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ORIGINAL RESEARCH

Extremality of Families of Sets and Set-Valued Optimization

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Abstract

The paper continues our recent work (Cuong et al. in Optimization 73(12):3593–3607, 2024) where another extension of the *extremal principle* has been established. We demonstrate its applicability to set-valued optimization problems with general preferences, weakening the assumptions of the known results and streamlining their proofs.

Keywords Extremal principle · Separation · Optimality conditions · Set-valued optimization

Mathematics Subject Classification 49J52 · 49J53 · 49K40 · 90C30 · 90C46

1 Introduction

The paper continues our recent work [7] where a new extremality model involving collections of arbitrary families of sets has been studied and another extension of the *extremal principle* has been established.

We consider applications of the latter result to set-valued optimization problems of the type

$$\text{minimize}\ \ F(x) \quad \text{subject to}\ \ x \in \Omega, \tag{P}$$

Dedicated to R. Tyrrell Rockafellar, a trailblazer in optimization and analysis, on his 90th birthday

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where $F : X \rightrightarrows Y$ is a set-valued mapping between normed vector spaces, Ω is a subset of X, and the space Y is equipped with a general preference relation determined by an abstract *level-set mapping* $L : Y \rightrightarrows Y$, as well as more general problems with set-valued constraints.

The conventional special case of importance is when Y is equipped with a partial order determined by a nontrivial pointed convex cone, and optimality is understood in the sense of *Pareto*.

Definition 1.1 (Pareto optimality) Let X and Y be normed spaces, $F : X \rightrightarrows Y$, $\Omega \subset X$, $\bar{x} \in \Omega$ and $\bar{y} \in F(\bar{x})$. The point $(\bar{x}, \bar{y})$ is a (local) Pareto solution to (P) with respect to a nontrivial pointed convex cone $K \subset Y$ if there is a $\delta \in (0, +\infty]$ such that $F(\Omega \cap B_\delta(\bar{x})) \cap (\bar{y} - K) = \{\bar{y}\}$.

The conventional extremal principle [12, 14, 16] covers a wide range of problems in optimization and variational analysis as demonstrated, e.g., in the books [4, 14, 15]. The advantages of employing the extremal principle as the main tool when deducing necessary optimality conditions in vector and set-valued optimization problems compared to the scalarization and other traditional techniques were emphasized in [15, Sect. 5.3 and 5.5.18]. At the same time, there exist multiobjective problems with more general preference relations, which "may go far beyond generalized Pareto/weak Pareto concepts of optimality" [15, p. 70] that cannot be covered by traditional techniques or within the framework of the conventional extremal principle using linear translations. The first example of this kind was identified in Zhu [23]. Fortunately, such problems can be handled with the help of a more flexible extended version of the extremal principle using nonlinear perturbations (deformations) of the sets defined by set-valued mappings. Such an extension was developed in Mordukhovich et al. [17] (see also [4, 15]) and applied to various multiobjective problems [2, 13, 21]. Below is our interpretation of the corresponding definitions from [15, 17] complying with the notation and terminology adopted in the current paper.

Definition 1.2 (Extremality: set-valued perturbations) Let $\Omega_1, \ldots, \Omega_n$ be subsets of a normed space X, $\bar{x} \in \bigcap_{i=1}^n \Omega_i$, and, for each $i = 1, \ldots, n$, $S_i : M_i \rightrightarrows X$ be a set-valued mapping from a metric space (M_i, d_i) to X and $S_i(\bar{s}_i) = \Omega_i$ for some $\bar{s}_i \in M_i$. The collection $\{\Omega_1, \ldots, \Omega_n\}$ is extremal at $\bar{x}$ with respect to $\{S_1, \ldots, S_n\}$ if there exists a $\rho \in (0, +\infty]$ such that, for any $\varepsilon > 0$, there exist $s_i \in M_i$ $(i = 1, \ldots, n)$ such that $\max_{1 \le i \le n} d_i(s_i, \bar{s}_i) < \varepsilon$, $\max_{1 \le i \le n} d(\bar{x}, S_i(s_i)) < \varepsilon$ and $\bigcap_{i=1}^n S_i(s_i) \cap B_\rho(\bar{x}) = \emptyset$.

The model in Definition 1.2 exploits non-intersection of perturbations of given sets $\Omega_1, \ldots, \Omega_n$. The perturbations are chosen from the respective families of sets $\Xi_i := \{S_i(s) \mid s \in M_i\}$ $(i = 1, \ldots, n)$ determined by given set-valued mappings $S_i : M_i \rightrightarrows X$ $(i = 1, \ldots, n)$ on metric spaces. In the particular case of linear translations, i.e., when, for all $i = 1, \ldots, n$, $(M_i, d_i) = (X, d)$ and $S_i(a) = \Omega_i - a$ $(a \in X)$, the model reduces to the conventional extremal principle. It was shown by examples in [15, 17] that the framework of set-valued perturbations is richer than that of linear translations. With minor modifications in the proof, the conventional extremal principle was extended to the set-valued setting producing a more advanced model.

Definition 1.2 talks about extremality of a collection of sets, but in fact it is about certain properties of a collection of set-valued mappings $S_i : M_i \rightrightarrows X$ $(i = 1, \ldots, n)$, loosely connected with the given sets. This model has been refined in [7], making it more flexible and, at the same time, simpler. Instead of the set-valued mappings $S_1, \ldots, S_n$, the refined model studies extremality and stationarity of nonempty families $\Xi_1, \ldots, \Xi_n$ of arbitrary sets and is applicable to a wider range of variational problems. The next definition and theorem are simplified versions of [7, Definition 3.2 and Theorem 3.4], respectively.

Definition 1.3 (Extremality and stationarity: families of sets) Let $\Xi_1, \ldots, \Xi_n$ be families of subsets of a normed space X, and $\bar{x} \in X$. The collection $\{\Xi_1, \ldots, \Xi_n\}$ is

(i) extremal at $\bar{x}$ if there is a $\rho \in (0, +\infty]$ such that, for any $\varepsilon > 0$, there exist $A_i \in \Xi_i$ $(i = 1, \ldots, n)$ such that $\max_{1\le i\le n} d(\bar{x}, A_i) < \varepsilon$ and $\bigcap_{i=1}^n A_i \cap B_\rho(\bar{x}) = \emptyset$;
(ii) stationary at $\bar{x}$ if, for any $\varepsilon > 0$, there exist a $\rho \in (0, \varepsilon)$ and $A_i \in \Xi_i$ $(i = 1, \ldots, n)$ such that $\max_{1\le i\le n} d(\bar{x}, A_i) < \varepsilon\rho$ and $\bigcap_{i=1}^n A_i \cap B_\rho(\bar{x}) = \emptyset$;
(iii) approximately stationary at $\bar{x}$ if, for any $\varepsilon > 0$, there exist a $\rho \in (0, \varepsilon)$, $A_i \in \Xi_i$ and $x_i \in B_\varepsilon(\bar{x})$ $(i = 1, \ldots, n)$ such that $\max_{1\le i\le n} d(x_i, A_i) < \varepsilon\rho$ and $\bigcap_{i=1}^n (A_i - x_i) \cap (\rho\mathbb{B}) = \emptyset$.

Theorem 1.1 *Let $\Xi_1, \ldots, \Xi_n$ be families of closed subsets of a Banach space X, and $\bar{x} \in X$. If $\{\Xi_1, \ldots, \Xi_n\}$ is approximately stationary at $\bar{x}$, then, for any $\varepsilon > 0$, there exist $A_i \in \Xi_i$, $x_i \in A_i \cap B_\varepsilon(\bar{x})$, and $x_i^* \in N^C_{A_i}(x_i)$ $(i = 1, \ldots, n)$ such that*

$$\left\| \sum_{i=1}^n x_i^* \right\| < \varepsilon \quad \textit{and} \quad \sum_{i=1}^n \|x_i^*\| = 1.$$

If X is Asplund, then N^C in the above assertion can be replaced by N^F.

The symbols N^C and N^F in the above theorem denote, respectively, the Clarke and Fréchet normal cones. Recall that a Banach space is *Asplund* if every continuous convex function on an open convex set is Fréchet differentiable on a dense subset [18], or equivalently, if the dual of each its separable subspace is separable. We refer the reader to [14, 18] for discussions about and characterizations of Asplund spaces. All reflexive, particularly, all finite dimensional Banach spaces are Asplund. Most assertions involving Fréchet normals, subdifferentials and coderivatives are only valid in Asplund spaces; see [16].

Remark 1.1 (i) Part (ii) of Definition 1.3 is the explicit form of [7, Definition 3.2 (iii)], while part (iii) is a particular case of [7, Definition 3.2 (ii)] with $\Omega_1 = \cdots = \Omega_n := X$, thus, representing the weakest version of the property in [7, Definition 3.2 (ii)].
(ii) It is easy to see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Definition 1.3. Hence, the necessary conditions in Theorem 1.1 are also valid for the stationarity and extremality.
(iii) Theorem 1.1 shows that approximate stationarity of a given collection of families of closed sets implies its fuzzy (up to ε) separation. Note that, unlike the general model discussed in [5], not only the points x_i and x_i^* $(i = 1, \ldots, n)$ usually involved in fuzzy separation statements depend on ε, but also the sets $A_1, \ldots, A_n$.
(iv) For each $i = 1, \ldots, n$, the sets A_i making the family Ξ_i in Definition 1.3 can be considered as perturbations of some given set Ω_i. With this interpretation in mind, Definition 1.3 (i) covers Definition 1.2. Note that the "perturbation" sets in Definition 1.2 are rather loosely connected with the given sets.

Example 1.1 Let Ξ_1 consist of a single one-point set $\{0\} \subset \mathbb{R}$, and Ξ_2 be a family of singletons $\{1/n\}$ for $n \in \mathbb{N}$. It is easy to see that $\{\Xi_1, \Xi_2\}$ is extremal at 0 in the sense of Definition 1.3 (i) (even with $\rho = +\infty$). The subsets of Ξ_1, Ξ_2 may be considered as "perturbations" of the sets $\Omega_1 = \Omega_2 := \mathbb{R}$ in the sense of Definition 1.2. The pair $\{\Omega_1, \Omega_2\}$ is clearly not extremal at 0 in the conventional sense of [12, 14].

The more general (and simpler) model in Definition 1.3 and Theorem 1.1 is capable of treating a wider range of applications. In this paper, we demonstrate the applicability of Theorem 1.1 to set-valued optimization problems of the type (P) and more general ones. This allows us to expand the range of set-valued optimization models studied in earlier publications, weaken their assumptions and streamline the proofs.

We study extremality/stationarity properties of the triple $\{F, \Omega, \Xi\}$, where $F : X \rightrightarrows Y$ is a set-valued mapping between normed spaces, Ω is a subset of X, and Ξ is a nonempty family of subsets of Y. The latter family may, in particular be determined by an abstract *level-set mapping* defining a *preference* relation on Y.

Members of Ξ do not have to be simply translations (deformations) of a fixed set (ordering cone). Extremality/stationarity properties of the triple $\{F, \Omega, \Xi\}$ reduce to the corresponding properties of the two special families of subsets of $X \times Y$:

$$\Xi_1 := \{\operatorname{gph} F\} \quad \text{and} \quad \Xi_2 := \{\Omega \times A \mid A \in \Xi\}. \tag{1.1}$$

The first family consists of the single set gph F, and the first component of each member of the second family is always the given set Ω; only the second component varies.

The properties are illustrated by examples. Application of Theorem 1.1 yields necessary conditions for approximate stationarity and, hence, also stationarity and extremality. Natural *qualification conditions* in terms of Clarke or Fréchet coderivatives and normal cones are provided, which allow one to write down the necessary conditions in the form of an abstract *multiplier rule*. The statements cover the corresponding results in [17, 20, 21].

Requirements on preference relations defined by level-set mappings, making them meaningful in optimization and applications, are discussed. A certain subset of properties, which are satisfied by most conventional and many other preference relations, is established. The properties are shown to be in general weaker than those used in [2, 10, 15], but still sufficient for the corresponding set-valued optimization problems to fall within the theory developed in the current paper. Several multiplier rules for problems with a single set-valued mapping, and then with multiple set-valued mappings are formulated.

The structure of the paper is as follows. Sect. 2 recalls some definitions and facts used throughout the paper. The applicability of Theorem 1.1 is illustrated in Sects. 3–5 considering set-valued optimization problems with general preference relations. A model with a single set-valued mapping is studied in Sect. 3. A particular case of this model when the family Ξ is determined by an abstract level-set mapping is considered in Sect. 4. A more general model with multiple set-valued mappings is briefly discussed in Sect. 5.

2 Preliminaries

Our basic notation is standard, see, e.g., [8, 9, 14, 19]. Throughout the paper, if not explicitly stated otherwise, X and Y are normed spaces. Products of normed spaces are assumed to be equipped with the maximum norm. The topological dual of a normed space X is denoted by X^*, while $\langle \cdot, \cdot \rangle$ denotes the bilinear form defining the pairing between the two spaces. The open ball with center x and radius $\delta > 0$ is denoted by $B_\delta(x)$. If $(x, y) \in X \times Y$, we write $B_\varepsilon(x, y)$ instead of $B_\varepsilon((x, y))$. The open unit ball is denoted by $\mathbb{B}$ with a subscript indicating the space, e.g., $\mathbb{B}_X$ and $\mathbb{B}_{X^*}$. Symbols $\mathbb{R}$ and $\mathbb{N}$ stand for the real line and the set of all positive integers, respectively.

The interior and closure of a set Ω are denoted by $\operatorname{int}\Omega$ and $\operatorname{cl}\Omega$, respectively. The distance from a point $x \in X$ to a subset $\Omega \subset X$ is defined by $d(x, \Omega) := \inf_{u \in \Omega} \|u - x\|$, and we use the convention $d(x, \emptyset) = +\infty$.

Given a subset Ω of a normed space X and a point $\bar{x} \in \Omega$, the sets (cf. [6, 11])

$$N_\Omega^F(\bar{x}) := \left\{x^* \in X^* \mid \limsup_{\Omega \ni x \to \bar{x},\, x \neq \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0\right\}, \tag{2.1}$$

$$N_\Omega^C(\bar{x}) := \left\{x^* \in X^* \mid \left\langle x^*, z\right\rangle \leq 0 \quad \text{for all} \quad z \in T_\Omega^C(\bar{x})\right\} \tag{2.2}$$

are the *Fréchet* and *Clarke normal cones* to Ω at $\bar{x}$, where $T_\Omega^C(\bar{x})$ stands for the *Clarke tangent cone* to Ω at $\bar{x}$:

$$T_\Omega^C(\bar{x}) := \big\{z \in X \mid \forall x_k \to \bar{x},\ x_k \in \Omega,\ \forall t_k \downarrow 0,\ \exists z_k \to z \quad \text{such that} \quad x_k + t_k z_k \in \Omega \quad \text{for all} \quad k \in \mathbb{N}\big\}.$$

The sets (2.1) and (2.2) are nonempty closed convex cones satisfying $N_\Omega^F(\bar{x}) \subset N_\Omega^C(\bar{x})$. If Ω is a convex set, they reduce to the normal cone in the sense of convex analysis:

$$N_\Omega(\bar{x}) := \left\{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq 0 \quad \text{for all} \quad x \in \Omega\right\}.$$

By convention, we set $N_\Omega^F(\bar{x}) = N_\Omega^C(\bar{x}) := \emptyset$ if $\bar{x} \notin \Omega$.

A set-valued mapping $F : X \rightrightarrows Y$ between two sets X and Y is a mapping, which assigns to every $x \in X$ a (possibly empty) subset $F(x)$ of Y. We use the notations $\operatorname{gph} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$ and $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ for the graph and the domain of F, respectively, and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists with possibly empty values at some y) is defined by $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$, $y \in Y$. Obviously $\operatorname{dom} F^{-1} = F(X)$.

If X and Y are normed spaces, the *Clarke coderivative* $D^{*C}F(x, y)$ of F at $(x, y) \in \operatorname{gph} F$ is a set-valued mapping defined by

$$D^{*C}F(x, y)(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_{\operatorname{gph} F}^C(x, y)\}, \quad y^* \in Y^*. \tag{2.3}$$

Replacing the Clarke normal cone in (2.3) by the Fréchet one, we obtain the definition of the *Fréchet coderivative*.

Definition 2.1 (Aubin property) A mapping $F : X \rightrightarrows Y$ between metric spaces has the Aubin property at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with constant $\tau > 0$ if there exists a $\delta > 0$ such that

$$d(y, F(x)) \leq \tau d(x, x') \quad \text{for all} \quad x, x' \in B_\delta(\bar{x}),\ y \in F(x') \cap B_\delta(\bar{y}). \tag{2.4}$$

Aubin property (sometimes referred to as the locally *Lipschitz-like* property) is among the most widely used properties of set-valued mappings in variational analysis (see, e.g., [1, 8, 9, 14, 19]). It is known, in particular, to be equivalent to the *metric regularity* of the inverse mapping. It also yields estimates for the normals to the graph of the (given) mapping.

Lemma 2.1 *Let X and Y be normed spaces, $F : X \rightrightarrows Y$, and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.*

(i) *If F has the Aubin property at $(\bar{x}, \bar{y})$ with constant $\tau > 0$, then there is a $\delta > 0$ such that*

$$\|x^*\| \leq \tau \|y^*\| \quad \text{for all} \quad (x, y) \in \operatorname{gph} F \cap B_\delta(\bar{x}, \bar{y}),\ (x^*, y^*) \in N_{\operatorname{gph} F}^F(x, y). \tag{2.5}$$

(ii) *If $\dim Y < +\infty$, then N^F in the above assertion can be replaced by N^C.*

Proof (i) is well known; see, e.g., [14, Theorem 1.43(i)]. (The latter theorem is formulated in [14] in the Banach space setting, but the proof is valid in arbitrary normed spaces.)

(ii) Suppose $\dim Y < +\infty$, and F has the Aubin property at $(\bar{x}, \bar{y})$ with constant $\tau > 0$, i.e., condition (2.4) is satisfied for some $\delta > 0$. Let $(x, y) \in B_\delta(\bar{x}, \bar{y}) \cap \operatorname{gph} F$ and $(x^*, y^*) \in N^C_{\operatorname{gph} F}(x, y)$. Take any sequences $(x_k, y_k) \in \operatorname{gph} F$ and $t_k > 0$ such that $(x_k, y_k) \to (x, y)$ and $t_k \downarrow 0$ as $k \to +\infty$. Fix an arbitrary $u \in X$. Without loss of generality, we can assume that $x_k, x_k + t_k u \in B_\delta(\bar{x})$ and $y_k \in B_\delta(\bar{y})$ for all $k \in \mathbb{N}$. By (2.4), for each $k \in \mathbb{N}$, there exists a point $y_k' \in F(x_k + t_k u)$ such that $\|y_k' - y_k\| \le \tau t_k \|u\|$. Set $v_k := (y_k' - y_k)/t_k$. Then $\|v_k\| \le \tau \|u\|$. Passing to subsequences, we can suppose that $v_k \to v \in Y$. Observe that $(u, v_k) \to (u, v)$ as $k \to +\infty$, and $(x_k, y_k) + t_k(u, v_k) \in \operatorname{gph} F$ for each $k \in \mathbb{N}$. Thus, $(u, v) \in T^C_{\operatorname{gph} F}(x, y)$, and $\|v\| \le \tau \|u\|$. By the definition of the Clarke normal cone, we have $\langle x^*, u\rangle \le -\langle y^*, v\rangle \le \tau \|y^*\| \|u\|$. Since vector u is arbitrary, it follows that $\|x^*\| \le \tau \|y^*\|$. $\square$

3 Set-Valued Optimization: A Single Mapping

Let X and Y be normed spaces, $\Omega \subset X$, $F : X \rightrightarrows Y$, $\bar{x} \in \Omega$ and $\bar{y} \in F(\bar{x})$. To model the setting of Definition 1.3, we consider a nonempty family Ξ of subsets of Y, and two families of subsets of $X \times Y$ given by (1.1). To emphasize the structure of the pair (1.1), when referring to the corresponding properties in Definition 1.3, we will talk about extremality/stationarity of the triple $\{F, \Omega, \Xi\}$.

Definition 3.1 The triple $\{F, \Omega, \Xi\}$ is extremal (resp., stationary, approximately stationary) at $(\bar{x}, \bar{y})$ if the pair (1.1) is extremal (resp., stationary, approximately stationary) at $(\bar{x}, \bar{y})$.

The next proposition is a direct consequence of Definitions 1.3 and 3.1.

Proposition 3.1 *The triple* $\{F, \Omega, \Xi\}$ *is*

(i) *extremal at* $(\bar{x}, \bar{y})$ *if and only if there is a* $\rho \in (0, +\infty]$ *such that, for any* $\varepsilon > 0$, *there exists an* $A \in \Xi$ *such that* $d(\bar{y}, A) < \varepsilon$, *and*

$$F(\Omega \cap B_\rho(\bar{x})) \cap A \cap B_\rho(\bar{y}) = \emptyset; \tag{3.1}$$

(ii) *stationary at* $(\bar{x}, \bar{y})$ *if and only if for any* $\varepsilon > 0$, *there exist a* $\rho \in (0, \varepsilon)$ *and an* $A \in \Xi$ *such that* $d(\bar{y}, A) < \varepsilon\rho$, *and condition* (3.1) *is satisfied*;

(iii) *approximately stationary at* $(\bar{x}, \bar{y})$ *if and only if, for any* $\varepsilon > 0$, *there exist a* $\rho \in (0, \varepsilon)$, *an* $A \in \Xi$, *and* $(x_1, y_1), (x_2, y_2) \in B_\varepsilon(\bar{x}, \bar{y})$ *such that* $d((x_1, y_1), \operatorname{gph} F) < \varepsilon\rho$, $d(x_2, \Omega) < \varepsilon\rho$, $d(y_2, A) < \varepsilon\rho$, *and*

$$F(x_1 + (\Omega - x_2) \cap (\rho\mathbb{B}_X)) \cap (y_1 + (A - y_2) \cap (\rho\mathbb{B}_Y)) = \emptyset.$$

Remark 3.1 Definition 3.1 gives rather general concepts of extremality/stationarity. In the particular case when F is single-valued and $\Xi := \{K + F(\bar{x}) - y \mid y \in Y\}$ for some subset $K \subset Y$ containing 0, thanks to Proposition 3.1 (i), the extremality in the sense of Definition 3.1 means that there is a $\rho \in (0, +\infty]$ and a sequence $\{y_k\} \subset Y$ such that $d(y_k, K) \to 0$ as $k \to +\infty$, and

$$F(x) - F(\bar{x}) \notin (K - y_k) \cap (\rho\mathbb{B}) \ \text{ for all } \ x \in \Omega \cap B_\rho(\bar{x}) \ \text{ and } \ k \in \mathbb{N}.$$

Clearly, $d(y_k, K) \to 0$ if $y_k \to 0$, in which case the above condition becomes a constrained (on Ω) localized (in the image space) version of the (F, K)-optimality in [15, Definition 5.53]. As commented in [15, p. 70], when K is a convex cone, the latter property covers the conventional notion of local Pareto optimality as well as local weak Pareto optimality if $\operatorname{int} K \neq \emptyset$.

The next example illustrates relations between the properties in Definition 3.1.

Example 3.1 Let $X = Y = \Omega := \mathbb{R}$, $\Xi := \{(-\infty, t] \mid t \in \mathbb{R}\}$, and $F_1, F_2, F_3, F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$F_1(x) := [0, +\infty) \quad \text{for all } x \in \mathbb{R}, \qquad F_2(x) := \begin{cases} [x+1, +\infty) & \text{if } x < -1, \\ [0, +\infty) & \text{if } x \geq -1, \end{cases}$$

$$F_3(x) := [-x^2, +\infty) \text{ for all } x \in \mathbb{R}, \qquad F_4(x) := \begin{cases} [x, +\infty) & \text{if } x < 0, \\ [-x^2, +\infty) & \text{if } x \geq 0. \end{cases}$$

Then $0 \in F_i(0)$ for all $i = 1, 2, 3, 4$. The following assertions hold true:

(i) $\{F_1, \mathbb{R}, \Xi\}$ is extremal at $(0, 0)$ with $\rho = +\infty$;
(ii) $\{F_2, \mathbb{R}, \Xi\}$ is extremal at $(0, 0)$ with some $\rho \in (0, +\infty)$ but not with $\rho = +\infty$;
(iii) $\{F_3, \mathbb{R}, \Xi\}$ is stationary but not extremal at $(0, 0)$;
(iv) $\{F_4, \mathbb{R}, \Xi\}$ is approximately stationary at $(0, 0)$ but not stationary at $(0, 0)$.

The assertions are straightforward. We only prove assertion (iv). Let $\varepsilon \in (0, 1)$. Choose any $\rho \in (0, \varepsilon)$ and $t \in (\varepsilon\rho, \rho)$. Then $-t \in \rho\mathbb{B}_{\mathbb{R}}$, $-t \in F_4(-t)$ and $-t \in A$ for any $A := (-\infty, -\eta] \in \Xi$ with $d(0, A) < \varepsilon\rho$ (i.e., for any $\eta < \varepsilon\rho$). By Proposition 3.1 (ii), $\{F_4, \mathbb{R}, \Xi\}$ is not stationary at $(0, 0)$.

Let $\varepsilon > 0$. Choose a $\rho \in (0, \min\{\varepsilon, 1\}/3)$ and points $(x_1, y_1) := (\rho, -\rho^2) \in \operatorname{gph} F_4 \cap (\varepsilon\mathbb{B}_{\mathbb{R}^2})$, $x_2 := 0 \in \varepsilon\mathbb{B}_{\mathbb{R}}$, $y_2 := 0 \in \varepsilon\mathbb{B}_{\mathbb{R}}$. Observe that $A := (-\infty, -3\rho^2] \in \Xi$ satisfies $d(0, A) < \varepsilon\rho$. Then

$$F_4(x_1 + (\rho\mathbb{B}_{\mathbb{R}})) = F_4(0, 2\rho) = (-4\rho^2, +\infty),$$

$$y_1 + (A - y_2) \cap (\rho\mathbb{B}_{\mathbb{R}}) = -\rho^2 + (-\rho, -3\rho^2] = (-\rho^2 - \rho, -4\rho^2],$$

and consequently, $F_4(x_1 + (\rho\mathbb{B}_{\mathbb{R}})) \cap (y_1 + (A - y_2) \cap (\rho\mathbb{B}_{\mathbb{R}})) = \emptyset$. By Proposition 3.1 (iii), $\{F_4, \mathbb{R}, \Xi\}$ is approximately stationary at $(0, 0)$.

The following example shows that the family of sets Ξ plays an important role in determining the properties.

Example 3.2 Let $X = Y = \Omega := \mathbb{R}$ and $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by

$$F(x) := \begin{cases} \{x\sqrt{2}\} & \text{if } x \text{ is rational}, \\ \{x\} & \text{otherwise}. \end{cases}$$

Then $0 \in F(0)$.

Let $\Xi := \{(-\infty, t] \mid t \in \mathbb{R}\}$ and $\varepsilon \in (0, 1)$. Choose any $\rho \in (0, \varepsilon)$, $(x_1, y_1) \in \operatorname{gph} F$, $x_2 \in \mathbb{R}$, $y_2 \in (-\varepsilon, 0]$ and $A := (-\infty, -t] \in \Xi$ with $d(y_2, A) < \varepsilon\rho$ (i.e., $t < \varepsilon\rho - y_2$). Then

$x_1 + (\Omega - x_2) \cap (\rho\mathbb{B}_X) = B_\rho(x_1)$, $A - y_2 = (-\infty, -\tau]$, where $\tau := y_2 + t < \varepsilon\rho$, and consequently, $y_1 + (A - y_2) \cap (\rho\mathbb{B}_Y) \supset (y_1 - \rho, y_1 - \varepsilon\rho)$. We next show that $F(B_\rho(x_1)) \cap (y_1 - \rho, y_1 - \varepsilon\rho) \neq \emptyset$. If x_1 is rational, then $y_1 = x_1\sqrt{2}$, and choosing a rational number $\hat{x} \in (x_1 - \rho/\sqrt{2}, x_1 - \varepsilon\rho/\sqrt{2}) \subset B_\rho(x_1)$, we get $\hat{y} := \hat{x}\sqrt{2} \in F(\hat{x}) \cap (y_1 - \rho, y_1 - \varepsilon\rho)$. If x_1 is irrational, then $y_1 = x_1$, and choosing an irrational number $\hat{x} \in (x_1 - \rho, x_1 - \varepsilon\rho) \subset B_\rho(x_1)$, we get $\hat{y} := \hat{x} \in F(\hat{x}) \cap (y_1 - \rho, y_1 - \varepsilon\rho)$. By Proposition 3.1 (iii), $\{F, \mathbb{R}, \Xi\}$ is not approximately stationary at $(0, 0)$.

Let $\Xi := \{\{-1/n\} \mid n \in \mathbb{N}\}$. Since $F(\mathbb{R})$ only contains irrational numbers, we have $F(\mathbb{R}) \cap A = \emptyset$ for all $A \in \Xi$. By Proposition 3.1 (i), $\{F, \mathbb{R}, \Xi\}$ is extremal at $(0, 0)$ (with $\rho = +\infty$).

Application of Theorem 1.1 yields necessary conditions for approximate stationarity and, hence, also stationarity and extremality.

Theorem 3.1 *Let X and Y be Banach spaces, the sets Ω, $\operatorname{gph} F$ and all members of Ξ be closed. If the triple $\{F, \Omega, \Xi\}$ is approximately stationary at $(\bar{x}, \bar{y})$, then, for any $\varepsilon > 0$, there exist $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $A \in \Xi$, $y_2 \in A \cap B_\varepsilon(\bar{y})$, $(x_1^*, y_1^*) \in N^C_{\operatorname{gph} F}(x_1, y_1)$, $x_2^* \in N^C_\Omega(x_2)$ and $y_2^* \in N^C_A(y_2)$ such that*

$$\|(x_1^*, y_1^*) + (x_2^*, y_2^*)\| < \varepsilon \quad \text{and} \quad \|(x_1^*, y_1^*)\| + \|(x_2^*, y_2^*)\| = 1.$$

If X and Y are Asplund, then N^C in the above assertion can be replaced by N^F.

The normalization condition $\|(x_1^*, y_1^*)\| + \|(x_2^*, y_2^*)\| = 1$ in Theorem 3.1 ensures that normal vectors (x_1^*, y_1^*) to $\operatorname{gph} F$ remain sufficiently large when $\varepsilon \downarrow 0$, i.e., x_1^* and y_1^* cannot go to 0 simultaneously. The case when vectors y_1^* are bounded away from 0 (hence, one can assume $\|y_1^*\| = 1$) is of special interest as it leads to a proper *multiplier rule*. A closer look at the alternative: either y_1^* are bounded away from 0 as $\varepsilon \downarrow 0$, or they are not (hence, vectors x_1^* remain large), allows one to formulate the following consequence of Theorem 3.1.

Corollary 3.1 *Let X and Y be Banach spaces, the sets Ω, $\operatorname{gph} F$ and all members of Ξ be closed. If the triple $\{F, \Omega, \Xi\}$ is approximately stationary at $(\bar{x}, \bar{y})$, then one of the following assertions holds true:*

(i) *there is an $M > 0$ such that, for any $\varepsilon > 0$, there exist $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $A \in \Xi$, $y_2 \in A \cap B_\varepsilon(\bar{y})$, and $y^* \in N^C_A(y_2) + \varepsilon\mathbb{B}_{Y^*}$ such that $\|y^*\| = 1$ and*

$$0 \in D^{*C} F(x_1, y_1)(y^*) + N^C_\Omega(x_2) \cap (M\mathbb{B}_{X^*}) + \varepsilon\mathbb{B}_{X^*}; \tag{3.2}$$

(ii) *for any $\varepsilon > 0$, there exist $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $x_1^* \in D^{*C} F(x_1, y_1)(\varepsilon\mathbb{B}_{Y^*})$ and $x_2^* \in N^C_\Omega(x_2)$ such that $\|x_1^* + x_2^*\| < \varepsilon$ and $\|x_1^*\| + \|x_2^*\| = 1$.*

If X and Y are Asplund, then N^C and D^{*C} in the above assertions can be replaced by N^F and D^{*F}, respectively.

Proof Let the triple $\{F, \Omega, \Xi\}$ be approximately stationary at $(\bar{x}, \bar{y})$. By Theorem 3.1, for any $j \in \mathbb{N}$, there exist $(x_{1j}, y_{1j}) \in \operatorname{gph} F \cap B_{1/j}(\bar{x}, \bar{y})$, $x_{2j} \in \Omega \cap B_{1/j}(\bar{x})$, $A_j \in \Xi$, $y_{2j} \in A_j \cap B_{1/j}(\bar{y})$, $(x_{1j}^*, y_{1j}^*) \in N^C_{\operatorname{gph} F}(x_{1j}, y_{1j})$, $x_{2j}^* \in N^C_\Omega(x_{2j})$ and $y_{2j}^* \in N^C_{A_j}(y_{2j})$ such that $\|(x_{1j}^*, y_{1j}^*)\| + \|(x_{2j}^*, y_{2j}^*)\| = 1$ and $\|(x_{1j}^*, y_{1j}^*) + (x_{2j}^*, y_{2j}^*)\| < 1/j$. We consider two cases.

Case 1. $\limsup_{j\to+\infty}\|y^*_{1j}\| > \alpha > 0$. Note that $\alpha < 1$. Set $M := 1/\alpha$. Let $\varepsilon > 0$. Choose a number $j \in \mathbb{N}$ so that $j^{-1} < \alpha\varepsilon$ and $\|y^*_{1j}\| > \alpha$. Set $x_1 := x_{1j}$, $y_1 := y_{1j}$, $x_2 := x_{2j}$, $A := A_j$, $y_2 := y_{2j}$, $y^* := -y^*_{1j}/\|y^*_{1j}\|$, $x^*_1 := x^*_{1j}/\|y^*_{1j}\|$, $x^*_2 := x^*_{2j}/\|y^*_{1j}\|$ and $y^*_2 := y^*_{2j}/\|y^*_{1j}\|$. Then $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $y_2 \in A \cap B_\varepsilon(\bar{y})$, $x^*_1 \in D^{*C}F(x_1, y_1)(y^*)$, $x^*_2 \in N^C_\Omega(x_2)$, $\|y^*\| = 1$, $y^*_2 \in N^C_A(y_2)$, $\|x^*_2\| < 1/\alpha = M$. Furthermore, $\|y^* - y^*_2\| = \|y^*_{1j} + y^*_{2j}\|/\|y^*_{1j}\| < 1/(\alpha j) < \varepsilon$, hence, $y^* \in N^C_A(y_2) + \varepsilon\mathbb{B}_{Y^*}$; and $\|x^*_1 + x^*_2\| = \|x^*_{1j} + x^*_{2j}\|/\|y^*_{1j}\| < 1/(\alpha j) < \varepsilon$, hence, condition (3.2) is satisfied. Thus, assertion (i) holds true.

Case 2. $\lim_{j\to+\infty}\|y^*_{1j}\| = 0$. Then $y^*_{2j} \to 0$, $x^*_{1j} + x^*_{2j} \to 0$ and $1 \ge \|x^*_{1j}\| + \|x^*_{2j}\| \to 1$ as $j \to +\infty$. Let $\varepsilon > 0$. Choose a number $j \in \mathbb{N}$ so that $\|x^*_{1j}\| + \|x^*_{2j}\| > 0$ and $\max\{j^{-1}, \|y^*_{1j}\|, \|x^*_{1j} + x^*_{2j}\|/(\|x^*_{1j}\| + \|x^*_{2j}\|)\} < \varepsilon$. Set $x_1 := x_{1j}$, $y_1 := y_{1j}$, $x_2 := x_{2j}$, $x^*_1 := x^*_{1j}/(\|x^*_{1j}\| + \|x^*_{2j}\|)$ and $x^*_2 := x^*_{2j}/(\|x^*_{1j}\| + \|x^*_{2j}\|)$. Then $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $x^*_1 \in D^{*C}F(x_1, y_1)(\varepsilon\mathbb{B}_{Y^*})$, $x^*_2 \in N^C_\Omega(x_2)$, $\|x^*_1\| + \|x^*_2\| = 1$, and $\|x^*_1 + x^*_2\| < \varepsilon$. Thus, assertion (ii) holds true.

If X and Y are Asplund, then N^C and D^{*C} in the above arguments can be replaced by N^F and D^{*F}, respectively. □

Remark 3.2 (i) Part (i) of Corollary 3.1 gives a kind of fuzzy multiplier rule with y^* playing the role of the vector of multipliers. If F is single-valued and Lipschitz continuous around $\bar{x}$, then $D^{*F}(x_1, F(x_1))(y^*) = \partial^F\langle y^*, F\rangle(x_1)$ for all $y^* \in Y^*$ and all x_1 sufficiently close to $\bar{x}$ (see, e.g., [14, Theorem 1.90]). If, additionally, $\Omega = X$, the Asplund space version of condition (3.2) becomes $0 \in \partial^F\langle y^*, F\rangle(x_1) + \varepsilon\mathbb{B}_{X^*}$.

(ii) Part (ii) corresponds to 'singular' behaviour of F on Ω. It involves 'horizontal' normals to the graph of F; the y^* component vanishes, and consequently, Ξ plays no role.

The following condition is the negation of the condition in Corollary 3.1 (ii).

$(QC)_C$ there is an $\varepsilon > 0$ such that $\|x^*_1 + x^*_2\| \ge \varepsilon$ for all $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $x^*_1 \in D^{*C}F(x_1, y_1)(\varepsilon\mathbb{B}_{Y^*})$ and $x^*_2 \in N^C_\Omega(x_2))$ such that $\|x^*_1\| + \|x^*_2\| = 1$.

It excludes the singular behavior mentioned in Remark 3.2 (ii) and serves as a qualification condition ensuring that only the condition in part (i) of Corollary 3.1 is possible. We denote by $(QC)_F$ the analogue of $(QC)_C$ with N^F and D^{*F} in place of N^C and D^{*C}, respectively.

Corollary 3.2 *Let X and Y be Banach spaces, Ω, $\operatorname{gph} F$ and all members of Ξ be closed. Suppose that the triple $\{F, \Omega, \Xi\}$ is approximately stationary at $(\bar{x}, \bar{y})$. If condition $(QC)_C$ is satisfied, then assertion* (i) *in Corollary* 3.1 *holds true.*

If X and Y are Asplund and condition $(QC)_F$ is satisfied, then assertion (i) *in Corollary* 3.1 *holds true with N^F and D^{*F} in place of N^C and D^{*C}, respectively.*

The next proposition provides two typical sufficient conditions for the fulfillment of conditions $(QC)_C$ and $(QC)_F$.

Proposition 3.2 *Let X and Y be normed spaces.*

(i) *If F has the Aubin property at $(\bar{x}, \bar{y})$, then $(QC)_F$ is satisfied. If, additionally,* $\dim Y < +\infty$, *then $(QC)_C$ is satisfied too.*
(ii) *If $\bar{x} \in \operatorname{int}\Omega$, then both $(QC)_C$ and $(QC)_F$ are satisfied.*

Proof (i) If F has the Aubin property at $(\bar{x}, \bar{y})$, then, by Lemma 2.1, condition (2.5) is satisfied with some $\tau > 0$ and $\delta > 0$, and, if $\dim Y < +\infty$, then the latter condition is also satisfied with N^C in place of N^F. Hence, $(QC)_F$ is satisfied with $\varepsilon := 1/(2\tau + 1)$, as well as $(QC)_C$ if $\dim Y < +\infty$. Indeed, if $(x_1, y_1) \in \operatorname{gph} F \cap B_\delta(\bar{x}, \bar{y})$ and $x_1^* \in D^*F(x_1, y_1)(y_1^*)$ (where D^* stands for either D^{*C} or D^{*F}), $x_2^* \in X^*$, $\|x_1^*\| + \|x_2^*\| = 1$ and $\|y_1^*\| < \varepsilon$, then $\|x_1^* + x_2^*\| \geq \|x_2^*\| - \|x_1^*\| = 1 - 2\|x_1^*\| > 1 - 2\tau\varepsilon = \varepsilon$.

(ii) If $\bar{x} \in \operatorname{int} \Omega$, then $N_\Omega^C(x_2) = N_\Omega^F(x_2) = \{0\}$ for all x_2 near $\bar{x}$, and consequently, for any normal vector x_2^* to Ω at x_2 and any $x_1^* \in X^*$, condition $\|x_1^*\| + \|x_2^*\| = 1$ yields $\|x_1^* + x_2^*\| = 1$. Hence, both $(QC)_C$ and $(QC)_F$ are satisfied with any sufficiently small ε. □

As a consequence of Corollary 3.1, we obtain dual necessary conditions for the (local) Pareto optimality covering [20, Theorems 3.1 and 4.1], [21, Corollary 3.1] and [17, Proposition 5.1].

Corollary 3.3 *Let X and Y be Banach spaces, Ω and $\operatorname{gph} F$ be closed. If $(\bar{x}, \bar{y})$ is a (local) Pareto solution to (P) with respect to a nontrivial pointed closed convex cone $K \subset Y$, then either there is an $M > 0$ such that, for any $\varepsilon > 0$, there exist $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, and $y^* \in Y^*$ such that $\langle y^*, y\rangle \geq 0$ for all $y \in K$, $\|y^*\| = 1$, and*

$$0 \in D^{*C} F(x_1, y_1)(y^* + \varepsilon\mathbb{B}_{Y^*}) + N_\Omega^C(x_2) \cap (M\mathbb{B}_{X^*}) + \varepsilon\mathbb{B}_{X^*},$$

or assertion (ii) in Corollary 3.1 holds true.

If X and Y are Asplund, then N^C and D^{*C} in the above assertion can be replaced by N^F and D^{*F}, respectively.

Proof (Sketch) Set $\Xi := \{K + \bar{y} - y \mid y \in Y\}$ and observe that the conditions in Definition 1.1 ensure that the triple $\{F, \Omega, \Xi\}$ is approximately stationary at $(\bar{x}, \bar{y})$. Deducing the conclusion from Corollary 3.1 requires straightforward renorming of the involved dual vectors. □

Remark 3.3 In view of Proposition 3.2 (ii), if $\bar{x} \in \operatorname{int} \Omega$, then only the first alternative in Corollary 3.3 is possible; cf. [20, Theorems 3.1 and 4.1].

The next example illustrates the verification of the necessary conditions for approximate stationarity in Corollaries 3.1 and 3.2 for the triple $\{F_4, \Omega, \Xi\}$, where F_4 is defined in Example 3.1.

Example 3.3 Let $X = Y = \Omega := \mathbb{R}$, and F_4 and Ξ be as in Example 3.1. Thus, the triple $\{F_4, \Omega, \Xi\}$ is approximately stationary at $(\bar{x}, \bar{y})$, and the conclusions of Corollary 3.1 must hold true. Moreover, the assumptions in both parts of Proposition 3.2 are satisfied, and consequently, condition $(QC)_F$ holds true. By Corollary 3.2, assertion (i) in Corollary 3.1 holds true with N^F and D^{*F} in place of N^C and D^{*C}, respectively. We now verify this assertion.

Let $M > 0$ and $\varepsilon > 0$. Choose a $t \in (0, \min\{\varepsilon, 1\})$. Set $(x_1, y_1) := (t/2, -t^2/4) \in \operatorname{gph} F_4 \cap (\varepsilon\mathbb{B}_{\mathbb{R}^2})$, $x_2 := 0 \in \Omega \cap (\varepsilon\mathbb{B}_\mathbb{R})$, $A := (-\infty, -t] \in \Xi$, $y_2 := -t \in A \cap (\varepsilon\mathbb{B}_\mathbb{R})$ and $y^* := 1$. Thus, $N_\Omega^F(x_2) = \{0\}$, $y^* \in N_A^F(y_2)$ and $D^{*F} F_4(x_1, y_1)(y^*) = \{-t\}$. Hence,

$$D^{*F} F_4(x_1, y_1)(y^*) + N_\Omega^F(x_2) \cap (M\mathbb{B}_\mathbb{R}) = \{-t\} \in \varepsilon\mathbb{B}_\mathbb{R},$$

i.e., assertion (i) in Corollary 3.1 holds true (in terms of Fréchet normals and coderivatives).

4 Abstract Level-Set Mapping

We now consider a particular case of the model in Sect. 3 when the family Ξ is determined by an abstract *level-set mapping* $L:Y\rightrightarrows Y$. The latter mapping defines a *preference* relation $\prec$ on Y: $v\prec y$ if and only if $v\in L(y)$; see, e.g., [10, p. 67].

Given a point $y\in Y$, we employ below the following notations:

$$L^\circ(y):=L(y)\setminus\{y\},\quad L^-(y):=L(y)\cup\{y\}. \tag{4.1}$$

Certain requirements are usually imposed on L in order to make the corresponding preference relation meaningful in optimization and applications; see, e.g., [10, 15, 17, 23]. In this section, we discuss the following properties of L at or near the reference point $\bar y$:

(O1) $\liminf\limits_{L^\circ(\bar y)\ni y\to\bar y} d(\bar y,L(y))=0$;
(O2) $\bar y\in \operatorname{cl} L^\circ(\bar y)$;
(O3) $\bar y\notin L(\bar y)$;
(O4) $y\in\operatorname{cl} L(y)$ for all y near $\bar y$;
(O5) if $y\in L^\circ(\bar y)$ and $v\in\operatorname{cl} L(y)$, then $v\in L^\circ(\bar y)$;
(O6) if $y\in L(\bar y)$ and $v\in\operatorname{cl} L(y)$, then $v\in L(\bar y)$.

Some characterizations of the properties and relations between them are collected in the next proposition.

Proposition 4.1 *Let $L:Y\rightrightarrows Y$, L° be given by* (4.1)*, and $\bar y\in Y$. The following assertions hold true.*

(i) (O1) $\Leftrightarrow$ $\{y\in Y\mid d(\bar y,L(y))<\varepsilon\}\cap L^\circ(\bar y)\cap B_\varepsilon(\bar y)\ne\emptyset$ *for all* $\varepsilon>0$.
(ii) (O1) $\Rightarrow$ (O2).
(iii) (O3) $\Leftrightarrow$ $[L(\bar y)=L^\circ(\bar y)]$.
(iv) (O2) & (O4) $\Rightarrow$ (O1).
(v) (O3) & (O4) $\Rightarrow$ (O2).
(vi) (O3) $\Rightarrow$ [(O5) $\Leftrightarrow$ (O6)].

Proof (i) (O1) $\Leftrightarrow$ $\inf_{y\in L^\circ(\bar y)\cap B_\varepsilon(\bar y)} d(\bar y,L(y))=0$ for any $\varepsilon>0$ $\Rightarrow$ for any $\varepsilon>0$, there is a $y\in L^\circ(\bar y)\cap B_\varepsilon(\bar y)$ such that $d(\bar y,L(y))<\varepsilon$. This proves the '$\Rightarrow$' implication. Conversely, let $\delta:=\inf_{y\in L^\circ(\bar y)\cap B_\varepsilon(\bar y)} d(\bar y,L(y))>0$ for some $\varepsilon>0$. Then $\{y\in Y\mid d(\bar y,L(y))<\delta\}\cap L^\circ(\bar y)\cap B_\varepsilon(\bar y)=\emptyset$, and consequently, $\{y\in Y\mid d(\bar y,L(y))<\varepsilon'\}\cap L^\circ(\bar y)\cap B_{\varepsilon'}(\bar y)=\emptyset$, where $\varepsilon':=\min\{\varepsilon,\delta\}$. The implication '$\Leftarrow$' follows.
(ii) (O1) $\Rightarrow$ there exists a sequence $\{y_k\}\subset L^\circ(\bar y)$ with $y_k\to\bar y$ $\Leftrightarrow$ (O2).
(iii) The assertion is a consequence of the definition of L° in (4.1).
(iv) Suppose conditions (O2) and (O4) are satisfied. Let $\varepsilon>0$. Thanks to (O4), we can choose a $\xi\in(0,\varepsilon)$ such that $y\in\operatorname{cl} L(y)$ for all $y\in B_\xi(\bar y)$. If $y\in B_\xi(\bar y)$, then $d(\bar y,L(y))=d(\bar y,\operatorname{cl} L(y))\le\|y-\bar y\|<\xi$. Thus, $B_\xi(\bar y)\subset\{y\in Y\mid d(\bar y,L(y))<\xi\}$. Thanks to (O2), we have

$$\{y\in Y\mid d(\bar y,L(y))<\varepsilon\}\cap L^\circ(\bar y)\cap B_\varepsilon(\bar y)\supset L^\circ(\bar y)\cap B_\xi(\bar y)\ne\emptyset.$$

Since ε is an arbitrary positive number, in view of (i), this proves (O1).
(v) (O4) $\Rightarrow$ $\bar y\in\operatorname{cl} L(\bar y)$. The conclusion follows thanks to (iii).
(vi) The assertion is a consequence of (iii). □

Remark 4.1 Properties (O4) and (O6) are components of the definition of *closed preference* relation (see [15, Definition 5.55], [2, p. 583], [10, p. 68]) widely used in vector and set-valued optimization. They are called, respectively, *local satiation* (around $\bar{y}$) and *almost transitivity*. Note that the latter property is actually stronger than the conventional transitivity. It is not satisfied for the preference defined by the *lexicographical order* (see [15, Example 5.57]) and some other natural preference relations important in vector optimization and its applications including those to welfare economics (see [10, Sect. 15.3]). Closed preference relations are additionally assumed in [2, 10, 15] to be *nonreflexive*, thus, satisfying, in particular, property (O3). In view of Proposition 4.1, if a preference relation satisfies properties (O3), (O4) and (O6), it also satisfies properties (O1), (O2) and (O5). In this section, we employ the weaker properties (O1) and (O5), which are satisfied by most conventional and many other preference relations. This makes our model applicable to a wider range of multiobjective and set-valued optimization problems compared to those studied in [2, 10, 15].

The next proposition addresses some reasonably conventional settings.

Proposition 4.2 *Let $L(y) := y - K$ for some $K \subset Y$ and all $y \in Y$. Let $\bar{y} \in Y$. Denote $K^\circ := K \setminus \{0\}$, $K^- := K \cup \{0\}$. Suppose that $0 \in \operatorname{cl} K^\circ$. Then*

(i) *$L^\circ(y) = y - K^\circ$ and $L^-(y) = y - K^-$ for all $y \in Y$;*
(ii) *properties (O1), (O2) and (O4) are satisfied;*
(iii) *if $0 \notin K$, then property (O3) is satisfied;*
(iv) *if K is an open convex cone and $K \neq Y$, then $K^\circ = K$ and properties (O5) and (O6) are satisfied;*
(v) *if K is a closed convex cone, then $K^- = K$ and property (O6) is satisfied.*

Proof (i) is obvious.

(ii) Let $y_k \in L^\circ(\bar{y})$ ($k \in \mathbb{N}$) and $y_k \to \bar{y}$ as $k \to +\infty$. By (i), $c_k := \bar{y} - y_k \in K$ ($k \in \mathbb{N}$) and $c_k \to 0$ as $k \to +\infty$. Then $v_k := y_k - c_k \in L(y_k)$ ($k \in \mathbb{N}$) and $v_k \to \bar{y}$ as $k \to +\infty$. This proves (O1). Since $0 \in \operatorname{cl} K^\circ$, it follows from (i) that $\bar{y} \in \operatorname{cl} L^\circ(\bar{y})$. This proves (O2). By the assumption, $0 \in \operatorname{cl} K$, and consequently, $y \in y - \operatorname{cl} K = \operatorname{cl} L(y)$ for all $y \in Y$. Property (O4) follows.

(iii) is a consequence of (i).

(iv) Let K be an open convex cone and $K \neq Y$. Then $0 \notin K$, and consequently, $K^\circ = K$. If $y \in L^\circ(\bar{y})$ and $v \in \operatorname{cl} L(y)$, then $\bar{y} - y \in K$ and $y - v \in \operatorname{cl} K$; hence, $\bar{y} - v \in K + \operatorname{cl} K = K = K^\circ$, i.e., $v \in L^\circ(\bar{y}) = L(\bar{y})$.

(v) Let K be a closed convex cone. Then $0 \in K$, and consequently, $K^- = K$. If $y \in L(\bar{y})$ and $v \in \operatorname{cl} L(y)$, then $\bar{y} - y \in K$ and $y - v \in K$; hence, $\bar{y} - v \in K + K = K$, i.e., $v \in L(\bar{y})$. $\square$

Corollary 4.1 *Let K be a nontrivial open convex cone, and $L(y) := y - K$ for all $y \in Y$. Let $\bar{y} \in Y$. Then properties (O1)–(O6) are satisfied.*

The next two examples illustrate some characterizations of the level-set mapping.

Example 4.1 Let $L(y) := \{y\}$ for all $y \in Y$. Then $L^\circ(y) = \emptyset$. Thus, properties (O4) and (O5) are obviously satisfied, while properties (O1) and (O2) are violated.

Example 4.2 Let $L : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined by

$$L(y_1, y_2) := \begin{cases} \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1 < y_1,\ v_2 < y_2\} & \text{if } (y_1, y_2) \neq (0,0), \\ \{(0,0)\} & \text{otherwise.} \end{cases}$$

Let $\bar{y} := (0,0)$. Then $L^\circ(y_1, y_2) = L(y_1, y_2)$ if $(y_1, y_2) \neq \bar{y}$ and $L^\circ(\bar{y}) = \emptyset$. As in Example 4.1, properties (O4) and (O5) are satisfied, while properties (O1) and (O2) are violated.

Given a level-set mapping L, a point $\bar{y} \in Y$ and a number $\delta > 0$, we are going to employ in our model the 'localized' family of sets

$$\Xi^\delta := \{\operatorname{cl} L(y) \mid y \in L^-(\bar{y}) \cap B_\delta(\bar{y})\}. \tag{4.2}$$

Note that members of Ξ^δ are not simply translations (deformations) of the fixed set $L(\bar{y})$ (or $L^\circ(\bar{y})$); they are defined by sets $L(y)$ where y does not have to be equal to $\bar{y}$.

Remark 4.2 Given a set K containing 0 one can naturally define the level-set mapping by $L(y) = y + K$ for all $y \in Y$. Then (4.2) defines the family of perturbations as the traditional collection of translations of $\operatorname{cl} K$, i.e., $\Xi^\delta = \{y + \bar{y} + \operatorname{cl} K \mid y \in K \cap (\delta B)\}$.

In the current setting, the properties in Proposition 3.1 take the following form.

Proposition 4.3 *Let $\delta > 0$, and Ξ^δ be given by (4.2). The triple $\{F, \Omega, \Xi^\delta\}$ is*

(i) *extremal at $(\bar{x}, \bar{y})$ if and only if there is a $\rho \in (0, +\infty]$ such that, for any $\varepsilon > 0$, there exists a $y \in L^-(\bar{y}) \cap B_\delta(\bar{y})$ such that $d(\bar{y}, L(y)) < \varepsilon$, and*

$$F(\Omega \cap B_\rho(\bar{x})) \cap \operatorname{cl} L(y) \cap B_\rho(\bar{y}) = \emptyset; \tag{4.3}$$

(ii) *stationary at $(\bar{x}, \bar{y})$ if and only if, for any $\varepsilon > 0$, there exist a $\rho \in (0, \varepsilon)$ and a $y \in L^-(\bar{y}) \cap B_\delta(\bar{y})$ such that $d(\bar{y}, L(y)) < \varepsilon\rho$, and condition (4.3) is satisfied;*

(iii) *approximately stationary at $(\bar{x}, \bar{y})$ if and only if, for any $\varepsilon > 0$, there exist a $\rho \in (0, \varepsilon)$, a $y \in L^-(\bar{y}) \cap B_\delta(\bar{y})$, and $(x_1, y_1), (x_2, y_2) \in B_\varepsilon(\bar{x}, \bar{y})$ such that $d((x_1, y_1), \operatorname{gph} F) < \varepsilon\rho$, $d(x_2, \Omega) < \varepsilon\rho$, $d(y_2, L(y)) < \varepsilon\rho$, and*

$$F(x_1 + (\Omega - x_2) \cap (\rho\mathbb{B}_X)) \cap (y_1 + (\operatorname{cl} L(y) - y_2) \cap (\rho\mathbb{B}_Y)) = \emptyset.$$

The statements of Theorem 3.1 and its corollaries can be easily adjusted to the current setting. For instance, Corollary 3.2 can be reformulated as follows.

Corollary 4.2 *Let X and Y be Banach spaces, Ω and $\operatorname{gph} F$ be closed, $\bar{x} \in \Omega$, $\bar{y} \in F(\bar{x})$, $\delta > 0$, and Ξ^δ be given by (4.2). Suppose condition $(QC)_C$ is satisfied. If the triple $\{F, \Omega, \Xi^\delta\}$ is approximately stationary at $(\bar{x}, \bar{y})$, then there is an $M > 0$ such that, for any $\varepsilon > 0$, there exist $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $y \in L^-(\bar{y}) \cap B_\delta(\bar{y})$, $y_2 \in \operatorname{cl} L(y) \cap B_\varepsilon(\bar{y})$, and $y^* \in N^C_{\operatorname{cl} L(y)}(y_2) + \varepsilon\mathbb{B}_{Y^*}$ such that $\|y^*\| = 1$, and condition (3.2) holds true.*

If X is Asplund and condition $(QC)_F$ is satisfied, then the above assertion holds true with N^F and D^{*F} in place of N^C and D^{*C}, respectively.

The properties in Definition 3.1 are rather general. They cover various optimality and stationarity concepts in vector and set-valued optimization. With Ω, F and L as above, and points $\bar{x} \in \Omega$ and $\bar{y} \in F(\bar{x})$, the next definition seems reasonable.

Definition 4.1 The point $(\bar{x}, \bar{y})$ is *extremal* for F on Ω if there is a $\rho \in (0, +\infty]$ such that

$$F(\Omega \cap B_\rho(\bar{x})) \cap L^\circ(\bar{y}) \cap B_\rho(\bar{y}) = \emptyset. \tag{4.4}$$

Definition 4.1 covers both local ($\rho < +\infty$) and global ($\rho = +\infty$) extremality. The above concept is applicable, in particular, to solutions of the set-valued minimization problem (P), and the conventional Pareto optimality implies the extremality in the sense of Definition 4.1. Indeed, if $(\bar{x}, \bar{y})$ is a (local) Pareto solution to (P) with respect to a nontrivial pointed convex cone $K \subset Y$, then, by Definition 1.1, there is a $\rho \in (0, +\infty]$ such that $F(\Omega \cap B_\rho(\bar{x})) \cap L^\circ(\bar{y}) = \emptyset$, where $L^\circ(\bar{y}) := (\bar{y} - K) \setminus \{\bar{y}\}$. The latter condition obviously implies condition (4.4). Hence, $(\bar{x}, \bar{y})$ is an extremal point for F on Ω.

Remark 4.3 (i) The concept in Definition 4.1 is broader than just (local) minimality as F is not assumed to be an objective mapping of an optimization problem. It can, for instance, be involved in modeling constraints.

(ii) The property in Definition 4.1 is similar to the one in the definition of *fully localized minimizer* in [3, Definition 3.1] (see also [10, p. 68]). The latter definition uses the larger set $\operatorname{cl} L(\bar{y}) \setminus \{\bar{y}\}$ in place of $L^\circ(\bar{y})$ in (4.4). Unlike many solution concepts in vector optimization, the above definition involves “image localization” (hence, is in general weaker). It has proved to be useful when studying locally optimal allocations of welfare economics; cf. [3, 10].

We next show that, under some mild assumptions on the level-set mapping L, the extremality in the sense of Definition 4.1 can be treated in the framework of the extremality in the sense of Definition 3.1 (or its characterization in Proposition 4.3 (i)).

Proposition 4.4 *Let* $\bar{x} \in \Omega$, $\bar{y} \in F(\bar{x})$, $\delta > 0$, *and* Ξ^δ *be given by* (4.2). *Suppose* L *satisfies conditions* (O1) *and* (O5). *If* $(\bar{x}, \bar{y})$ *is extremal for* F *on* Ω, *then the triple* $\{F, \Omega, \Xi^\delta\}$ *is extremal at* $(\bar{x}, \bar{y})$.

Proof In view of (O1), it follows from Proposition 4.1 (i) that

$$\{y \in Y \mid d(\bar{y}, L(y)) < \varepsilon\} \cap L^\circ(\bar{y}) \cap B_\varepsilon(\bar{y}) \neq \emptyset \text{ for all } \varepsilon > 0. \tag{4.5}$$

Suppose $\{F, \Omega, \Xi^\delta\}$ is not extremal at $(\bar{x}, \bar{y})$. Let $\rho \in (0, +\infty]$. By Proposition 4.3 (i), there exists an $\varepsilon > 0$ such that, for any $y \in L^-(\bar{y}) \cap B_\delta(\bar{y})$ with $d(\bar{y}, L(y)) < \varepsilon$, it holds

$$F(\Omega \cap B_\rho(\bar{x})) \cap \operatorname{cl} L(y) \cap B_\rho(\bar{y}) \neq \emptyset. \tag{4.6}$$

In view of (4.5), there is a point $y \in L^\circ(\bar{y}) \cap B_\delta(\bar{y}) \subset L^-(\bar{y}) \cap B_\delta(\bar{y})$ with $d(\bar{y}, L(y)) < \varepsilon$, and we can choose a point $\hat{y}$ belonging to the set in (4.6). Thus, $y \in L^\circ(\bar{y})$ and $\hat{y} \in \operatorname{cl} L(y)$. Thanks to (O5), we have $\hat{y} \in L^\circ(\bar{y})$, and consequently, $\hat{y} \in F(\Omega \cap B_\rho(\bar{x})) \cap L^\circ(\bar{y}) \cap B_\rho(\bar{y})$. Since $\rho \in (0, +\infty]$ is arbitrary, $(\bar{x}, \bar{y})$ is not extremal for F on Ω. □

Thanks to Proposition 4.4, if the level-set mapping L satisfies conditions (O1) and (O5), then extremal points of problem (P) satisfy the necessary conditions in Theorem 3.1 and its corollaries. In particular, the next statement holds true.

Corollary 4.3 *Let* X *and* Y *be Banach spaces,* Ω *and* $\operatorname{gph} F$ *be closed,* $\bar{x} \in \Omega$, $\bar{y} \in F(\bar{x})$, *and* $\delta > 0$. *Suppose that condition* $(QC)_C$ *is satisfied as well as conditions* (O1) *and* (O5) *for* *some mapping* $L : Y \rightrightarrows Y$. *If* $(\bar{x}, \bar{y})$ *is extremal for* F *on* Ω, *then there is an* $M > 0$ *such that, for any* $\varepsilon > 0$, *there exist* $(x_1, y_1) \in \operatorname{gph} F \cap B_\varepsilon(\bar{x}, \bar{y})$, $x_2 \in \Omega \cap B_\varepsilon(\bar{x})$, $y \in L^-(\bar{y}) \cap B_\delta(\bar{y})$, $y_2 \in \operatorname{cl} L(y) \cap B_\varepsilon(\bar{y})$, *and* $y^* \in N^C_{\operatorname{cl} L(y)}(y_2) + \varepsilon \mathbb{B}_{Y^*}$ *such that* $\|y^*\| = 1$, *and condition* (3.2) *holds true.*

If X *is Asplund and condition* $(QC)_F$ *is satisfied, then the above assertion holds true with* N^F *and* D^{*F} *in place of* N^C *and* D^{*C}, *respectively.*

5 Set-Valued Optimization: Multiple Mappings

It is not difficult to upgrade the model used in Definition 3.1 and the subsequent statements to make it directly applicable to constraint optimization problems: instead of a single mapping $F : X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$ for some $\bar{x} \in \Omega \subset X$ and a single family Ξ of subsets of Y, one can consider finite collections of mappings $F_i : X \rightrightarrows Y_i$ between normed spaces together with points $\bar{y}_i \in F_i(\bar{x})$, and nonempty families Ξ_i of subsets of Y_i $(i = 1, \ldots, n)$.

This more general setting can be viewed as a structured particular case of the set-valued optimization model considered in Sect. 3 if one sets

$$Y := Y_1 \times \cdots \times Y_n, \quad F := (F_1, \ldots, F_n), \quad \bar{y} := (\bar{y}_1, \ldots, \bar{y}_n) \quad \text{and} \quad \Xi := \Xi_1 \times \cdots \times \Xi_n.$$

Thus, $\bar{y} \in F(\bar{x})$, and $A \in \Xi$ means that $A = A_1 \times \cdots \times A_n$ and $A_i \in \Xi_i$ $(i = 1, \ldots, n)$. To shorten the notation, we keep talking in this section about extremality/stationarity of the triple $\{F, \Omega, \Xi\}$ at $(\bar{x}, \bar{y})$.

Definition 5.1 The triple $\{F, \Omega, \Xi\}$ is extremal (resp., stationary, approximately stationary) at $(\bar{x}, \bar{y})$ if the collection of $n + 1$ families of sets:

$$\widehat{\Xi}_i := \{\Omega_i\} \ (i = 1, \ldots, n) \quad \text{and} \quad \widehat{\Xi}_{n+1} := \{\Omega \times A \mid A \in \Xi\}.$$

is extremal (resp., stationary, approximately stationary) at $(\bar{x}, \bar{y})$, where $\Omega_i := \{(x, y_1, \ldots, y_n) \in X \times Y_1 \times \cdots \times Y_n \mid y_i \in F_i(x)\}$ $(i = 1, \ldots, n)$.

With the notation introduced above, Definitions 1.3 and 5.1 lead to characterizations of the extremality and stationarity of the triple $\{F, \Omega, \Xi\}$ given in parts (i) and (ii) of Proposition 3.1. The corresponding characterization of the approximate stationarity is a little different. It is formulated in the next proposition.

Proposition 5.1 *The triple* $\{F, \Omega, \Xi\}$ *is approximately stationary at* $(\bar{x}, \bar{y})$ *if and only if, for any* $\varepsilon > 0$, *there exist a* $\rho \in (0, \varepsilon)$, $A_i \in \Xi_i$ $(i = 1, \ldots, n)$, $x_i \in B_\varepsilon(\bar{x})$ $(i = 1, \ldots, n + 1)$, *and* $y_i, v_i \in B_\varepsilon(\bar{y}_i)$ $(i = 1, \ldots, n)$ *such that* $d((x_i, y_i), \operatorname{gph} F_i) < \varepsilon\rho$, $d(v_i, A_i) < \varepsilon\rho$ $(i = 1, \ldots, n)$, $d(x_{n+1}, \Omega) < \varepsilon\rho$ *and, for each* $x \in \Omega \cap B_\rho(x_{n+1})$, *there is an* $i \in \{1, \ldots, n\}$ *such that*

$$F_i(x_i + x - x_{n+1}) \cap \big(y_i + (A_i - v_i) \cap (\rho \mathbb{B}_Y)\big) = \emptyset.$$

Application of Theorem 1.1 in the current setting produces necessary conditions for approximate stationarity and, hence, also stationarity and extremality extending Theorem 3.1 and its corollaries. Condition $(QC)_C$ can be extended as follows:

$(\widehat{QC})_C$ there is an $\varepsilon > 0$ such that $\|\sum_{i=1}^{n+1} x_i^*\| \geq \varepsilon$ for all $(x_i, y_i) \in \operatorname{gph} F_i \cap B_\varepsilon(\bar{x}, \bar{y}_i)$, $x_i^* \in D^{*C} F_i(x_i, y_i)(\varepsilon \mathbb{B}_{Y_i^*})$ $(i = 1, \ldots, n)$, $x_{n+1} \in \Omega \cap B_\varepsilon(\bar{x})$ and $x_{n+1}^* \in N_\Omega^C(x_{n+1})$ such that $\sum_{i=1}^{n+1} \|x_i^*\| = 1$,

while the corresponding extension $(\widehat{QC})_F$ of condition $(QC)_F$ is obtained by replacing N^C and D^{*C} in $(\widehat{QC})_C$ by N^F and D^{*F}, respectively. An extension of Corollary 4.2 takes the following form.

Theorem 5.1 *Let $X, Y_1, \ldots, Y_n$ be Banach spaces, Ω and, for each $i = 1, \ldots, n$, the graph* $\operatorname{gph} F_i$ *and all members of Ξ_i be closed. Suppose $\{F, \Omega, \Xi\}$ is approximately stationary at $(\bar{x}, \bar{y})$. If condition $(\widehat{QC})_C$ is satisfied, then there is an $M > 0$ such that, for any $\varepsilon > 0$, there exist $(x_i, y_i) \in \operatorname{gph} F_i \cap B_\varepsilon(\bar{x}, \bar{y}_i)$, $A_i \in \Xi_i$, $v_i \in A_i \cap B_\varepsilon(\bar{y}_i)$, $y_i^* \in N_{A_i}^C(v_i) + \varepsilon \mathbb{B}_{Y_i^*}$ $(i = 1, \ldots, n)$, and $x_{n+1} \in \Omega \cap B_\varepsilon(\bar{x})$ such that $\sum_{i=1}^n \|y_i^*\| = 1$ and*

$$0 \in \sum_{i=1}^n D^{*C} F_i(x_i, y_i)(y_i^*) + N_\Omega^C(x_{n+1}) \cap (M\mathbb{B}_{X^*}) + \varepsilon \mathbb{B}_{X^*}.$$

If X is Asplund and condition $(\widehat{QC})_F$ is satisfied, then the above assertion holds true with N^F and D^{*F} in place of N^C and D^{*C}, respectively.

Remark 5.1 (i) Proposition 3.2 (with $F = (F_1, \ldots, F_n)$ in part (i)) gives two typical sufficient conditions for the fulfillment of conditions $(\widehat{QC})_C$ and $(\widehat{QC})_F$.
(ii) Theorem 5.1 covers [21, Theorems 3.1 and 3.2]. In view of the previous item, it also covers [21, Corollary 3.2].
(iii) Theorem 5.1 is a consequence of the dual necessary conditions for approximate stationarity of a collection of sets in Theorem 1.1. The latter theorem can be extended to cover a more general quantitative notion of approximate α-stationarity (with a fixed $\alpha > 0$), leading to corresponding extensions of Theorem 5.1 and its corollaries covering, in particular, dual conditions for ε-Pareto optimality in [22, Theorems 4.3 and 4.5].

Employing the multiple-mapping model studied in this section, one can consider a more general than (P) optimization problem with set-valued constraints:

$$\text{minimize } \; F_0(x) \quad \text{subject to } \; F_i(x) \cap K_i \neq \emptyset \; (i = 1, \ldots, n), \;\; x \in \Omega, \tag{$\mathcal{P}$}$$

where $F_i : X \rightrightarrows Y_i$ $(i = 0, \ldots, n)$ are mappings between normed spaces, $\Omega \subset X$, $K_i \subset Y_i$ $(i = 1, \ldots, n)$, and Y_0 is equipped with a level-set mapping L. The "functional" constraints in $(\mathcal{P})$ can model a system of equalities and inequalities as well as more general operator-type constraints.

Using the set of *admissible solutions*

$$\widehat{\Omega} := \{x \in \Omega \mid F_i(x) \cap K_i \neq \emptyset, \; i = 1, \ldots, n\},$$

we say that $(\bar{x}, \bar{y}_0) \in X \times Y_0$ is an extremal point of problem $(\mathcal{P})$ if it is extremal for F_0 on $\widehat{\Omega}$. This means, in particular, that $\bar{x} \in \Omega$, $\bar{y}_0 \in F_0(\bar{x})$, and there exist $\bar{y}_i \in F_i(\bar{x}) \cap K_i$ $(i = 1, \ldots, n)$.

We are going to employ the model studied in the first part of this section with $n+1$ objects in place of n. There are $n+1$ mappings $F_0, \ldots, F_n$ and n sets $K_1, \ldots, K_n$ in $(\mathcal{P})$. As in (4.2), we define $\Xi_0^\delta := \{\operatorname{cl} L(y) \mid y \in L_\delta(\bar{y}_0)\}$ $(\delta > 0)$, where $L_\delta(\bar{y}_0) = (L(\bar{y}_0) \cap B_\delta(\bar{y})) \cup \{\bar{y}\}$. Now, set

$$Y := Y_0 \times \cdots \times Y_n, \quad F := (F_0, \ldots, F_n), \quad \bar{y} := (\bar{y}_0, \ldots, \bar{y}_n) \quad \text{and}$$

$$\Xi^\delta := \Xi_0^\delta \times K_1 \times \cdots \times K_n.$$

Using the same arguments, one can prove the next extension of Proposition 4.4.

Proposition 5.2 *Let $\bar{x} \in \Omega$, $\bar{y}_0 \in F_0(\bar{x})$, $\bar{y}_i \in F_i(\bar{x}) \cap K_i$ $(i = 1, \ldots, n)$, $\delta > 0$, and F, $\bar{y}$ and Ξ^δ be defined as above. Suppose L satisfies conditions (O1) and (O5). If $(\bar{x}, \bar{y}_0)$ is an extremal point of problem ($\mathcal{P}$), then $\{F, \Omega, \Xi^\delta\}$ is extremal at $(\bar{x}, \bar{y})$.*

Condition $(\widehat{QC})_C$ in the current setting is reformulated as follows:

$(\widehat{QC})'_C$ there is an $\varepsilon > 0$ such that $\|\sum_{i=0}^{n+1} x_i^*\| \geq \varepsilon$ for all $(x_i, y_i) \in \operatorname{gph} F_i \cap B_\varepsilon(\bar{x}, \bar{y}_i)$, $x_i^* \in D^{*C} F_i(x_i, y_i)(\varepsilon \mathbb{B}_{Y_i^*})$ $(i = 0, \ldots, n)$, $x_{n+1} \in \Omega \cap B_\varepsilon(\bar{x})$ and $x_{n+1}^* \in N_\Omega^C(x_{n+1})$ such that $\sum_{i=0}^{n+1} \|x_i^*\| = 1$,

while the corresponding reformulation $(\widehat{QC})'_F$ of condition $(\widehat{QC})_F$ is obtained by replacing N^C and D^{*C} in $(\widehat{QC})'_C$ by N^F and D^{*F}, respectively. In view of Proposition 5.2, Theorem 5.1 yields the following statement.

Corollary 5.1 *Let $X, Y_0, \ldots, Y_n$ be Banach spaces, the sets Ω, $\operatorname{gph} F_i$ $(i = 0, \ldots, n)$ and K_i $(i = 1, \ldots, n)$ be closed, and $\delta > 0$. Suppose L satisfies conditions (O1) and (O5). If $(\bar{x}, \bar{y}_0)$ is an extremal point of problem ($\mathcal{P}$) and condition $(\widehat{QC})'_C$ is satisfied, then there is an $M > 0$ such that, for any $\varepsilon > 0$, there exist $(x_i, y_i) \in \operatorname{gph} F_i \cap B_\varepsilon(\bar{x}, \bar{y}_i)$ $(i = 0, \ldots, n)$, $x_{n+1} \in \Omega \cap B_\varepsilon(\bar{x})$, $y \in B_\delta(\bar{y}_0)$, $v_0 \in \operatorname{cl} L(y) \cap B_\varepsilon(\bar{y}_0)$, $y_0^* \in N_{\operatorname{cl} L(y)}^C(v_0) + \varepsilon \mathbb{B}_{Y_0^*}$, $v_i \in K_i \cap B_\varepsilon(\bar{y}_i)$ and $y_i^* \in N_{K_i}^C(v_i) + \varepsilon \mathbb{B}_{Y_i^*}$ $(i = 1, \ldots, n)$ such that $\sum_{i=0}^n \|y_i^*\| = 1$ and*

$$0 \in \sum_{i=0}^n D^{*C} F(x_i, y_i)(y_i^*) + N_\Omega^C(x_{n+1}) \cap (M \mathbb{B}_{X^*}) + \varepsilon \mathbb{B}_{X^*}.$$

If X is Asplund and condition $(\widehat{QC})'_F$ is satisfied, then the above assertion holds true with N^F and D^{*F} in place of N^C and D^{*C}, respectively.

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