{F} : \, \sigma_{1;\boldsymbol{s}} \le \xi^{1/q_1}, \; \sigma_{2;\boldsymbol{s}} \le 2^{-(\alpha+1/2)k}\xi^{\vartheta}\right\} & \text{if } \alpha > 1/q_2 - 1/2, \end{cases}$$

with $\vartheta := 1/q_1 + (1/q_1 - 1/q_2)/(2\alpha)$.

Since $\delta_k v$ belongs to the subspace $V_k + V_{k-1}$ of dimension at most $2^k + 2^{k-1}$ and the number of sampling points in the sampling algorithms $I_{\Lambda_k(\xi)}$ is

$$n_k := \sum_{\boldsymbol{s}\in\Lambda} \sum_{\boldsymbol{e}\in E_{\boldsymbol{s}}} |\pi_{\boldsymbol{s}-\boldsymbol{e}}|,$$

the computational complexity $\mathrm{Comp}\,(\mathcal{I}_\xi)$ of the operator $\mathcal{I}_\xi$ can be defined as

$$\mathrm{Comp}\,(\mathcal{I}_\xi) := \sum_{k=0}^{\lfloor \log_2 \xi \rfloor} \left(2^k + 2^{k-1}\right) n_k,$$

Theorem 4.1. *Let v satisfy Assumption 3.1 except the item 4. Let $q_i := p_i/(1-p_i)$ for $i = 1, 2$. Let*

$$\beta := \left(\frac{1}{q_1} - \frac{1}{2}\right) \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{1}{q_1} - \frac{1}{q_2}.$$

Then for each $n \in \mathbb{N}$ there exists a number ξ_n such that

$$\mathrm{Comp}\,(\mathcal{I}_{\xi_n}) \;\le\; n,$$

and

$$\|v - \mathcal{I}_{\xi_n} v\|_{L_2(\mathbb{R}^\infty, X;\gamma)} \ \le \ C \begin{cases} n^{-\alpha} & \text{if } \alpha \le 1/q_2 - 1/2, \\ n^{-\beta} & \text{if } \alpha > 1/q_2 - 1/2, \end{cases}$$

where the constant C is independent of v and n.

Proof. The operator $\mathcal{I}_\xi$ can be rewritten in the form

$$\mathcal{I}_\xi \ = \ \mathcal{I}_{G(\xi)} := \sum_{(k,\boldsymbol{s}) \in G(\xi)} \Delta^{\rm I}_{\boldsymbol{s}} \delta_k, \tag{4.4}$$

where

$$G(\xi) := \begin{cases} \left\{(k,\boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : \ 2^k \sigma_{2;\boldsymbol{s}}^{q_2} \le \xi \right\} & \text{if } \alpha \le 1/q_2 - 1/2; \\ \left\{(k,\boldsymbol{s}) \in \mathbb{N}_0 \times \mathbb{F} : \ \sigma_{1;\boldsymbol{s}}^{q_1} \le \xi, \ 2^{(\alpha+1/2)k} \sigma_{2;\boldsymbol{s}} \le \xi^\vartheta \right\} & \text{if } \alpha > 1/q_2 - 1/2. \end{cases}$$

On the other hand, it holds the following statement which can be proven in a way similar to the proof of [21, Theorem 3.8] (see also [22] for some proof corrections) with some slight modification. For each $n \in \mathbb{N}$ there exists a number ξ_n such that

$$\operatorname{Comp}\left(\mathcal{I}_{G(\xi_n)}\right) \ \le \ n,$$

and

$$\|v - \mathcal{I}_{G(\xi_n)} v\|_{L_2(\mathbb{R}^\infty, X;\gamma)} \ \le \ C \begin{cases} n^{-\alpha} & \text{if } \alpha \le 1/q_2 - 1/2, \\ n^{-\beta} & \text{if } \alpha > 1/q_2 - 1/2, \end{cases}$$

where the constant C is independent of v and n. From this statement and (4.4) we prove the theorem. ∎

5 Applications to parametric elliptic PDEs

In this section, we prove that the parametric solution u to the parametric elliptic PDEs (1.1) on a bounded polygonal domain with log-normal inputs (1.3) is a $(\boldsymbol{b}_j, \xi, \delta, V)$-holomorphic function on $\mathbb{R}^\infty$ satisfying Assumption 3.1 with $\boldsymbol{b}_j$ as in (5.4), $j = 1, 2$. This allows to prove convergence rates of a fully discrete multi-level collocation algorithm S_n. The spatial components in S_n are based on finite element Lagrange interpolations associated with triangulations of the spatial domain D. While the parametric components are based on Hermite-Lagrange GPC interpolation as in Theorem 5.3 in the case of small spatial regularity $\alpha \le 1/p_2 - 3/2$, and on extended least squares sampling algorithms as in Theorem 3.2 in the cases of higher spatial $\alpha > 1/p_2 - 3/2$.

We rewrite the solution to the equation (1.1) as the mapping $a \mapsto \mathcal{U}(a)$ from $\mathcal{W}^{r-1}_\infty(D)$ to $\mathcal{K}^r_{\varkappa+1}(D)$ satisfying the equation

$$-\operatorname{div}(a \nabla \mathcal{U}(a)) = f \quad \text{in } D, \qquad \mathcal{U}(a) = 0 \quad \text{on } \partial D. \tag{5.1}$$

The following result has been proven in [24, Theorem 7.8]

Lemma 5.1. *Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain and $r \in \mathbb{N}$, $r \geq 2$. Then there exist $\varkappa > 0$ and $C_r > 0$ depending on D and r such that for all $a \in W^1_\infty(D) \cap \mathcal{W}^{r-1}_\infty(D)$ and all $f \in \mathcal{K}^{r-2}_{\varkappa-1}(D)$ the weak solution $\mathcal{U} \in H^1_0(D)$ of* (5.1) *belongs to the space $\mathcal{K}^r_{\varkappa+1}(D)$ and satisfies with $N_r := \frac{r(r-1)}{2}$*

$$\|\mathcal{U}\|_{\mathcal{K}^r_{\varkappa+1}(D)} \leq C_r \frac{1}{\rho(a)} \left(\frac{\|a\|_{\mathcal{W}^{r-1}_\infty(D)} + \|a\|_{W^1_\infty(D)}}{\rho(a)} \right)^{N_r} \|f\|_{\mathcal{K}^{r-2}_{\varkappa-1}(D)}.$$

We recall a concept of Lagrange finite element. Let $r \in \mathbb{N}$ be given and $\mathcal{T}$ a triangulation of D with triangles T. Let $V(\mathcal{T})$ be the finite element space which consists of those continuous functions on D that restrict to polynomials of order at most r on each triangle $T \in \mathcal{T}$. For any function $v \in C(D)$, define $Iv := I(\mathcal{T})v$ as the interpolating function associated to v, the interpolation nodes $\boldsymbol{x}_1, ..., \boldsymbol{x}_{m(\mathcal{T})} \in D$ being obtained by taking points with baricentric coordinates $r^{-1}\mathbb{Z}$. The operator I is called Lagrange interpolation operator associated with $\mathcal{T}$. Thus, for any continuous function $v \in C(\bar{D})$, the operator I is uniquely determined by the conditions that $Iv(\boldsymbol{x}_i) = v(\boldsymbol{x}_i)$ for any interpolation node $\boldsymbol{x}_i$, $i = 1, ..., m(\mathcal{T})$, and $Iv \in V(\mathcal{T})$. From the definitions we can see that

$$(Iv)(\boldsymbol{x}) := \sum_{i=1}^{m(\mathcal{T})} v(\boldsymbol{x}_i)\phi_i(\boldsymbol{x}), \tag{5.2}$$

where ϕ_i, $i = 1, ..., m(\mathcal{T})$, are the nodal basic of $V(\mathcal{T})$. If $\Sigma := \{\sigma_1, ..., \sigma_{m(\mathcal{T})}\}$ are the linear forms such that $\sigma_i(v) := v(\boldsymbol{x}_i)$, $i = 1, ..., m(\mathcal{T})$, for $v \in V(\mathcal{T})$, then the triple $(D, V(\mathcal{T}), \Sigma)$ is a Lagrange finite element.

We consider a special class of the so-called shape-regular triangulation $\mathcal{T}$ which satisfies the condition $\frac{R_T}{r_T} \leq c$ for every $T \in \mathcal{T}$, where c is a constant and r_T (R_T) is the radius of the largest (smallest) ball contained in (containing) T. The following result is well-known (see, e.g., [10, 15]).

Lemma 5.2. *Let $r \in \mathbb{N}$ and $I := I(\mathcal{T})$, $m := m(\mathcal{T})$. Assume that the triangulation $\mathcal{T}$ is shape-regular. Then there exists a constant $C = C_{r,a,c}$ such that for any $v \in H^r(D)$*

$$\|v - Iv\|_{H^1(D)} \leq Cm^{-\frac{r-1}{2}} \|v\|_{H^r(D)}.$$

For the reader's convenience, we recall the definition of the class $\mathcal{C}$ of triangulations $\mathcal{T}$ of D introduced in [11, Definition 4.1]. For simplicity of presentation we assume that D is a triangle all of whose angles are acute. The general case can be considered in a similar way by first dividing D in triangles with acute angles. We also assume that D is an open set. Let l be the length of shortest edge of D. Let D_δ be the union of three isosceles triangles that have equal sides of length δ. The complement $D \setminus D_\delta$ is a hexagon when $\delta < l$. Fix $r \in \mathbb{N}$ and let $\varkappa \in (0, 1]$, $h > 0$, $\epsilon, b \in (0, 1)$, and $a \in (0, \pi/2)$ be parameters. We define $\mathcal{C} := \mathcal{C}(r, h, \varkappa, \epsilon, a, b)$ to be the set of triangulations $\mathcal{T}$ defined as follows. Choose m such that

$$\epsilon^{\varkappa m} \leq M(l, \epsilon l/8, a)h^r.$$

We decompose D as the union of $\Omega_0 := D \setminus D_{l/4}$, $\Omega_1 := D_{l/4} \setminus D_{\epsilon l/4}$,..., $\Omega_m := D_{\epsilon^{m-1}l/4} \setminus D_{\epsilon^m l/4}$, and $\tilde{\Omega}_{m+1} := D_{\epsilon^m l/4}$. For each $j = 0, ..., m$, we triangulate Ω_j with triangles with all angles $\geq a$, and edges of length at most

$$h_{m,j} := h\epsilon^{(1-\varkappa/r)j}$$

and at least $bh_{m,j}$. Then $\mathcal{T}$ is the union of the triangles appearing in the triangulations Ω_j, $j \le m$, and of three triangles forming $\tilde{\Omega}_{m+1}$.

Notice that the finite element space $V(\mathcal{T})$ with $\mathcal{T} \in \mathcal{C}$, r is the degree of the polynomials used in the approximation, h is the largest admissible length of the sides of the triangles in the partition, ϵ controls the decay of the triangles as they approach a vertex, a is the minimum admissible angle of a triangle in the partition , $0 < \varkappa < \pi/a_1$, where a_1 is the largest angle of the polygon, and b controls the ratio of the sizes of close triangles. The constants $r, h, \varkappa, \epsilon, a, b$ must satisfy certain conditions for the class $\mathcal{C} := \mathcal{C}(r, h, \varkappa, \epsilon, a, b)$ to be non-empty. The following result [11, Theorem 0.3] is therefore relevant. For any polygon D, there exist $0 < \varkappa \le 1$, $a > 0$, $1 > b > 0$ and a sequence $h_k = h_0 2^{-k}$ such that, if $\epsilon = 2^{-r/\varkappa}$, the class $\mathcal{C}_k := \mathcal{C}(r, h_k, \varkappa, \epsilon, a, b)$ is not empty, where $h_0 > 0$ is a certain constant.

If $\mathcal{T}_k \in \mathcal{C}_k$, there are positive constants C and C' such that

$$Cn \le h_n^{-2} \le C'n, \quad \dim V_k = m_k \le 2^k.$$

where we denote $V_k := V(\mathcal{T}_k)$ and $m_k := m(\mathcal{T}_k)$.

Let $\mathcal{T}_k \in \mathcal{C}_k$, $k \in \mathbb{N}_0$. Let $P_k := I(\mathcal{T}_k)$ be the Lagrange interpolation operator defined as in (5.2):

$$(P_k v)(\boldsymbol{x}) := \sum_{i_k=1}^{m_k} v(\boldsymbol{x}_{k,i_k})\phi_{k,i_k}(\boldsymbol{x}), \tag{5.3}$$

where $\boldsymbol{x}_{k,1}, ..., \boldsymbol{x}_{k,m_k} \in D$ are interpolation nodes and $\phi_{k,1}, ..., \phi_{k,m_k} \in C(D)$ are the nodal basis of V_k whose restriction to every $T \in \mathcal{T}_k$ is a polynomial of degree at most r. From [11, Theorem 4.4] one can derive the following

Lemma 5.3. *Let $r \in \mathbb{N}$. Then there exists a constant $C > 0$ such that we have for every $k \in \mathbb{N}_0$,*

$$\sup_{0 \ne v \in \mathcal{K}^r_{\varkappa+1}(D)} \frac{\|v - P_k v\|_{\mathcal{K}^1_1(D)}}{\|v\|_{\mathcal{K}^r_{\varkappa+1}(D)}} \ \le\ C 2^{-\frac{r-1}{2}k}.$$

Throughout the rest of this section, we consider the equation (5.1) with the parametric diffusion coefficient (1.3) satisfying the condition $\psi_j \in W^1_\infty(D) \cap \mathcal{W}^{r-1}_\infty(D)$, $j \in \mathbb{N}$, and $f \in \mathcal{K}^{r-2}_{\varkappa-1}(D)$. Denote

$$b_{1,j} := \|\psi_j\|_{L^\infty}, \quad b_{2,j} := \max\left\{ \|\psi_j\|_{W^1_\infty(D)}, \|\psi_j\|_{\mathcal{W}^{r-1}_\infty(D)} \right\} \tag{5.4}$$

and $\boldsymbol{b}_1 := (b_{1,j})_{j\in\mathbb{N}}$, $\boldsymbol{b}_2 := (b_{2,j})_{j\in\mathbb{N}}$.

Lemma 5.4. *Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain and $r \in \mathbb{N}$, $r \ge 2$. Let $\alpha = \frac{r-1}{2}$. Let $\psi_k \in W^1_\infty(D) \cap \mathcal{W}^{r-1}_\infty(D)$, $k \in \mathbb{N}$, and $f \in \mathcal{K}^{r-2}_{\varkappa-1}(D)$. Let $0 < p_1, p_2 < 1$. For $i = 1, 2$, let the sequences $\boldsymbol{b}_i := (b_{i,j})_{j\in\mathbb{N}}$ be defined as in (5.4) with $\boldsymbol{b}_i \in \ell^{p_i}(\mathbb{N})$. Then there exist $\xi > 0$ and $\delta > 0$ such that for the parametric solution to (5.1) with inputs (1.3)*

$$u(\boldsymbol{y}) := \mathcal{U}\left(\exp\left(\sum_{j\in\mathbb{N}} y_j \psi_j \right)\right)$$

and for every $k \in \mathbb{N}$,

(1) u *is* $(\boldsymbol{b}_1, \xi, \delta, V)$*-holomorphic,*

(2) $P_k u$ *is* $(\boldsymbol{b}_1, \xi, \delta, V)$*-holomorphic,*

(3) $u - P_k u$ *is* $(\boldsymbol{b}_1, \xi, \delta, V)$*-holomorphic,*

(4) $u - P_k u$ *is* $(\boldsymbol{b}_2, \xi, \delta 2^{-\alpha k}, V)$*-holomorphic,*

(5) *For every* $m \in \mathbb{N}_0$*, the function* $w := P_m u$ *possesses the properties* (1)–(4) *with the parameter* δ *replaced by* $C\delta$*, the same parameters* $\alpha, \eta, p_1, p_2, \xi$ *and sequences* $\boldsymbol{b}_1, \boldsymbol{b}_2$*.*

Proof. The proof of this lemma is a modification of the proof of [24, Proposition 7.12]. We apply [24, Theorem 4.11] on holomorphy of composite functions with $E = L^\infty(D)$ and $X = V$ to prove the claims (1)–(5).

Step 1. We verify the claim (1). By [24, Proposition 7.12] there exist $\xi_1 > 0$ and $\delta_1 > 0$ such that $u(\boldsymbol{y})$ is $(\boldsymbol{b}_1, \xi_1, \delta_1, V)$-holomorphic on the open set

$$O_1 = \{a \in L^\infty(D; \mathbb{C}) \,:\, \rho(a) > 0\} \subset L^\infty(D; \mathbb{C})$$

for some constants $\xi_1 > 0$ and $\delta_1 > 0$ depending on O_1.

Step 2. Concerning the claim (2), by assumption, $b_{1,j} = \|\psi_j\|_{L^\infty}$ satisfies $\boldsymbol{b}_1 = (b_{1,j})_{j\in\mathbb{N}} \in \ell^{p_1}(\mathbb{N}) \subseteq \ell^1(\mathbb{N})$, which corresponds to the assumption (iv) of Theorem [24, Theorem 4.11]. It remains to verify the assumptions (i), (ii) and (iii) of [24, Theorem 4.11] for $P_k u$, $k \in \mathbb{N}_0$.

(i) $P_k\mathcal{U} : O_1 \to V$ is holomorphic since the map $a \mapsto P_k\mathcal{U}(a)$ is a composition of holomorphic functions.

(ii) For all $a \in O_1$, we have by Lemma 5.2 and the well-known estimate $\|\mathcal{U}(a)\|_V \leq \frac{\|f\|_{V'}}{\rho(a)}$,

$$\begin{aligned} \|P_k\mathcal{U}(a)\|_V &\leq \|\mathcal{U}(a)\|_V + \|\mathcal{U}(a) - P_k\mathcal{U}(a)\|_V \\ &\leq C\,\|\mathcal{U}(a)\|_V \leq C\frac{\|f\|_{V'}}{\rho(a)} = C\frac{\|f\|_{H^{-1}(D)}}{\rho(a)}. \end{aligned} \tag{5.5}$$

(iii) For all a, $b \in O_1$ we have by Lemma 5.2 and [24, (4.21)],

$$\|P_k\mathcal{U}(a) - P_k\mathcal{U}(b)\|_V \leq C\,\|\mathcal{U}(a) - \mathcal{U}(b)\|_V \leq \|f\|_{H^{-1}(D)} \frac{1}{\min\{\rho(a), \rho(b)\}^2} \|a - b\|_{L^\infty}. \tag{5.6}$$

According to [24, Theorem 4.11] the map

$$\mathcal{U} \mapsto P_k\mathcal{U} \in L^2(U, V; \gamma)$$

is $(\boldsymbol{b}_1, \xi_2, \delta_2, V)$-holomorphic, for some fixed constants $\xi_2 > 0$ and $\delta_2 > 0$ depending on O_1 but independent of k.

Step 3. The claim (3) follows directly from Steps 1 and 2 that the difference $u - P_k u$ is $(\boldsymbol{b}_1, \xi_3, \delta_3, V)$-holomorphic for some constants $\xi_3 > 0$ and $\delta_3 > 0$ depending on O_1 but independent of k.

Step 4. To show the claim (4), we set

$$O_2 = \{a \in W^1_\infty(D) \cap \mathcal{W}^{s-1}_\infty(D) \,:\, \rho(a) > 0\},$$

and verify again the assumptions (i), (ii) and (iii) of [24, Theorem 4.11] with $E = W^1_\infty(D) \cap \mathcal{W}^{s-1}_\infty(D)$. First, observe that with

$$b_{2,j} := \max\left\{ \|\psi_j\|_{\mathcal{W}^{s-1}_\infty(D)}, \|\psi_j\|_{W^1_\infty}(D) \right\},$$

by assumption

$$\boldsymbol{b}_2 = (b_{2,j})_{j\in\mathbb{N}} \in \ell^{p_2}(\mathbb{N}) \hookrightarrow \ell^1(\mathbb{N})$$

which corresponds to the assumption (iv) of Theorem [24, Theorem 4.11].

(i) For every $k \in \mathbb{N}$, the mapping $\mathcal{U} - P_k\mathcal{U} : O_2 \to V$ is holomorphic: Since O_2 can be considered a subset of O_1 (and O_2 is equipped with a stronger topology than O_1), Fréchet differentiability follows by Fréchet differentiability of

$$\mathcal{U} - P_k\mathcal{U} : O_1 \to V,$$

which holds by Step 3.

(ii) For every $a \in O_2$, by the embedding inequality (1.4) and Lemmata 5.3 and 5.1,

$$\|(\mathcal{U} - P_k\mathcal{U})(a)\|_V \le \delta_k \frac{(\|a\|_{W^1_\infty(D)} + \|a\|_{\mathcal{W}^{r-1}_\infty(D)})^{N_r+1}}{\rho(a)^{N_r+2}},$$

where $\delta_k := K2^{-\alpha k}\, \|f\|_{\mathcal{K}^{r-2}_{\varkappa-1}}$.

(iii) For every $a,\, b \in O_2 \subseteq O_1$, by (5.6) and [24, (4.21)],

$$\begin{aligned}\|(\mathcal{U} - P_k\mathcal{U})(a) - (\mathcal{U} - P_k\mathcal{U})(b)\|_V &\le \|\mathcal{U}(a) - \mathcal{U}(b)\|_V + \|P_k\mathcal{U}(a) - P_k\mathcal{U}(b)\|_V \\ &\le C\,\|f\|_{H^{-1}(D)} \frac{2}{\min\{\rho(a), \rho(b)\}^2} \|a - b\|_{L_\infty(D)}.\end{aligned}$$

We conclude with [24, Theorem 4.11] that there exist ξ_4 and C_4 depending on O_2, D but independent of k such that $u - P_k u$ is $(\boldsymbol{b}_2, \xi_4, C_4\delta_k, V)$-holomorphic.

Summing up, the claims (1)–(4) hold with

$$\xi := \min\{\xi_1, \xi_2, \xi_3, \xi_4, \} \qquad \text{and} \qquad \delta := \max\left\{\tilde{C}_1, C_2, C_3, C_4K\,\|f\|_{\mathcal{K}^{r-2}_{\varkappa-1}}\right\}.$$

Step 5. The claim (5) can be proven with the same arguments as ones in Steps 1–4 by using the inequality (5.5):

$$\|(P_m\mathcal{U})(a)\|_V \;\le\; C\,\|\mathcal{U}(a)\|_V$$

for every $m \in \mathbb{N}_0$ and every $a \in O_2$, ◻

Let $\mathcal{T}_k \in \mathcal{C}_k$. Recall that we denote $V_k := V(\mathcal{T}_k)$ and $m_k := m(\mathcal{T}_k)$ and $P_k := I(\mathcal{T}_k)$ the interpolation operator defined as in (5.3). Notice that there are positive constants C and C' such that

$$Cn \le h_n^{-2} \le C'n, \quad \dim V_k = m_k \le 2^k.$$

Let the operators δ_k, $k \in \mathbb{N}_0$, be defined in (5.7) for the sequence $(P_k)_{k\in\mathbb{N}}$. Then we have

$$(\delta_k v)(\boldsymbol{x}) := \sum_{i_k=1}^{m_k} v(\boldsymbol{x}_{k,i_k})\phi_{k,i_k}(\boldsymbol{x}) - \sum_{i_{k-1}=1}^{m_{k-1}} v(\boldsymbol{x}_{k-1,i_{k-1}})\phi_{k-1,i_{k-1}}(\boldsymbol{x}), \quad v \in C(D),$$

which can be rewritten as

$$(\delta_k v)(\boldsymbol{x}) := \sum_{i_k=1}^{\bar{m}_k} v(\bar{\boldsymbol{x}}_{k,i_k})\bar{\phi}_{k,i_k}(\boldsymbol{x}), \quad v \in C(D), \tag{5.7}$$

where $\bar{m}_k := m_k + m_{k-1}$; $\bar{\boldsymbol{x}}_{k,i_k} := \boldsymbol{x}_{k,i_k}$, $\bar{\phi}_{k,i_k} := \phi_{k,i_k}$ for $i_k = 1, ..., m_k$, and $\bar{\boldsymbol{x}}_{k,i_k} := \boldsymbol{x}_{k-1,i_{k-1}}$, $\bar{\phi}_{k,i_k} := -\phi_{k-1,i_{k-1}}$ for $i_k = m_k + 1, ..., \bar{m}_k$.

Recall that for $n_k \in \mathbb{N}_0$, the operator

$$S^V_{c_p n_k} v := \sum_{j_k=1}^{c_p n_k} v\left(\boldsymbol{y}_{k,j_k}\right) h_{k,j_k}$$

has been defined in (2.26) with $X = V$ for $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in U$ and $h_{k,1}, ..., h_{k,n_k} \in L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)$, and $c_p > 0$ is the constant as in Lemma 2.4. Then for all $n \ge 2$, with δ_k as in (5.7) the operator $\bar{\mathcal{S}}^V_n := \mathcal{S}^V_{\lceil n/\log n\rceil}$ defined as in (2.24) and (2.26) can be rewritten as

$$\bar{\mathcal{S}}^V_n v \;=\; \sum_{k=0}^{k_{\lceil n/\log n\rceil}} \sum_{i_k=1}^{\bar{m}_k} \sum_{j_k=1}^{c_p n_k} v\left(\bar{\boldsymbol{x}}_{k,i_k}, \boldsymbol{y}_{k,j_k}\right) \Phi_{k,i_k,j_k}(\boldsymbol{x}, \boldsymbol{y}), \tag{5.8}$$

where $\Phi_{k,i_k,j_k}(\boldsymbol{x}, \boldsymbol{y}) := \bar{\phi}_{k,i_k}(\boldsymbol{x}) h_{k,j_k}(\boldsymbol{y})$ and $k_{\lceil n/\log n\rceil}$ is defined by (2.25) for a sequence $(\xi_n)_{n\in\mathbb{N}}$ of increasing positive numbers ξ_n satisfying the condition (2.28). The operator $\bar{\mathcal{S}}^V_n$ is a linear sampling algorithm in the space $L_2(\mathbb{R}^\infty, V; \gamma)$ defined for functions $v(\boldsymbol{x}, \boldsymbol{y})$ on the spatial-parametric domain $D \times \mathbb{R}^\infty$ and based on the sampling points

$$\mathrm{SamplePts}\left(\bar{\mathcal{S}}^V_n\right) \;=\; \left\{\left(\boldsymbol{x}_{k,i_k}, \boldsymbol{y}_{k,j_k}\right) : \; i_k = 1, ..., \bar{m}_k, \; j_k = 1, ..., c_p n_k, \; k = 0, ..., k_{\lceil n/\log n\rceil}\right\}.$$

Assumption 5.1. *$D \subset \mathbb{R}^2$ is a bounded polygonal domain and $r \in \mathbb{N}$, $r \ge 2$; $f \in \mathcal{K}^{r-2}_{\varkappa-1}(D)$ and $\psi_k \in W^1_\infty(D) \cap \mathcal{W}^{r-1}_\infty(D)$, $k \in \mathbb{N}$. For $i = 1, 2$, the sequences $\boldsymbol{b}_i := (b_{i,j})_{j\in\mathbb{N}}$ defined as in* (5.4) *satisfies the condition $\boldsymbol{b}_i \in \ell^{p_i}(\mathbb{N})$ with $0 < p_1 \le p_2 < 1$ and $p_1 < 2/3$.*

Theorem 5.2. *Let Assumption 5.1 hold. Let the numbers α and β be defined by*

$$\alpha := \frac{r-1}{2}, \quad \beta := \left(\frac{1}{p_1} - 1\right)\frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{1}{p_1} - \frac{1}{p_2}. \tag{5.9}$$

Then for all $n \geq 2$ *there exist* $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in \mathbb{R}^\infty$ *and* $h_{k,1}, ..., h_{k,n_k} \in L_2(\mathbb{R}^\infty, \mathbb{C}; \gamma)$, $k = 0, ..., k_{\lceil n/\log n \rceil}$, *such that for the linear sampling algorithm defined by* (5.8), *we have that*

$$\left|\operatorname{SamplePts}\left(\bar{\mathcal{S}}_n^V\right)\right| \leq n,$$

and for the parametric solution u *to the equation* (1.2) *with log-normal random inputs,*

$$\left\|u - \bar{\mathcal{S}}_n^V u\right\|_{L_2(\mathbb{R}^\infty, V;\gamma)} \leq \begin{cases} C_\varepsilon n^{-\alpha} (\log n)^{\alpha+1+\varepsilon} & \text{if } \alpha < 1/q_2, \\ C_\varepsilon n^{-\alpha} (\log n)^{\alpha+2+\varepsilon} & \text{if } \alpha = 1/q_2, \\ C n^{-\beta} (\log n)^{\beta+\alpha/(\alpha-\beta)} & \text{if } \alpha > 1/q_2. \end{cases}$$

for an arbitrarily small number $\varepsilon > 0$, *where the constants* C *and* C_ε *is independent of* u *and* n.

Proof. Under the hypothesis of this theorem, all the assumptions of Lemma 5.4 are satisfied. This yields that Assumption 3.1 holds for $u \in L_2(\mathbb{R}^\infty, V; \gamma)$. Hence, Theorem 3.2 is true for u. To complete the proof it is sufficient to notice that

$$\left|\operatorname{SamplePts}\left(\bar{\mathcal{S}}_n^V\right)\right| = \operatorname{Comp}\left(\bar{\mathcal{S}}_n^V\right) \leq n.$$

∎

Let the operators δ_k, $k \in \mathbb{N}_0$, be defined in (5.7) for the sequence $(P_k)_{k\in\mathbb{N}}$ given by (5.3). Then by the formulas (4.2) and (5.7) we can represent the operator $\mathcal{I}_\xi$ defined in (4.3), as

$$\mathcal{I}_\xi v(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=0}^{\lfloor \log_2 \xi \rfloor} \sum_{i_k=1}^{\bar{m}_k} \sum_{(\boldsymbol{s}_k, \boldsymbol{e}_k, \boldsymbol{j}_k) \in \Gamma_k(\xi)} (-1)^{|\boldsymbol{e}_k|_1} v(\bar{\boldsymbol{x}}_{k,i_k}, \boldsymbol{y}_{\boldsymbol{s}_k - \boldsymbol{e}_k; \boldsymbol{j}_k}) \bar{\phi}_{k,i_k}(\boldsymbol{x}) L_{\boldsymbol{s}_k - \boldsymbol{e}_k; \boldsymbol{j}_k}(\boldsymbol{y}), \tag{5.10}$$

where

$$\Gamma_k(\xi) := \{(\boldsymbol{s}_k, \boldsymbol{e}_k, \boldsymbol{j}_k) \in \mathbb{F} \times \mathbb{F} \times \mathbb{F} : \boldsymbol{s}_k \in \Lambda_k(\xi),\ \boldsymbol{e}_k \in E_{\boldsymbol{s}_k},\ \boldsymbol{j}_k \in \pi_{\boldsymbol{s}_k - \boldsymbol{e}_k}\}.$$

We rewrite $\mathcal{I}_\xi$ in (5.10) in the form of sampling algorithm in $L_2(\mathbb{R}^\infty, V; \gamma)$

$$\mathcal{I}_\xi v(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=0}^{\lfloor \log_2 \xi \rfloor} \sum_{i_k=1}^{\bar{m}_k} \sum_{(\boldsymbol{s}_k, \boldsymbol{e}_k, \boldsymbol{j}_k) \in \Gamma_k(\xi)} v(\bar{\boldsymbol{x}}_{k,i_k}, \boldsymbol{y}_{\boldsymbol{s}_k - \boldsymbol{e}_k; \boldsymbol{j}_k}) \Phi_{i_k, \boldsymbol{s}_k, \boldsymbol{e}_k, \boldsymbol{j}_k}(\boldsymbol{x}, \boldsymbol{y}),$$

where

$$\Phi_{i_k, \boldsymbol{s}_k, \boldsymbol{e}_k, \boldsymbol{j}_k}(\boldsymbol{x}, \boldsymbol{y}) := (-1)^{|\boldsymbol{e}_k|_1} \bar{\phi}_{k,i_k}(\boldsymbol{x}) L_{\boldsymbol{s}_k - \boldsymbol{e}_k; \boldsymbol{j}_k}(\boldsymbol{y}).$$

In a similar way to the proof of Theorem 5.2, from Theorem 4.1 and Lemma 5.4 we derive

Theorem 5.3. *Let Assumption 5.1 hold. Let the numbers* α *and* β *be defined by*

$$\alpha := \frac{r-1}{2}, \quad \beta := \left(\frac{1}{p_1} - \frac{3}{2}\right) \frac{\alpha}{\alpha + \delta}, \quad \delta := \frac{1}{p_1} - \frac{1}{p_2}. \tag{5.11}$$

Then for each $n \in \mathbb{N}$ *there exists a number* ξ_n *such that*

$$\left|\operatorname{SamplePts} \mathcal{I}_{\xi_n}\right| \leq n,$$

and for the parametric solution u to the equation (1.2) *with log-normal random inputs* (1.3)*,*

$$\left\|u - \mathcal{I}_{\xi_n} u\right\|_{L_2(\mathbb{R}^\infty, V;\gamma)} \ \le\ C \begin{cases} n^{-\alpha} & \text{if } \alpha \le 1/p_2 - 3/2, \\ n^{-\beta} & \text{if } \alpha > 1/p_2 - 3/2, \end{cases}$$

where the constant C is independent of u and n.

By combining Theorems 5.2 and 5.3 we obtain the following final results.

Theorem 5.4. *Let Assumption 5.1 hold. Let the numbers α and β be defined by* (5.11)*. Then for all $n \ge 2$ there exist $(\boldsymbol{x}_1, \boldsymbol{y}_1), ..., (\boldsymbol{x}_n, \boldsymbol{y}_n) \in D \times \mathbb{R}^\infty$ and $\varphi_1, ..., \varphi_n \in V$, $h_1, ..., h_n \in L_2(\mathbb{R}^\infty, \mathbb{R}; \gamma)$ such that for the linear sampling algorithm S_n on the spatial-parametric domain $D \times \mathbb{R}^\infty$ defined for functions v on $D \times \mathbb{R}^\infty$ by*

$$S_n(v)(\boldsymbol{x}, \boldsymbol{y}) := \sum_{i=1}^n v(\boldsymbol{x}_i, \boldsymbol{y}_i)\varphi_i(\boldsymbol{x}) h_i(\boldsymbol{y}), \quad \boldsymbol{x} \in D, \quad \boldsymbol{y} \in \mathbb{R}^\infty,$$

and for the parametric solution u to the equation (1.2) *with log-normal random inputs* (1.3)*, it holds the error bounds*

$$\left\|u - S_n u\right\|_{L_2(\mathbb{R}^\infty, V;\gamma)} \ \le\ \begin{cases} C n^{-\alpha} & \text{if } \alpha \le 1/p_2 - 3/2, \\ C_\varepsilon n^{-\alpha} (\log n)^{\alpha+1+\varepsilon} & \text{if } 1/p_2 - 3/2 < \alpha < 1/p_2 - 1, \\ C_\varepsilon n^{-\alpha} (\log n)^{\alpha+2+\varepsilon} & \text{if } \alpha = 1/p_2 - 1, \\ C n^{-\beta} (\log n)^{\beta + \alpha/(\alpha-\beta)} & \text{if } \alpha > 1/p_2 - 1, \end{cases}$$

for an arbitrarily small number $\varepsilon > 0$, where the constants C and C_ε are independent of u and n.

6 Extensions of least squares sampling algorithms

In this section, we discuss various least squares sampling algorithms for functions in the reproducing kernel Hilbert space $H_{\mathbb{C},\boldsymbol{\sigma}}$, and inequalities between sampling n-widths and Kolmogorov n-widths of the unit ball $B_{\mathbb{C},\boldsymbol{\sigma}}$ of this space. We explain then how to apply these inequalities to obtain corresponding convergence rates of multi-level linear sampling recovery in abstract Bochner spaces and of fully discrete multi-level collocation approximation of the parametric solution u to the parametric elliptic PDEs (1.1) on a bounded polygonal domain with log-normal inputs (1.3).

Recall that in Subsection 2.2, a notion of weighted least squares sampling algorithm $S_{cn}^{\mathbb{C}}$ in $L_2(U, \mathbb{C}; \mu)$ is introduced as in (2.13)–(2.15). Extensions of $S_{cn}^{\mathbb{C}}$ to $L_2(U, X; \mu)$ is defined as in (2.16). By the help of extended weighted least squares sampling algorithms with the special choice of sample points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{cn}$ and weights $\omega_1, \ldots, \omega_{cn}$, and the bounds of the approximation error as in Lemma 2.4, we constructed efficient multi-level least squares sampling algorithms in the Bochner space $L_2(U, X; \mu)$ for functions in H_X^α, and proved the convergence rates by them as in Theorem 2.2. The choice of sample points $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{cn}$, weights $\omega_1, \ldots, \omega_{cn}$, and approximation space Φ_m is crucial for the error of the least squares sampling algorithm. We recall three choices presented in [7], with a trade-off between constructiveness and tightness of the error bound. The third choice has been considered in Lemma 2.4.

Assumption 6.1. *Let $n \in \mathbb{N}$, $n \geq 90$, $c_1 \geq 1$, $c_2 > 1 + \frac{1}{n}$, and $c_3 \geq 3284$. Let the probability measure ν be defined as in* (2.17)*.*

(1) *Let $m := \lfloor n/(20 \log n) \rfloor$. Let further $\boldsymbol{y}_1, \dots \boldsymbol{y}_{c_1 n} \in U$ be points drawn i.i.d. with respect to ν and $\omega_i := (\varrho(\boldsymbol{y}_i))^{-1}$.*

(2) *Let $m := n$ and $\lceil 20n \log n \rceil$ points be drawn i.i.d. with respect to ν and subsampled using [8, Algorithm 3] to $c_2 n \asymp m$ points. Denote the resulting points by $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{c_2 n} \in U$ and $\omega_i = \frac{c_2 n}{\lceil 20n \log n \rceil} (\varrho(\boldsymbol{y}_i))^{-1}$.*

(3) *Let $m := n$ and $\lceil 20n \log n \rceil$ points be drawn i.i.d. with respect to ν. Let further $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{c_3 n} \in U$ be the subset of points fulfilling [23, Theorem 1] with $c_3 n \asymp m$ and $\omega_i := \frac{c_3 n}{\lceil 20n \log n \rceil} (\varrho(\boldsymbol{y}_i))^{-1}$.*

Lemma 6.1. *Let $0 < p < 2$. Let $d_n := d_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(U, \mathbb{C}; \mu))$. For $c, n, m \in \mathbb{N}$ with $cn \geq m$, let $S_{cn}^{\mathbb{C}}$ be the least squares sampling algorithm defined as in* (2.13)*–*(2.15)*. There are constants $c_1, c_2, c_3 \in \mathbb{N}$ depending on p such that for all $n \geq 2$ we have the following.*

(1) *The points from Assumption 6.1*(1) *fulfill with high probability*

$$\varrho_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(\mathbb{R}^\infty, \mathbb{C}; \mu)) \;\leq\; \sup_{v \in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\| v - S_{c_1 n}^X v \right\|_{L_2(U,\mathbb{C};\mu)} \;\leq\; \left(\frac{\log n}{n} \sum_{j \geq n/\log n} d_j^p \right)^{1/p}$$

(2) *The points from Assumption 6.1*(2) *fulfill with high probability*

$$\varrho_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(\mathbb{R}^\infty, \mathbb{C}; \mu)) \;\leq\; \sup_{v \in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\| v - S_{c_2 n}^X v \right\|_{L_2(U,\mathbb{C};\mu)} \;\leq\; \left(\frac{\log n}{n} \sum_{j \geq n} d_j^p \right)^{1/p}.$$

(3) *The points from Assumption 6.1*(3) *fulfill with high probability*

$$\varrho_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(\mathbb{R}^\infty, \mathbb{C}; \mu)) \;\leq\; \sup_{v \in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\| v - S_{c_2 n}^X v \right\|_{L_2(U,\mathbb{C};\mu)} \;\leq\; \left(\frac{1}{n} \sum_{j \geq n} d_j^p \right)^{1/p}.$$

This lemma has been formulated in [7]. As mentioned there, the claims (1) and (3) have been proven in [29, Theorem 8] and [23, Theorem 1], respectively. The claim (2) can be proven a similar way based on the result [8, Theorem 6.7].

As commented in [7], regarding the constructiveness of the linear least squares sampling algorithms in Lemma 6.1, the bound Lemma 6.1(1) is the most coarse bound, but the points construction requires only a random draw, which is computationally inexpensive. The sharper bound in Lemma 6.1(2) uses an additional constructive subsampling step. This was implemented and numerically tested in [8] for up to 1000 basis functions. For larger problem sizes the current algorithm is to slow as its runtime is cubic in the number of basis functions. The sharpest bound in

Lemma 6.1(3) is a pure existence result. So, up to now, the only way to obtain this point set is to brute-force every combination, which is computational infeasible.

As mentioned above, the linear least squares sampling algorithms in Lemma 2.4(3) are based on the sample points in Assumption 6.1(3) which give the best convergence rate among the sample points in Assumption 6.1(1)–(3), but are least constructive. Hence, the parametric components in the linear sampling algorithms in Theorems 2.1, 2.2, 3.2 and 5.2 are based on such points. The linear sampling algorithms in Lemma 2.4(1)–(2) based on the sample points in Assumption 6.1(1)–(2), respectively, are pure least squares algorithms or least squares algorithms with constructive subsampling, and therefore, constructive. But they give slightly worse error bounds.

Again, let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of increasing positive numbers whose values will be selected later, such that $\xi_n \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$ and $j = 1,2,3$, we consider the multilevel least squares operator $\mathcal{S}^X_{(j),n}$ defined by

$$\mathcal{S}^X_{(j),n} v \ = \ \sum_{k=0}^{k_n} S^X_{c_j n_{j,k}}(\delta_k v),$$

where

$$n_{j,k} := \lfloor c_j^{-1} n 2^{-k} \rfloor, \quad k_n := \lfloor \log \xi_n \rfloor,$$

and

$$S^X_{c_j n_{j,k}} v := \sum_{i=1}^{c_j n_k} v\left(\boldsymbol{y}^{(j)}_{k,i}\right) h^{(j)}_{k,i}$$

the extended least squares approximations with sample points $\boldsymbol{y}^{(j)}_{k,1},...,\boldsymbol{y}^{(j)}_{k,n_k} \in U$ as in Assumption 6.1(j), $h^{(j)}_{k,1},...,h^{(j)}_{k,n_k} \in L_2(\mathbb{R}^\infty,\mathbb{C};\mu)$, and $c_j > 0$ are the constants as in Lemma 6.1, respectively. We define the operators:

$$\bar{\mathcal{S}}^X_{(j),n} := \ \mathcal{S}^X_{(j),\lfloor n/\log n\rfloor}. \tag{6.1}$$

Theorem 2.2 gives the error bounds of approximation of $v \in B^{\alpha,\tau}_X$ by the operators $\bar{\mathcal{S}}^X_{(3),n} = \bar{\mathcal{S}}^X_n$ based on the parametric sample points in Assumption 6.1(3) and proven with the help of Lemma 6.1(3). Below we formulate its counterparts based on the parametric sample points in Assumption 6.1(1)–(2) and proven with the help of Lemma 6.1(1)–(2), respectively, which can be proven in a similar way with slight modifications.

Theorem 6.2. *Let the assumptions of Theorem 2.2 hold. Then for $j = 1,2$ and all $n \geq 2$, there exist $\boldsymbol{y}_{k,1},...,\boldsymbol{y}_{k,n_k} \in U$ and $h_{k,1},...,h_{k,n_k} \in L_2(\mathbb{R}^\infty,\mathbb{C};\mu)$, $k = 0,...,k_{\lceil n/\log n\rceil}$, such that for the operator $\bar{\mathcal{S}}^X_{(j),n}$ defined as in* (6.1),

$$\operatorname{Comp}\left(\bar{\mathcal{S}}^X_{(1),n}\right) \ \leq \ n,$$

and

$$\sup_{v\in B^\alpha_X} \|v - \bar{\mathcal{S}}^X_{(1),n} v\|_{L_2(U,X;\mu)} \ \leq \ \begin{cases} C_\varepsilon n^{-\alpha}(\log n)^{\alpha+1+1/q_2+\varepsilon} & \text{if } \alpha < 1/q_2, \\ C_\varepsilon n^{-\alpha}(\log n)^{\alpha+2+1/q_2+\varepsilon} & \text{if } \alpha = 1/q_2, \\ C n^{-\beta}(\log n)^{2\beta+\alpha/(\alpha-\beta)} & \text{if } \alpha > 1/q_2, \end{cases}$$

for an arbitrarily small number $\varepsilon > 0$.

Theorem 6.3. *Let the assumptions of Theorem 2.2 hold. Then for $j = 1, 2$ and all $n \geq 2$, there exist $\boldsymbol{y}_{k,1}, ..., \boldsymbol{y}_{k,n_k} \in U$ and $h_{k,1}, ..., h_{k,n_k} \in L_2(\mathbb{R}^\infty, \mathbb{C}; \mu)$, $k = 0, ..., k_{\lceil n/\log n \rceil}$, such that for the operator $\bar{\mathcal{S}}^X_{(2),n}$ defined as in* (6.1)*,*

$$\mathrm{Comp}\left(\bar{\mathcal{S}}^X_{(2),n}\right) \;\leq\; n,$$

and

$$\sup_{v \in B^\alpha_X} \|v - \bar{\mathcal{S}}^X_{(2),n} v\|_{L_2(U,X;\mu)} \;\leq\; C_\varepsilon \begin{cases} n^{-\alpha} (\log n)^{\alpha + 3/2 + \varepsilon} & \text{if } \alpha < 1/q_2, \\ n^{-\alpha} (\log n)^{\alpha + 5/2 + \varepsilon} & \text{if } \alpha = 1/q_2, \\ n^{-\beta} (\log n)^{3\beta/2 + \alpha/(\alpha - \beta) + \varepsilon} & \text{if } \alpha > 1/q_2, \end{cases}$$

for an arbitrarily small number $\varepsilon > 0$.

From Theorems 6.2 and 6.3 one can derive respective results similar to Theorem 5.2 and the others in Sections 3 and 5, based on the least squares sampling algorithms defined as in (2.13)–(2.15) with the choice of sample points and weights as in Assumption 6.1(1) or (2).

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