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# Interval-valued fuzzy matrices for ranking from imprecise preferences: Models and applications

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ABSTRACT

This paper presents novel ranking methods for decision-making under uncertainty or incomplete information, drawing inspiration from human cognitive processes. It integrates interval-valued fuzzy sets to model uncertain preferences and employs masking matrices to generate weight vectors that quantify the strength of each option. We focus on ranking the greatest tennis players of all time, based on head-to-head win-loss data from the ATP and WTA websites. The challenge is that win-loss scores do not fully reflect comparative preferences for players from different eras. We propose an original Bayesian-statistics-based method to construct interval-valued fuzzy preference relations, enabling the proposed methods to derive interval-valued weight vectors and use them to create different ranking lists. The proposed method uses the Frobenius norm to measure the deviation between input interval-valued fuzzy matrices and their consistent counterparts, and we introduce two innovative ranking models. These models leverage pseudoinverse matrices, built on an established masking-matrix framework, to compute optimal interval-valued weight vectors. Additionally, we present properties of the masking-matrix representations and outline an algorithm for applying pseudoinverse matrices, thereby enhancing practical applicability.

## 1. Introduction

Ranking problems are common in many fields and pose a major challenge because multiple aspects must be considered simultaneously. This is the core issue in Multi-Criteria Decision Analysis (MCDA).

In MCDA, decisions are modelled by evaluating alternatives (options) against various criteria (e.g., cost, environmental impact, benefits), often using weights to reflect priorities. This produces a ranked or scored list that supports transparent and justifiable decisions. Both MCDA and related approaches deal with incorporating human preferences into decision processes, but they differ in philosophy, data handling, and focus: MCDA emphasizes interactive, human-centered preference construction. Hüllermeier and Słowiński [1] provide a comprehensive overview of techniques for integrating preference information with multiple criteria decision aiding. Consequently, the task transforms into a problem of ranking the alternatives.

When developing an MCDA model for a real-world ranking problem, preference scores among available options are often established through consultations with decision-makers (DMs) or subject matter experts. Experts can express their preferences when evaluating option pairs. For each pair of options $(i, j)$ and an expert $k$, the preference score can be numerically represented by $\mathbf{P}_k(i, j)$, indicating the degree of priority of one option over the other. This is the common input form for all MCDA models.

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The Analytic Hierarchy Process (AHP), introduced by Saaty [2], is one of the most popular approaches to MCDA. It provides a systematic method for ranking options by deriving a weight vector from pairwise comparison matrices (PCMs), which capture the relative preferences between pairs of options through numerical or interval judgments. The weights derived from this process are used to rank the options. Nonetheless, the reliability and consistency of the AHP method rely strongly on the quality of the data; inaccurate or incomplete information can adversely affect PCM calculations, which may result in inconsistent and less than optimal outcomes. Standard ranking algorithms that use preference relations typically involve three primary steps:

**Step 1:** Convert raw data into a pairwise comparison format, for example, a preference matrix **P** for input purposes;

**Step 2:** Select a suitable optimization solver for the pairwise comparison;

**Step 3:** Identify the optimal weight vector **w** for output and order the alternatives according to the optimal weights.

Besides AHP, there are many other approaches to the ranking problem. Masson and Denoeux [3] presented a ranking model based on belief functions. Barrientos et al. [4] introduced a Bayesian framework for quantifying uncertainty in conclusions about the orderings of NBA players and lineups. If the expert's opinions are consistent, then all ranking methods should return the same ranking. However, the fundamental differences between those approaches are based on the conflict resolution methods in the case of nonlinear relations. The belief function uses the plausibility functions, defined on all linear extensions. The proposed method in this paper tries to minimize the deviation between the input matrix **P** and the weight vector **w**.

The approaches based on Bayes statistics and Dempster-Shafer theory offer the complete methods to the ranking problem as they are considering the set of all possible rankings (permutations). However, those methods can require significant data to build a robust model or can be complex to set up, especially when dealing with a large number of variables or a complex decision structure.

AHP-based methods have simple ranking algorithms, which is an advantage. However, the quality of the final ranking depends heavily on how accurate and reliable the input preferences are. A useful feature of AHP is its consistency indexes, which measure how much the comparison matrix deviates from perfect consistency. In real-world decisions—which are often complex, uncertain, and subjective—asking people to give exact numerical judgments is usually not enough to capture reality properly.

A more intuitive and robust approach, similar to rough set theory [5], involves using intervals, ranging from lower to upper bounds, for PCM elements [6]. This method provides a more nuanced and accurate representation of human judgments. As a PCM corresponds to a fuzzy preference relation (FPR) [7], an interval-valued PCM can likewise be transformed into an equivalent form: the interval-valued fuzzy preference relation (IV-FPR).

Several ranking methods exist for determining the weight vector corresponding to FPRs, including the Eigenvector Method [2,8,9], the Least Squares Method [10], and the Logarithmic Least Squares Method [9,11,12]. While all these methods converge when the comparison matrix is consistent (i.e., of rank one), they typically produce differing rank-one approximations in the presence of inconsistency. Despite numerous studies on these methods [10,13], determining the optimal approach remains a significant challenge. However, in practice, due to a lack of information or knowledge, using a single numerical value to compare the preference between two objects is often imprecise. In sports, for example, the outcome of a match between two players - whether they win or lose - is influenced by numerous factors, including their ability, skill, technique, current performance, or other objective elements. Therefore, using interval values to compare the preference levels of two alternatives will yield more promising results. Consequently, when pairwise comparisons are specified as an interval within [0,1], the resulting preference relation is referred to as an IV-FPR. The problem of ranking options using IV-FPRs has been researched more and more [14–19].

Imprecise or missing data can lead to incomplete information, hindering comparisons and resulting in blank entries within the comparison matrix. If the comparison matrix has missing data, then the algorithm's constructed input is an incomplete FPR (similar to an incomplete PCM) or an incomplete IV-FPR. Many studies have investigated strategies for the efficient analysis of PCMs with incomplete data [9,20–27]. The survey papers [28] and Yuan et al. [29] address incomplete preference relations, highlighting them as a common challenge in group decision-making (GDM), and divide the existing GDM strategies for incomplete PR into two main categories: iterative methods and optimization methods.

The method for transforming raw data depends on whether the input is an incomplete FPR or IV-FPR. For example, Bozóki et al. [23] used incomplete PCMs from head-to-head win-loss ratios of female tennis players to derive weight vectors for rankings. Match outcomes depend on many factors, so interval values are more objective than precise probabilities. Vo and Nguyen [30] used a larger WTA dataset with Bernoulli and Bayesian methods to create an incomplete IV-FPR. They then applied the optimization model from Vo and Nguyen [31] to obtain IV-WVs, showing IV-WVs better reflect rankings than traditional weights.

Methods that force a single crisp ranking from incomplete preference matrices (using point-weight vectors) ignore the inherent uncertainty of missing data. IV-WVs handle this imprecision better: they remain flexible and can yield multiple rankings based on different interval comparison criteria.

This paper makes two principal contributions.

First, we introduce a novel methodology for deriving IV-WVs from incomplete IV-FPRs. The approach uses a masking matrix to explicitly encode missing pairwise comparisons and employs **L2**-norm optimization, extending the **L1**-norm framework previously developed for complete IV-FPRs [31]. By combining pseudoinverse techniques applied to the masking matrix with logarithmic goal programming [32], we formulate two ranking models: **AAI**, which produces additive-consistent IV-WVs, and **MAI**, which yields multiplicative-consistent IV-WVs. It is noteworthy that the proposed method can also be applied to incomplete FPR, which is a special case of incomplete IV-FPR. In this case, the results are equivalent to the method presented in Bozóki et al. [9]. Furthermore, a significant contribution of this work is the proposal of an identifiability condition for the data, ensuring the uniqueness of the resulting ranking.

Second, we propose an integrated, data-driven procedure for generating incomplete IV-FPRs from raw head-to-head match statistics, with a concrete application to all-time tennis player rankings (WTA and ATP). Unlike most studies in the field, which focus primarily on ranking algorithms assuming complete or pre-processed preference information, this work covers the full pipeline: from historical pairwise results spanning five decades, we model win probabilities using the Bernoulli distribution and apply Bayesian maximum a posteriori (MAP) estimation to construct incomplete IV-FPRs. These relations then serve as input to the proposed **AAI** and **MAI** models to derive IV-WVs and corresponding rankings. To assess the reliability of the approach, the obtained rankings are compared against established expert rankings, providing empirical validation of both the preference-relation generation step and the overall methodology. This application demonstrates the practical utility of the framework in real-world head-to-head competitive domains beyond tennis.

The structure of this paper is as follows: Section 2 outlines the core ideas of our ranking algorithm, employing interval-valued fuzzy logic to model pairwise comparisons, which in turn leads to the generation of IV-WVs. Section 3 delves into the use of masking matrices for representing IV-FPRs and introduces new models based on this structure. In Section 4, we detail an innovative solution method that uses pseudoinverse matrices and examine their fundamental characteristics. Section 5 involves applying the proposed models to the practical task of ranking the greatest ATP and WTA tennis players in history. Lastly, Section 6 encapsulates the primary conclusions and wraps up the discussion.

## 2. Preliminaries

In ranking algorithms, experts articulate their preferences when assessing option pairs. These preferences are typically represented numerically to show one attribute's priority, or they may simply reflect indifference. We will start by considering the pairwise comparison of alternatives.

Multiple-criteria decision-making often involves comparing $n$ alternatives $X = \{x_1, \dots, x_n\}$. However, the relative preference between certain pairs of alternatives $(x_i, x_j)$ may remain ambiguous. An uncertain PR is described by a function $\mu : X \times X \to D$, where $D$ indicates how strong the preference is. This function may follow some generalized conditions for PR, like transitivity, antisymmetry, or reflexivity. The preferences can be shown as an $n \times n$ matrix $P = (p_{ij})$, where $p_{ij} = \mu(x_i, x_j) \in D$.

In this paper, $D$ may represent Boolean values ($\{0, 1\}$), real numbers ($\mathbb{R}$), or intervals ($[a, b]$). We denote the sets of all $n \times n$ binary, real, and interval-valued matrices by $\mathbf{B}_n$, $\mathbf{M}_n$, and $\mathbf{I}_n$, respectively. In the literature, if $D$ is a set of numerical values, PRs are represented by a matrix, often referred to as a PCM or an FPR. If $D$ is an interval within $[0, 1]$, PRs are referred to IV-FPR.

### *2.1. Pairwise comparison matrix or fuzzy preference relation*

Let us recall the formal definitions of PCM and FPR. Saaty [2] specified that PCM satisfies the reciprocal property (i.e $p_{ij} = 1/p_{ji}$, and if we associate the alternatives $x_1, \dots, x_n$ with the weights $w_1, \dots, w_n$ then $p_{ij} = w_i/w_j$.

**Definition 1** (PCM). A matrix $P = (p_{ij}) \in \mathbf{R}_+^{n\times n}$, where $p_{ij}$ represents the relative importance of element $i$ over element $j$, is called a pairwise comparison matrix (PCM), if $p_{ii} = 1$ and $p_{ij} = \dfrac{1}{p_{ji}}$ for all $i, j = 1, \dots, n$.

**Definition 2** (FPR). A matrix $R = (r_{ij}) \in \mathbf{M}_n$, where $r_{ij} \in [0, 1]$, is called a fuzzy preference relation (FPR) if $r_{ii} = 0.5$ and $r_{ij} = 1 - r_{ji}$ for all $i, j = 1, \dots, n$.

The conditions defining an FPR generalize reflexivity and antisymmetry. Its transitivity is typically characterized by a *normalized fuzzy weight vector* (normalized FWV) $w = (w_1, \dots, w_n) \in [0, 1]^n$ satisfying $w_1 + \dots + w_n = 1$.

An FPR $R = (r_{ij})$ is *a-consistent* if there exists a corresponding normalized FWV $w = (w_1, \dots, w_n)$ such that $r_{ij} = (w_i - w_j + 1)/2$ for all $i, j = 1, \dots, n$. Similarly, $R$ is *m-consistent* if $r_{ij} = w_i/(w_i + w_j)$ for all $i, j = 1, \dots, n$. A PCM $P = (p_{ij})$ can be transformed into an FPR $R = (r_{ij})$ using $r_{ij} = \frac{p_{ij}}{1+p_{ij}}$, and vice versa with $p_{ij} = \frac{r_{ij}}{1-r_{ij}}$. Thus, FPRs and PCMs are equivalent, and we use FPR to refer to real-valued pairwise comparisons.

Most of FPRs are neither a-consistent nor m-consistent. Thus, the ranking problem for a given FPR $R = (r_{ij})$ involves finding a normalized FWV $w$, which minimizes the difference between $R$ and the *a-consistent* or *m-consistent* matrix corresponding to $w$. When some pairwise comparisons are unavailable, the preference relation is deemed incomplete, specifically an incomplete FPR.

### *2.2. Ranking methods from incomplete preference relations*

In AHP, a decision problem with $n$ elements (criteria or alternatives) is modelled using a PCM $P = (p_{ij})$ in Definition 1. Based on Saaty's assumption of a precise mathematical relationship between preference values and the priority vector [33], using optimization to directly determine priority weights, three main methods have been developed to build indirect completion models: least squares, logarithmic least squares [11], and the eigenvector [2]. In the eigenvector approach, the normalized priority vector $w = (w_1, w_2, \dots, w_n)$ is typically the principal right eigenvector of $A$, corresponding to the largest eigenvalue $\lambda_{\max}$. For a perfectly consistent matrix, $a_{ij} = w_i/w_j$ and $\lambda_{\max} = n$. In practice, $\lambda_{\max} \geq n$, and consistency is measured via the consistency index $\mathrm{CI} = (\lambda_{\max} - n)/(n - 1)$.

Incomplete preferences (missing $p_{ij}$) make the problem harder. Ureña et al. [28] reviewed this common issue in decision making. They stress that incomplete data should not be of low quality and recommend estimating missing values instead of discarding data. They also survey existing methods and suggest future directions. Two main approaches exist: (1) Estimate missing preferences, and (2) Use only the available (incomplete) data. Research shows reliable results are still possible even with up to 50% missing values. The second approach is gaining popularity because it's faster and avoids estimation errors. Many recent methods exploit consistency to handle incomplete preferences effectively.

*Imputation technique.* Harker's method estimates the priority vector $w$ using available comparisons, modelled as a directed graph $D(P) = (V, E)$, where vertices $V = \{1, 2, \ldots, n\}$ are elements, and edges $(i, j) \in E$ mean $p_{ij}$ is known. The graph must be weakly connected to ensure a unique $w$. To estimate a missing $p_{ij}$, Harker [25] used all elementary paths (sequences of edges from $i$ to $j$ without repeating nodes). If $\pi = \{(i, k_1), (k_1, k_2), \ldots, (k_m, j)\}$ is an elementary path from $i$ to $j$, then the intensity of this path is the product of comparison ratios along it, i.e. $Int(\pi) = p_{i,k_1} p_{k_1,k_2} \cdots p_{k_m,j}$.

The missing $p_{ij}$ is estimated as the geometric mean of these path intensities:

$$p_{ij} \approx \sqrt[N_{ij}]{\prod_{\text{paths } \pi \text{ from } i \to j} Int(\pi)}$$

where $N_{ij}$ is the number of paths. Since computing all paths is complex, Harker simplified by sampling random spanning trees (connected subgraphs with $n - 1$ edges) and using their path intensities. The method proceeds as follows:

1. **Start:** Start with a partial matrix $P$ and construct the graph $D(P)$. If needed, gather more comparisons to achieve a weakly connected $D(P)$.
2. **Impute Missing Entries:** Estimate missing $p_{ij}$ using the geometric mean of path intensities from sampled spanning trees, then compute the eigenvector $w$ of the filled matrix.
3. **Select Next Comparison:** Calculate the sensitivity of $w$ to missing $p_{ij}$ (using derivatives $\partial w / \partial p_{ij}$ ) and ask for the comparison with the largest impact, i.e. $\max_{(i,j) \notin E} \left\| \frac{\partial w}{\partial a_{ij}} \right\|_2$.
4. **Stop or Continue:** Stop if criteria (e.g., small change in $w$) are met; otherwise, add the new comparison to $P$ and repeat from step 1.

This approach efficiently estimates priorities with fewer comparisons while maintaining consistency.

*Logarithmic least squares method technique (LLSM).* This method derives ratio scales from a PCM and equates to the geometric mean method for one decision-maker. If not all factor pairs are provided, the geometric least squares method is applied. Kwiesielewicz [26] used the LLSM for incomplete matrices, proposing a weighting approach with generalized pseudoinverse matrices via spectral decomposition.

Assume that there are $m$ decision-makers. Let $P_k = (p_{ij}^{(k)})$ be the PCM of the $k$th decision-maker ($k = 1, \ldots, m$) and for $E_k \subset \{1, \ldots, n\}^2$ be the set of indices of known entries of $P_k$. The weights $w_1, \ldots, w_n$ are estimated by minimizing the following difference, using the LLSM:

$$\sum_{k=1}^{m} \sum_{(i,j) \in E_k} \left( \ln (p_{ij}^{(k)}) - \ln(w_i / w_j) \right)^2. \tag{1}$$

By substituting $x_i = \ln(w_i)$ and $y_{ij} = \ln(p_{ij})$, the optimization problem (1) is converted into a quadratic form. Using differential calculus, this problem can be transformed into a system of equations in the matrix form $Ax = b$, where $A$ is a symmetric, but singular matrix. Applying the theorem in Pyle [34], the general solution is $x = A^+ b + (I - A^+ A)y$, where $A^+$ denotes the Moore-Penrose inverse and $y$ is an arbitrary vector.

*Eigenvalue optimization technique.* Bozóki et al. [9] modelled each incomplete PCM $P$ as a family of complete PCMs, denoted $P(\mathbf{x})$, where $\mathbf{x}$ represents the vector of missing entries in the upper triangular part of $P$. The matrix $P(\mathbf{x})$ is formed by substituting the unknown values in $P$ with the corresponding entries from $\mathbf{x}$. The objective is to address the eigenvalue optimization problem:

$$\lambda = \min_{\mathbf{x} > 0} \lambda_{\max}(P(\mathbf{x})),$$

where $\lambda_{\max}(P(\mathbf{x}))$ denotes the principal eigenvalue of $P(\mathbf{x})$. The corresponding eigenvector $\mathbf{w}$ to $\lambda$ provides the resulting weight vector.

### 2.3. Interval-valued fuzzy preference relation (IV-FPR)

Sometimes, due to time constraints, lack of knowledge, or the complexity of the decision problem, decision-makers may provide incomplete preferences, leading to an incomplete IV-FPR as the algorithm's input.

**Definition 3** (IV-FPR). A matrix $\mathbf{P} = (p_{ij} = [p_{ij}^l; p_{ij}^r]) \in \mathbf{I}_n$ is defined as an interval-valued fuzzy preference relation (IV-FPR), where $p_{ji} = [1 - p_{ij}^r, 1 - p_{ij}^l] = [p_{ji}^l, p_{ji}^r]$ and $p_{ij} = [p_{ij}^l, p_{ij}^r] \subset [0, 1]$ for all $i, j = 1, \ldots, n$.

Unlike FPR, the definitions of transitivity for IV-FPR are more complicated. From the perspective of rough set theory, lower and upper matrices are used to define the boundaries of imprecise FPR. Therefore, each IV-FPR $\mathbf{P} = ([p^l_{ij}, p^r_{ij}])_{n\times n}$ can be represented as a pair of lower and upper bounds, $(L_\mathbf{P} = (p^l_{ij}), U_\mathbf{P} = (p^r_{ij}))$.

The following relation defines a partial order on the set of all FPRs:

$$X = (x_{ij})_{n\times n} \preceq Y = (y_{ij})_{n\times n} \iff x_{ij} \le y_{ij}, \text{ for all } 1 \le i < j \le n.$$

Then, the IV-FPR $\mathbf{P} = (p_{ij})_{n\times n}$ represents a family $\mathbb{R}_\mathbf{P}$ of FPRs defined by

$$\mathbb{R}_\mathbf{P} = \{P : L_\mathbf{P} \preceq P \preceq U_\mathbf{P}\}.$$

If $L_\mathbf{P} = U_\mathbf{P}$, then $\mathbf{P}$ becomes an FPR, indicating that FPR is a special case of IV-FPR. Thus, it is necessary to extend the concepts *a-consistent* and *m-consistent* for IV-FPR. Below, we recall some extended definitions of those concepts in the context of ranking techniques for incomplete IV-FPR.

*Imputation technique for incomplete IV-FPR:* According to Xu et al. [22], the matrix of intervals $\mathbf{P} = ([l_{ij}, r_{ij}])_{n\times n}$ is called an *a-consistent* IV-FPR if

$$l_{ij} + l_{jk} = l_{ik} + 0.5 \quad \text{and} \quad r_{ij} + r_{jk} = r_{ik} + 0.5 \quad \text{for all } i < j < k. \tag{2}$$

Xu et al. [22] proposed a method to construct *a-consistent* IV-FPRs based on acceptable incomplete IV-FPRs. The idea was based on filling missing values using either *a-consistent* properties or weighted averaging from known values.

After estimation of missing values, the authors utilized the normalizing rank aggregation method to derive a final priority vector.

*Optimal satisfactory and approximation models:* We can also extend the transitivity for IV-FPR in a weaker way: $\mathbf{P}$ is *a-consistent* (or *m-consistent*) if there is an $R \in \mathbb{R}_\mathbf{P}$ such that $R$ is *a-consistent* (or *m-consistent*) FPR.

The ranking model is looking for a normalized weight vector $w = (w_1, \dots, w_n)$, which minimizes the total violation for one of the following conditions:

$$L_\mathbf{P} \preceq A(w) \preceq U_\mathbf{P} \quad \text{or} \quad L_\mathbf{P} \preceq M(w) \preceq U_\mathbf{P}. \tag{3}$$

It is conceivable that many weight vectors may fulfill Eq. (3), and among these vectors, some might correspond to various rankings of the alternatives.

Therefore, a more robust approach employs Interval-Valued Weight Vectors (IV-WVs), defined as $\mathbf{w} = ([l_1, r_1], \dots, [l_n, r_n])$ where $[l_i, r_i] \subseteq [0, 1]$. Compared to crisp weight vectors, IV-WVs are better suited for handling the imprecise and incomplete nature of real-world data [31].

From any IV-WV $\mathbf{w} = ([l_1, r_1], \dots, [l_n, r_n])$, Vo and Nguyen [31] proposed to construct two pairs of matrices: $A^l(\mathbf{w}) = (a^l_{ij})$, $A^r(\mathbf{w}) = (a^r_{ij})$ and $M^l(\mathbf{w}) = (m^l_{ij})$, $M^r(\mathbf{w}) = (m^r_{ij})$, where $a^l_{ii} = a^r_{ii} = m^l_{ii} = m^r_{ii} = 0.5$ for $i = 1, \dots, n$, and

$$a^l_{ij} = \frac{l_i - r_j + 1}{2}, \quad a^r_{ij} = \frac{r_i - l_j + 1}{2}, \quad m^l_{ij} = \frac{l_i}{l_i + r_j}, \quad m^r_{ij} = \frac{r_i}{r_i + l_j}, \tag{4}$$

for all $i \ne j$.

For a given IV-FPR $\mathbf{P} = (p_{ij})_{n\times n} = ([p^l_{ij}; p^r_{ij}])_{n\times n} = [L_\mathbf{P}, U_\mathbf{P}]$, the vector $\mathbf{w}$ is called *a-compatible* with $\mathbf{P}$ if

$$L_\mathbf{P} \preceq A^l(\mathbf{w}) \quad \text{and} \quad A^r(\mathbf{w}) \preceq U_\mathbf{P}, \tag{5}$$

and is called *m-compatible* with $\mathbf{P}$ if

$$L_\mathbf{P} \preceq M^l(\mathbf{w}) \quad \text{and} \quad M^r(\mathbf{w}) \preceq U_\mathbf{P}. \tag{6}$$

Ranking algorithms based on IV-FPRs typically involve two main steps:

(1) determining the appropriate optimization solver for a given IV-FPR;
(2) selecting an interval comparison method.

There are two types of ranking models using IV-WVs:

**Optimal satisfactory models:** Search for an IV-WV $\mathbf{w}$ that minimizes the total violation from conditions in Eqs. (5) and (6).

**Optimal approximation models:** Search for an IV-WV $\mathbf{w}$ that minimizes the differences between the given IV-FPR and the IV-FPR produced by $\mathbf{w}$, for instance, by minimizing the following **L1** norms:

$$\mathcal{A}_3(l_1, \dots, l_n, r_1, \dots, r_n) = \|L_\mathbf{P} - A^l(\mathbf{w})\|_1 + \|A^r(\mathbf{w}) - U_\mathbf{P}\|_1. \tag{7}$$

$$\mathcal{M}_3(l_1, \dots, l_n, r_1, \dots, r_n) = \|L_\mathbf{P} - M^l(\mathbf{w})\|_1 + \|M^r(\mathbf{w}) - U_\mathbf{P}\|_1. \tag{8}$$

Vo and Nguyen [31] summarized the ranking methods and labelled the additive and multiplicative ranking models as follows:

- **A2**, **M2**: for optimal satisfactory models related to Eqs. (5) and (6);
- **A3**, **M3**: for optimal approximation models related to Eqs. (7) and (8).

Since these models rely on the Linear Programming method, they can also be employed to extract IV-WV from a partial IV-FPR by limiting the constraints solely to the known elements of $\mathbf{P}$.

**Table 1**
Glossary of abbreviations and symbols.

| Notations | Meaning |
|---|---|
| FPR, PCM | Fuzzy Preference Relation, Pairwise Comparison Matrix |
| IV-FPR | Interval-Valued Fuzzy Preference Relation |
| IV-WV | Interval-Valued Weight Vector |
| PR, DM | Preference Relation, Decision-Maker |
| MAP | Maximum a Posteriori |
| WTA, ATP | Women's Tennis Association, Association of Tennis Professionals |
| LLSM | Logarithmic Least Square Method |
| $[A^l(\mathbf{w}), A^r(\mathbf{w})]$ | The a-consistent IV-FPR derived from an IV-FW $\mathbf{w}$. |
| $[M^l(\mathbf{w}), M^r(\mathbf{w})]$ | The m-consistent IV-FPR derived from an IV-FW $\mathbf{w}$, |
| $L_{\mathbf{P}}, U_{\mathbf{P}}$ | Lower and boundary matrices of IV-FPR $\mathbf{P}$, |
| $\mathbf{M}_n, \mathbf{B}_n, \mathbf{I}_n$ | Sets of all real, binary and interval-valued $n \times n$ matrices |

### 2.4. Ordering interval-valued weight vectors (IV-WVs)

Depending on the ranking criteria of each problem, we will choose an appropriate method for ranking the interval values. Comparing two intervals can be done using centers, lengths, bounds, combined center-length, or expected values. Each criterion may yield a different ranking of alternatives.

For any interval $w = [l_w, r_w]$, its center is denoted as $c(w) = (l_w + r_w)/2$ and its diameter as $d(w) = r_w - l_w$. To compare two intervals, we can define a score function for each. The simplest score function uses the centers, defined as $S_0(w) = c(w)$. To address the decision maker (DM) risk factor, Lin and Wang [35] defined a parameterized score.

$$S_\lambda(w) = (2c(w) - 1)(d(w) + 1) + \lambda d^2(w), \text{ where } \lambda \in [-1, 1]. \tag{9}$$

Zhang and Xu [36] proposed a pair of scores $S_2(w) = (L(w), H(w))$, where

$$L(w) = \frac{r_w}{1 + d(w)}; \quad H(a) = 1 - d(w);$$

and used $S_2$ to compare two intervals in the following way:

$$w < u \Longleftrightarrow L(w) < L(u) \text{ or } (L(w) = L(w) \text{ and } H(w) < H(u)).$$

Wang et al. [32] suggested comparing intervals by calculating a "preference degree" between interval $a$ and interval $b$ with the following formula:

$$p(a \geq b) = \frac{\max(0, r_a - l_b) - \max(0, l_a - r_b)}{w(a) + w(b)}. \tag{10}$$

If $p(a \geq b) > 0.5$, it indicates that $a$ is preferred over $b$ to the degree $p(a \geq b)$, which is expressed as $a \overset{p(a\geq b)}{\geq} b$.

The next section will delve into the representation of incomplete FPRs and incomplete IV-FPRs through the application of mask matrices. Subsequently, we will present a novel ranking model that employs a pseudo-inverse matrix, grounded in this representation.

For clarity, a summary of acronyms and notations is provided in Table 1.

## 3. Masking matrix representation and ranking model utilizing incomplete IV-FPRs

An incomplete IV-FPR is represented by a pair of $n \times n$ matrices $(\mathbf{P}, E)$, where $\mathbf{P} = (p_{ij}) = ([p^l_{ij}, p^r_{ij}]) \in \mathbf{I}_n$ is a standard IV-FPR and $E = (e_{ij}) \in \mathbf{B}_n$ is the *masking (or existence) matrix* satisfying $e_{ii} = 1$ and $e_{ij} = e_{ji}$ for all $i, j = 1, \dots, n$. Here, $e_{ij} = 1$ indicates that the preference relationship between the $i$th and $j$th options is known, while $e_{ij} = 0$ indicates that $p_{ij}$ is unknown.

The **L2** norm squares larger deviations and is more efficient than **L1** in gradient optimization. So, we replace **L1** with **L2** in Eqs. (7), (8), yielding new additive and multiplicative models by minimizing:

$$\mathcal{A}_4(\mathbf{l}, \mathbf{r}) = \sum_{i,j=1}^{n} e_{ij} \left[ \left( p^l_{ij} - \frac{l_i - r_j + 1}{2} \right)^2 + \left( p^r_{ij} - \frac{r_i - l_j + 1}{2} \right)^2 \right], \tag{11}$$

$$\mathcal{M}_4(\mathbf{l}, \mathbf{r}) = \sum_{i,j=1}^{n} e_{ij} \left[ \left( p^l_{ij} - \frac{l_i}{l_i + r_j} \right)^2 + \left( p^r_{ij} - \frac{r_i}{r_i + l_j} \right)^2 \right], \tag{12}$$

where $\mathbf{l} = (l_1, \dots, l_n)^T$, $\mathbf{r} = (r_1, \dots, r_n)^T$ are the vectors of left and right bounds, respectively, of the target IV-WV $\mathbf{w} = ([l_1, r_1], \dots, [l_n, r_n])$.

It is important to highlight that for a full IV-FPR, i.e. $e_{ij} = 1$ for all $1 \leq i, j \leq n$, if $\min\left(\mathcal{A}_4(\mathbf{l}, \mathbf{r})\right) = 0$, then matrix $\mathbf{P}$ satisfies the condition in Eq. (2). Thus $\mathbf{P}$ exhibits additive consistency, as per [22].

In the subsequent sections, we will delve into both additive and multiplicative approaches for handling incomplete IV-FPRs.

### 3.1. Additive model for incomplete IV-FPR

Note that $e_{jk} = e_{kj}$ for $k, j = 1, 2, \ldots, n$. The first partial derivatives of $\mathcal{A}_4$ in Eq. (11) with respect to $l_k$ and $r_k$ for $k = 1, 2, \ldots, n$ are as follows:

$$\frac{\partial \mathcal{A}_4}{\partial l_k} = l_k \sum_{j=1}^{n} e_{kj} - \sum_{j=1}^{n} e_{kj} r_j - \sum_{j=1}^{n} e_{kj}(p^l_{kj} - p^r_{jk}).$$

Since $p^r_{jk} = 1 - p^l_{kj}$, we have

$$\frac{\partial \mathcal{A}_4}{\partial l_k} = l_k \sum_{j=1}^{n} e_{kj} - \sum_{j=1}^{n} e_{kj} r_j - \sum_{j=1}^{n} e_{kj}(2p^l_{kj} - 1), \text{ and}$$
$$\frac{\partial \mathcal{A}_4}{\partial r_k} = r_k \sum_{j=1}^{n} e_{kj} - \sum_{j=1}^{n} e_{kj} l_j - \sum_{j=1}^{n} e_{kj}(2p^r_{kj} - 1).$$

The system of $2n$ simultaneous equations $\nabla \mathcal{A}_4 = 0$ can be expressed as

$$\begin{cases} S \cdot \mathbf{l} - E \cdot \mathbf{r} = \mathbf{v}_l \\ S \cdot \mathbf{r} - E \cdot \mathbf{l} = \mathbf{v}_r \end{cases} \tag{13}$$

where $S = diag(\sigma_1, \ldots, \sigma_n)$ with $\sigma_k = \sum_{j=1}^{n} e_{kj}$, and

$$\mathbf{v}_l = \begin{pmatrix} \sum_{j=1}^{n} e_{1j}(2p^l_{1j} - 1) \\ \vdots \\ \sum_{j=1}^{n} e_{nj}(2p^l_{nj} - 1) \end{pmatrix}, \quad \mathbf{v}_r = \begin{pmatrix} \sum_{j=1}^{n} e_{1j}(2p^r_{1j} - 1) \\ \vdots \\ \sum_{j=1}^{n} e_{nj}(2p^r_{nj} - 1) \end{pmatrix}. \tag{14}$$

On the other hand, if we define a matrix

$$B_E = \left( \begin{array}{c|c} S & -E \\ -E & S \end{array} \right) = \begin{pmatrix} \sigma_1 & \ldots & 0 & -e_{11} & \ldots & -e_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \sigma_n & -e_{n1} & \ldots & -e_{nn} \\ -e_{11} & \ldots & -e_{1n} & \sigma_1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -e_{n1} & \ldots & -e_{nn} & 0 & \ldots & \sigma_n \end{pmatrix}, \tag{15}$$

the system (13) can be written in matrix form as:

$$B_E \cdot \begin{pmatrix} \mathbf{l} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix}. \tag{16}$$

A key feature of Eq. (16) is that its left-hand side depends only on the masking matrix $E = (e_{ij})$, and not on the actual comparison values $p^l_{ij}$ and $p^r_{ij}$. Since $B_E$ is symmetric and has zero row and column sums, it is singular: $\mathrm{rank}(B_E) < 2n$ and $\det(B_E) = 0$. Furthermore, the condition The condition $\sum_{i,j=1}^{n} e_{ij}(2p^l_{ij} - 1 + 2p^r_{ij} - 1) = 0$ implies that Eq. (16) has infinitely many solutions.

The general solution to the linear system in Eq. (16) can be obtained using Moore–Penrose pseudoinverse matrices. This method will be discussed after the presentation of the multiplicative model for incomplete IV-FPR.

### 3.2. Multiplicative model for incomplete IV-FPR

This section addresses the minimum of the function $\mathcal{M}_4(\mathbf{l}, \mathbf{r})$ in (12), subject to the constraint $\prod_{i=1}^{n} l_i \times \prod_{i=1}^{n} r_i = 1$. We also utilize the function

$$\phi : (0, 1) \to \mathbb{R} \qquad \phi(x) = \ln \frac{x}{1 - x}, \implies \phi^{-1}(x) = \frac{\exp(x)}{1 + \exp(x)},$$

which is continuous and increasing on the interval $(0, 1)$ (Fig. 1). Notice that: $\phi(0.5) = 0$, $\phi(1 - x) = -\phi(x)$, and $\phi\left(\frac{w_i}{w_i + w_j}\right) = \ln w_i - \ln w_j$.

Considering the optimization problem for $\mathcal{M}_4$ (Eq. (12)), we propose a logarithmic goal programming methodology [32] due to its inherent complexity. Thus, instead of $\mathcal{M}_4$, we will search for the minimum of the following function:

$$\begin{aligned} \mathcal{M}_4^* &= \sum_{i,j=1}^{n} e_{ij} \left[ \phi(\frac{l_i}{l_i + r_j}) - \phi(p^l_{ij}) \right]^2 + \sum_{i,j=1}^{n} e_{ij} \left[ \phi(\frac{r_i}{r_i + l_j}) - \phi(p^r_{ij}) \right]^2 \\ &= \sum_{i,j=1}^{n} e_{ij} \left( \ln l_i - \ln r_j - \phi(p^l_{ij}) \right)^2 + \sum_{i,j=1}^{n} e_{ij} \left( \ln r_i - \ln l_j - \phi(p^r_{ij}) \right)^2. \end{aligned}$$

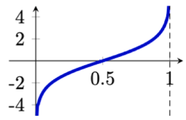



**Fig. 1.** The function $\phi$ that maps values from $(0,1)$ into $\mathbb{R}$.

Define $x_i = \ln l_i$ and $y_i = \ln r_i$ for $i = 1, \dots, n$, with the goal of minimizing:

$$\mathcal{L}_4(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{i,j=1}^{n} e_{ij}[\left(x_i - y_j - \phi(p^l_{ij})\right)^2 + \left(y_i - x_j - \phi(p^r_{ij})\right)^2].$$

Taking the derivative of $\mathcal{L}_4$ with respect to the variables $l_k$, we have:

$$\begin{aligned}\frac{\partial \mathcal{L}_4}{\partial l_k} &= 4l_k \sum_{j=1}^{n} e_{kj} - 4\sum_{j=1}^{n} e_{kj} r_j - 2\sum_{j=1}^{n} e_{kj}(\phi(p^l_{kj}) - \phi(p^r_{jk})) \\ &= 4l_k \sum_{j=1}^{n} e_{kj} - 4\sum_{j=1}^{n} e_{kj} r_j - 4\sum_{j=1}^{n} e_{kj}(\phi(p^l_{kj})).\end{aligned}$$

Similarly, $\frac{\partial \mathcal{L}_4}{\partial r_k} = 4r_k \sum_{j=1}^{n} e_{kj} - 4\sum_{j=1}^{n} e_{kj} l_j - 4\sum_{j=1}^{n} e_{kj}\phi(p^r_{kj})$. Therefore,

$$\nabla \mathcal{L}_4 = 4B_E \cdot \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} - 4\begin{pmatrix}\mathbf{u}_l\\ \mathbf{u}_r\end{pmatrix}, \tag{17}$$

where $B_E$ is defined in Eq. (16), $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$ are the unknown vectors, while $\mathbf{u}_l$ and $\mathbf{u}_r$ are the known vectors defined by:

$$\mathbf{u}_l = \begin{pmatrix}\sum_{j=1}^{n} e_{1j}\phi(p^l_{1j})\\ \dots \\ \sum_{j=1}^{n} e_{nj}\phi(p^l_{nj})\end{pmatrix}; \qquad \mathbf{u}_r = \begin{pmatrix}\sum_{j=1}^{n} e_{1j}\phi(p^r_{1j})\\ \dots \\ \sum_{j=1}^{n} e_{nj}\phi(p^r_{nj})\end{pmatrix}. \tag{18}$$

In the following section, we employ the pseudo-inverse matrix method to find the solutions of the models.

## 4. Ranking models based on pseudoinverse matrix

Penrose [37] proved that for any matrix $A \in \mathbb{R}_{m\times n}$, there exists a unique matrix $X \in \mathbb{R}_{n\times m}$ satisfying the following system of matrix equations:

$$\begin{array}{lll} 1.(AX)A = A & 3.(AX)^T = AX & (AX \text{is Hermitian});\\ 2.X(AX) = X & 4.(XA)^T = XA & (XA \text{is Hermitian});\end{array} \tag{19}$$

The unique solution $X = A^+$ is called the *Moore-Penrose inverse* or *pseudoinverse matrix* of $A$. The equation $A\mathbf{w} = b$ has a solution if and only if $AA^+b = b$ and the general solution is $\mathbf{w} = A^+b + (I - A^+A)y$, where $y$ is an arbitrary vector [34]. Moreover, Kwiesielewicz [26] proved that if $A$ is an $n \times n$ matrix of rank $n-1$, then the equation $A\mathbf{x} = b$ has a general solution given by:

$$\mathbf{x}^* = A^+ \cdot b + (c, \dots, c)^T, \text{where } c \in \mathbb{R} \text{ is an arbitrary constant.}$$

Computing the Moore–Penrose pseudoinverse of a dense $n \times n$ matrix requires $O(n^3)$ arithmetic operations (using SVD or similar methods).

In practice, calculating the exact inverse matrix is often unnecessary, as an approximation typically suffices. According to recent researches—such as those published in Soto-Quiros [38]—computation times have been significantly improved. For $n = 5000$, the calculation requires only 4.4183 seconds with an error margin of approximately $5.55 \times 10^{-17}$; for $n = 20,000$, the same error margin has been achieved in 210.45 seconds.

Notice that the matrix $B_E$ is a block matrix with a unique structural composition with many zero values. Consequently, we believe its pseudoinverse can be computed even more efficiently. Exploring these optimizations remains a primary objective for our future research.

### 4.1. Existence and uniqueness of solutions for the ranking models

Observe that the Additive Model generally admits infinitely many solutions. When $rank(B_E) < 2n-1$, there may exist many opposite rankings satisfying the constraints given by IV-FPR. However, when the matrix $B_E$ has rank $2n-1$, all solutions differ only by an additive constant and are therefore equivalent to ranking the alternatives. Therefore, in order to select a unique value for the weight vector from the family of solutions, one additional constraint must be imposed on the weights.

A common rule is that all real weights should be nonnegative and sum to 1. When weights are intervals (ranges), we generalize this rule: the sum of the midpoints (centers) of the interval weights should equal 1, i.e. $\sum_{i=1}^{n} \frac{l_i+r_i}{2} = 1$, which is equivalent to $\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} r_i = 2$, where $[l_i, r_i]$ denotes the interval-valued weight of alternative $i$.

Based on the results mentioned above in Kwiesielewicz [26], we can prove the following theorem on the conditions that ensure Eq. (16) has a unique solution.

**Theorem 1.** *If $rank(B_E) = 2n-1$, the system (16) has a unique solution:*

$$\begin{pmatrix} \mathbf{l} \\ \mathbf{r} \end{pmatrix} = B_E^{+} \cdot \begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix} + (1/n, \dots, 1/n)^T \tag{20}$$

*satisfying the condition: $\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} r_i = 2$.*

**Proof.** The proof is based on the fact that if $B_E$ has rank $2n-1$, the system of Eq. (16) has a general solution given by:

$$\begin{pmatrix} \mathbf{l} \\ \mathbf{r} \end{pmatrix} = B_E^{+} \cdot \begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix} + (c, \dots, c)^T,$$

where $c \in \mathbb{R}$ is an arbitrary constant [26].

It is also observed that the sum of the entries of the vector $\begin{pmatrix} \mathbf{l}_0 \\ \mathbf{r}_0 \end{pmatrix} = B_E^{+} \cdot \begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix}$ is zero. Consequently, the condition $\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} r_i = 2$ leads to $c = 1/n$. □

However, simply setting the constant shift to $1/n$ (i.e., centering the weights around the uniform distribution) does not guarantee that all derived interval weights remain nonnegative ($l_i \geq 0$ and $r_i \geq 0$ for all $i$). In this paper, we propose a different normalization approach by imposing the following constraint on the interval bounds: $\min\{l_1, \dots, l_n, r_1, \dots, r_n\} = 0$.

Let $\begin{pmatrix} \mathbf{l}_0 \\ \mathbf{r}_0 \end{pmatrix} = B_E^{+} \cdot \begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix}$, and $c_{\min} = -\min\{l_1^0, \dots, l_n^0, r_1^0, \dots, r_n^0\}$, we obtain:

$$\begin{pmatrix} \mathbf{l} \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} \mathbf{l}_0 \\ \mathbf{r}_0 \end{pmatrix} + c_{\min}(1, \dots, 1)^T. \tag{21}$$

Furthermore, inspecting the structure of the $\mathcal{A}_4$ function (Eq. (11)):

$$\mathcal{A}_4(\mathbf{l}, \mathbf{r}) = \|E(L_{\mathbf{P}} - A^l(\mathbf{w}))\|_2^2 + \|E(A^r(\mathbf{w}) - U_{\mathbf{P}})\|_2^2 + \frac{1}{2}\sum_{i=1}^{n}(r_i - l_i)^2,$$

we observe that the final term acts as a regularization factor, analogous to Ridge regression. Given these properties, it is no longer necessary to strictly enforce the constraint $l_i, r_i \leq 1$ for $i = 1, \dots, n$.

In the case of a multiplicative model, solving the Eq. (17) gives us:

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = B_E^{+} \cdot \begin{pmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{pmatrix} + (c, \dots, c)^T \implies l_i = \exp(x_i), r_i = \exp y_i.$$

One can notice the fact that if $c = 0$, then $\sum x_i + \sum y_i = 0$. It implies that $l_i, r_i > 0$ for all $i \in \{1, \dots, n\}$ and $l_1 \cdot l_2 \dots l_n \cdot r_1 \dots r_n = 1$. Thus, $c = 0$ is also a good normalization condition for the multiplicative model.

Another critical consideration involves the acceptance criteria for the final solution. It is evident that not all incomplete IV-FPR can be successfully transformed into a valid ranking, even when a marginal level of inconsistency is permitted. Furthermore, minimizing the objective function ($\mathcal{A}_4(\mathbf{l}, \mathbf{r})$ or $\mathcal{M}_4(\mathbf{l}, \mathbf{r})$) does not strictly guarantee the requirement $l_i \leq r_i$ for all $i \in \{1, \dots, n\}$. To address these constraints, we propose the following dual acceptance criteria:

- Rank condition: $Rank(B_E) = 2n - 1$
- Minimization of the objective function: The value of $\mathcal{A}_4(\mathbf{l}, \mathbf{r})$ or $\mathcal{M}_4(\mathbf{l}, \mathbf{r})$ must remain below a defined threshold to ensure overall model fit.
- Feasibility of interval bounds: The number of indices $i$ where the logical violation $l_i > r_i$ occurs must be minimal (or zero), ensuring the resulting weights represent valid intervals.

### 4.2. The additive and multiplicative ranking models for incomplete IV-FPR

The additive model for the incomplete IV-FPR is presented as follows:

**Model AAI**: Additive Approximation model for Incomplete IV-FPR

*Input* : An incomplete IV-FPR $(\mathbf{P},E), \epsilon_{\min}, n_{\min}$

1. Create the $B_E$ using Eq. (15);
2. **if** $rank(B_E) < 2n-1$ **then return**("no unique solution");
3. Calculate its pseudoinverse $B_E^+$;
4. Calculate vector $\mathbf{v}_l$ and $\mathbf{v}_r$ using Eq. (14);
5. Set $\begin{pmatrix}\mathbf{l}_0\\ \mathbf{r}_0\end{pmatrix} = B_E^+ \cdot \begin{pmatrix}\mathbf{v}_l\\ \mathbf{v}_r\end{pmatrix}$ and $c_{\min} = -\min\{l_1^0, \dots, l_n^0, r_1^0, \dots, r_n^0\}$;
6. Set $\begin{pmatrix}\mathbf{l}\\ \mathbf{r}\end{pmatrix} = \begin{pmatrix}\mathbf{l}_0\\ \mathbf{r}_0\end{pmatrix} + c_{\min}(1, \dots, 1)^T$;
7. **if** $\mathcal{A}_4(\mathbf{l},\mathbf{r}) \le \epsilon_{\min}$ and $|\{i : l_i > r_i\}| \le n_{\min}$ **then return** $(\mathbf{l},\mathbf{r})$ **otherwise return**("no solution");

Similarly to **AAI**, the multiplicative model **MAI** is as follows:

**Model MAI**: Multiplicative Approximation model for Incomplete IV-FPR

*Input* : An incomplete IV-FPR $(\mathbf{P}, E), \epsilon_{\min}, n_{\min}$

1. Create $B_E$ using Eq. (16)
2. **if** $rank(B_E) < 2n-1$ **then return**("no unique solution");
3. Calculate its pseudoinverse $B_E^+$;
4. Calculate vector $\mathbf{u}_l$ and $\mathbf{u}_r$ using Eq. (18);
5. Set $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix} = B_E^+ \cdot \begin{pmatrix}\mathbf{u}_l\\ \mathbf{u}_r\end{pmatrix}$ and $l_i = \exp(x_i), r_i = \exp(y_i)$;
6. **if** $\mathcal{M}_4(\mathbf{l},\mathbf{r}) \le \epsilon_{\min}$ and $|\{i : l_i > r_i\}| \le n_{\min}$ **then return** $(\mathbf{l},\mathbf{r})$ **otherwise return**("no solution");

### 4.3. Illustrated example

For example, the incomplete pairwise comparison of 5 options $A_1, A_2, A_3, A_4, A_5$ is given in the following incomplete IV-FPR $\mathbf{P}$.

$$\mathbf{P} = \begin{pmatrix} [0.50, 0.50] & [0.38, 0.77] & [0.45, 0.81] & \square & \square \\ [0.23, 0.62] & [0.50, 0.50] & [0.25, 0.90] & \square & \square \\ [0.19, 0.55] & [0.10, 0.77] & [0.50, 0.50] & [0.39, 0.64] & [0.35, 0.58] \\ \square & \square & [0.36, 0.61] & [0.50, 0.50] & \square \\ \square & \square & [0.42, 0.65] & \square & [0.50, 0.50] \end{pmatrix}$$

In this matrix, the intervals indicate the set of possible degrees to which one object is preferred over another, while the empty cells (or missing entries) denote unknown comparison values. The lower bound and upper bound of $\mathbf{P}$ are:

$$L_{\mathbf{P}} = \begin{pmatrix} 0.50 & 0.38 & 0.45 & 0 & 0 \\ 0.23 & 0.50 & 0.25 & 0 & 0 \\ 0.19 & 0.10 & 0.50 & 0.39 & 0.35 \\ 0 & 0 & 0.36 & 0.50 & 0 \\ 0 & 0 & 0.42 & 0 & 0.50 \end{pmatrix}; U_{\mathbf{P}} = \begin{pmatrix} 0.50 & 0.77 & 0.81 & 0 & 0 \\ 0.62 & 0.50 & 0.90 & 0 & 0 \\ 0.55 & 0.77 & 0.50 & 0.64 & 0.58 \\ 0 & 0 & 0.61 & 0.50 & 0 \\ 0 & 0 & 0.65 & 0 & 0.50 \end{pmatrix}$$

respectively. Then, the masking matrix is

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \Longrightarrow S = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

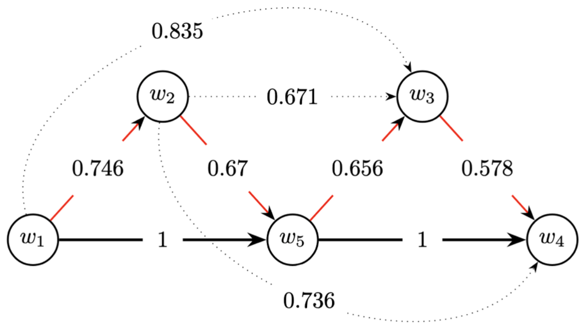



**Fig. 2.** An example of illustration the probabilistic comparisons between the intervals.

The matrix $B_E$ is presented in Fig. 3. Its pseudoinverse is

$$B_E^+ = \begin{pmatrix} 0.351 & 0.017 & -0.04 & -0.113 & -0.113 & 0.076 & 0.076 & 0.0 & -0.127 & -0.127 \\ 0.017 & 0.351 & -0.04 & -0.113 & -0.113 & 0.076 & 0.076 & 0.0 & -0.127 & -0.127 \\ -0.04 & -0.04 & 0.181 & -0.054 & -0.054 & 0.0 & 0.0 & -0.021 & 0.014 & 0.014 \\ -0.113 & -0.113 & -0.054 & 0.558 & -0.109 & -0.127 & -0.127 & 0.014 & 0.202 & -0.131 \\ -0.113 & -0.113 & -0.054 & -0.109 & 0.558 & -0.127 & -0.127 & 0.014 & -0.131 & 0.202 \\ 0.076 & 0.076 & 0.0 & -0.127 & -0.127 & 0.351 & 0.017 & -0.04 & -0.113 & -0.113 \\ 0.076 & 0.076 & 0.0 & -0.127 & -0.127 & 0.017 & 0.351 & -0.04 & -0.113 & -0.113 \\ 0.0 & 0.0 & -0.021 & 0.014 & 0.014 & -0.04 & -0.04 & 0.181 & -0.054 & -0.054 \\ -0.127 & -0.127 & 0.014 & 0.202 & -0.131 & -0.113 & -0.113 & -0.054 & 0.558 & -0.109 \\ -0.127 & -0.127 & 0.014 & -0.131 & 0.202 & -0.113 & -0.113 & -0.054 & -0.109 & 0.558 \end{pmatrix}$$

In this case, the vectors $\mathbf{v}_l$ and $\mathbf{v}_r$ are as follows:

$$\begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix} = (-0.34, -1.04, -1.94, -0.28, -0.16, 1.16, 1.04, 1.08, 0.22, 0.3)^T.$$

Using the model **(AAI)**, i.e., $\begin{pmatrix} \mathbf{l} \\ \mathbf{r} \end{pmatrix} = B_E^+ \cdot \begin{pmatrix} \mathbf{v}_l \\ \mathbf{v}_r \end{pmatrix} + (c, \dots, c)^T$ the output interval-valued weight vector is calculated as follows:

$$\mathbf{l} = (0.092, -0.141, -0.288, -0.137, -0.03)^T + (c, \dots, c)^T;$$
$$\mathbf{r} = (0.273, 0.233, 0.114, -0.104, -0.011)^T + (c, \dots, c)^T.$$

With $c = 0.288$, we found the final interval weight vectors:

$$w_1 = [0.38; 0.561], \quad w_2 = [0.147, 0.521], \quad w_3 = [0, 0.402]$$
$$w_4 = [0.151, 0.183], \quad w_5 = [0.257, 0.277].$$

It is evident that $w_1 > w_5 > w_4$ because the intervals do not overlap. Based on the centers of these intervals, it follows that $w_2$ should be positioned between $w_1$ and $w_5$, and $w_3$ between $w_5$ and $w_4$. We can illustrate the probabilistic comparisons between those intervals (using Eq. (10)) in the diagram provided in Fig. 2. And the final ranking is as follows:

$$w_1 \overset{0.746}{\geq} w_2 \overset{0.67}{\geq} w_5 \overset{0.656}{\geq} w_3 \overset{0.578}{\geq} w_4.$$

It is important to find the condition when $rank(B_E) = 2n - 1$. Let us discuss this in the next section. Before proving the key result, let us review some essential properties of matrices.

### *4.4. The strong connection property of the block matrix $B_E$*

This definition relates to the strong connection (SC) property of a given matrix (Definition 6.2.7, Horn and Johnson [39]).

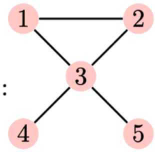
**Definition 4** (Strong Connection property). A matrix $A = (a_{ij}) \in \mathbf{M}_n$ has the SC property if for any distinct $p, q \in \{1, \dots, n\}$, there exists a sequence of distinct integers $k_1 = p, k_2, \dots, k_m = q$ such that $a_{k_i k_{i+1}} \neq 0$ for all $i = 1, \dots, m-1$.

A matrix $M = (m_{ij}) \in \mathbf{M}_n$ represents a directed graph $\mathbf{G}(M) = (V_M, E_M)$ with vertices $V_M = \{1, \dots, n\}$ and edges $E_M = \{(i, j) : m_{ij} \neq 0\}$. A matrix $M$ exhibits the SC property if and only if its corresponding graph $\mathbf{G}(M)$ is strongly connected, which means that there is a path in both directions between every pair of vertices in $\mathbf{G}(M)$. Furthermore, a masking matrix $E$ is the adjacency matrix for both $\mathbf{G}(E)$ and $\mathbf{G}(A_E)$.

Following (Definition 6.1.9 [39]), a matrix $A = (a_{ij}) \in \mathbf{M}_n$ is **diagonally slightly dominant** if: (1) $|a_{ii}| \geq R_i(A)$ for all rows $i$, where $R_i(A) = \sum_{j \neq i} |a_{ij}|$ ; and (2) $|a_{kk}| > R_k(A)$ for at least one row $k$.

$$E = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$\mathbf{G}(E)$ :




$$B_E^{10} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$\mathbf{G}(B_E)$ :

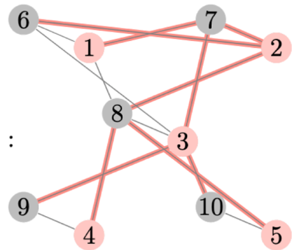



**Fig. 3.** An example of matrices $E$, $B_E$, $B_E^{10}$, graph $\mathbf{G}(E)$, and the spanning tree of $\mathbf{G}(B_E)$.

**Lemma 1** (Corollary 6.2.9, Horn and Johnson [39]). *If $A = (a_{ij}) \in \mathbf{M}_n$ satisfies property SC and the condition of being diagonally slightly dominant, then $A$ is nonsingular.*

The preceding results lead to the following theorem.

**Theorem 2.** *If the masking matrix $E$ has the SC property, then $rank(B_E) = 2n - 1$.*

**Proof.** Recall that $B_E = (b_{ij}) \in \mathbf{M}_{2n}$ (see Eq. (16)) has the following properties:

**P1:** $b_{i,i+n} = b_{i+n,i} = -1$ for all $i = 1, \dots, n$;

**P2:** if $e_{ij} = 1$ for some $i, j$ then $b_{i,j+n} = b_{j,i+n} = b_{i+n,j} = b_{j+n,i} = -1$.

Due to the property that the entries in each row and each column of $B_E$ sum to zero, it follows that the determinant of $B_E$ is zero, and its rank is less than 2n. The properties **P1** and **P2** imply that if $e_{xy} = 1$ for some $x \neq y$, then $b_{x,x+n} = b_{x+n,y} = -1$, indicating that $[x, x+n, y]$ forms a path in $\mathbf{G}(B_E)$.

Given a symmetric matrix $E$ with the SC property, its associated graph $\mathbf{G}(E)$ is undirected and connected. Moreover, $\mathbf{G}(B_E)$ is also an **undirected** and **connected** graph.

Since $\mathbf{G}(E)$ is connected, for any $i, j \in \{1, \dots, n\}$, there exists a path of length $l \leq n$:

$$[i = p_0 \to p_1 \to \cdots \to p_{l-1} \to p_l = j]$$

such that $e_{p_k, p_{k+1}} = 1$ for all $k = 0, \dots, l-1$ and $p_1, \dots, p_{l-1} \in \{1, \dots, n\}$. We can notice that the path

$$[\overset{i}{p_0} \to \overset{i+n}{p_0 + n} \to p_1 \to p_1 + n \cdots p_{l-1} + n \to \overset{j}{p_l} \to \overset{j+n}{p_l + n}]$$

connects the node $i$ with node $j + n$, passing through nodes $i + n$ and $j$, implying $\mathbf{G}(B_E)$ is a connected graph.

Let $T_v$ be a spanning tree of $\mathbf{G}(B_E)$ and let $v$ be a leaf of $T$. Without loss of generality, we assume that $v = 2n$. Let $B_E^{2n} \in \mathbf{M}_{2n-1}$ be the submatrix of $B_E$ obtained by removing the $2n$th row and column. We will show that the determinant of $B_E^{2n}$ is non-zero and that it has the following properties:

- **$B_E^{2n}$ has property SC:** As the elimination of any leaf from a tree maintains its structural integrity, if we denote the tree resulting from the removal of a node $v$ and its connecting edges by $T^v$, then $T^{2n}$ serves as the spanning tree for $\mathbf{G}(B_E^{2n})$. Fig. 3 illustrates this idea.
- **$B_E^{2n}$ is a diagonally slightly dominant matrix:** this is true since the row sums of $B_E = (b_{ij})$ are zero; hence,

$$b_{ii} = R_i(B_E) = \sum_{j \neq i}^{n} |b_{ij}| = \sum_{j=1}^{n} e_{ij} \geq \sum_{j \neq n} e_{ij} = R_i(B_E^{2n}), \text{ for all } i = 1, \dots, n.$$

Additionally, the $2n^{th}$ column of $E$ contains at least one non-zero entry. Assume $e_{kn} = 1$ for some $k \in \{1, \dots, n-1\}$. Then,

$$R_k(B_E^{2n}) = R_k(B_E) - e_{kn} < R_k(B_E) = b_{kk}.$$

According to Lemma 1, $B_E^{2n}$ is nonsingular; therefore, $rank(B_E) = 2n - 1$.
□

In the next section, we will discuss the special case where the model's input (16) is an incomplete FPR.

### 4.5. Optimal models for incomplete FPRs

If the model's input is an FPR (meaning IV-FPR $\mathbf{P} = ([p^l_{ij}, p^r_{ij}]) \in \mathbf{I}_n$, where $p^l_{ij} = p^r_{ij}$). The incomplete FPR is denoted by a pair $(\mathbf{R}, E)$, where $E$ is a masking matrix, and $\mathbf{R} = (p_{ij})$ satisfies the conditions:

$$p^l_{ij} = p^r_{ij} = p_{ij} \in [0,1], \quad p_{ii} = 0.5, \quad p_{ij} = 1 - p_{ji} \quad \text{for all } i, j = 1, \dots, n.$$

We now consider the additive and multiplicative models for this special case.

#### 4.5.1. Additive model for incomplete FPR

In this case, the two vectors $\mathbf{v}_l$ and $\mathbf{v}_r$ are the same. The system (13) can be expressed in the equivalent form:

$$\begin{cases} (S+E)(\mathbf{l}-\mathbf{r}) = 0 \\ (S-E)(\mathbf{l}+\mathbf{r}) = 2\mathbf{v}_l. \end{cases} \tag{22}$$

Let $A_E = S - E$, where $S = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ and $\sigma_k = \sum_{j=1}^n e_{kj}$ is the sum of entries in the $k$th row of $E$, i.e.

$$A_E = S - E = \begin{pmatrix} \sum_{j\neq 1} e_{1j} & \dots & -e_{1n} \\ \dots & \ddots & \dots \\ -e_{n1} & \dots & \sum_{j\neq n} e_{nj} \end{pmatrix}. \tag{23}$$

Because $A_E$ is a singular, symmetric matrix with zero row and column sums, and $\mathbf{v}_l$ is a real vector with entries summing to zero, system (22) has infinitely many solutions.

**Corollary 1.** *If $p^l_{ij} = p^r_{ij}$ for all $i, j = 1, \dots, n$ such that $e_{ij} = 1$, then the optimal solution of (11) is given by*

$$\mathbf{l}^* = \mathbf{r}^* = A_E^+ \cdot \mathbf{v}_l + (c, \dots, c)^T,$$

*where $A_E = S - E$.*

**Proof.**

Recall that the Levy–Desplanques theorem [39] states that a strictly diagonally dominant matrix is non-singular. In other words, given an $n \times n$ matrix $A$, if for every row $i$, $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, then $\det A \neq 0$.

Let $S + E = (a_{ij})$, then $a_{ij} = e_{ij} \in \{0, 1\}$ for all $i \neq j$, and

$$a_{ii} = \sum_{j\neq i} e_{ij} + 2 > \sum_{j\neq i} e_{ij} = \sum_{j\neq i} |a_{ij}|, \text{ for all } i = 1, \dots, n.$$

Since $S + E$ is diagonally strictly dominant, $det(S + E) \neq 0$. Thus, $\mathbf{l} = \mathbf{r}$ and $(S - E)\mathbf{l} = \mathbf{v}_l$. Therefore, the general solution is

$$\mathbf{l}^* = \mathbf{r}^* = A_E^+ \cdot \mathbf{v}_l + (c, \dots, c)^T,$$

where $c \in \mathbb{R}$ is an arbitrary constant [26]. Clearly, the vector $\mathbf{w}_0 = A_E^+ \cdot \mathbf{v}_l = (w^0_1, \dots, w^0_n)$ has a zero sum for its entries. As a result, the condition $w_1, \dots, w_n \geq 0$ implies that $c = -\min\{w^0_1, \dots, w^0_n\}$. □

The above corollary is equivalent to Theorem 4 in [9] and represents a special case of Theorem 1, where both the input matrix and output vector consist of real numbers only. In other words, Theorem 1 serves as a valid and consistent extension of Theorem 4 in [9] to the case of incomplete IV-FPRs and IV-WVs.

The additive model for incomplete FPRs is as follows.

| **Model AAI***: the additive approximation model for incomplete FPR $(R, E)$ |
|---|
| 1. Create the matrix $A_E = S - E$ |
| 2. Calculate its pseudoinverse $A_E^+$ and and vector $\mathbf{v}_l$ using Eq. (14); |
| 3. Return $\mathbf{w} = A_E^+ \cdot \mathbf{b}_{R,E} + (c, \dots, c)^T$, using appropriate value of $c$. |

Moreover, if $\mathbf{R}$ is a complete FPR, then $E = \mathbf{1}$ is the $n \times n$ all matrix with all entries equal 1. In this case we can prove that the proposed model $\mathbf{AAI}^*$ derives the optimal WV $\mathbf{w} = (w^*_1, \dots, w^*_n))$ defined by

$$w^*_k = \frac{2R_k(\mathbf{R}) + 1}{n} - 1 \quad \text{for } k = 1, \dots, n. \tag{24}$$

where $R_k(\mathbf{R}) = \sum_{j=1}^n p_{kj}$ denote the sum of values in the $k$th row of the matrix $\mathbf{R}$. This fact means the alternatives can be ranked by $R_k(\mathbf{R})$.

#### 4.5.2. relationship between **AAI** and $\mathbf{AAI}^*$

The ranking of any sequence of intervals can be approached through various methods, but the basis of all comparisons lies in the intervals' centers and widths.

**Theorem 3.** *For a given incomplete IV-FPR* $(\mathbf{P}, E)$*, the ranking list obtained by solving the system (13) and sorting the resulting intervals by their centers is equivalent to the ranking method* $\mathbf{AAI}^*$ *for the incomplete FPR* $(\mathbf{R}, E)$*, where* $\mathbf{R} = (r_{ij})$ *is defined by* $r_{ij} = \dfrac{p^l_{ij} + p^r_{ij}}{2}$.

**Proof.** For each interval $[l_i, r_i]$, we denote its center as $m_i = \dfrac{l_i + r_i}{2}$ and its width as $w_i = r_i - l_i$. The system described by Eq. (13) is equivalent to

$$(S - E) \cdot (\mathbf{l} + \mathbf{r}) = \mathbf{v}_l + \mathbf{v}_r \implies A_E \cdot \mathbf{m} = \frac{\mathbf{v}_l + \mathbf{v}_r}{2} \tag{25}$$

$$(S + E) \cdot (\mathbf{r} - \mathbf{l}) = \mathbf{v}_r - \mathbf{v}_l \implies (S + E) \cdot \mathbf{w} = \mathbf{v}_r - \mathbf{v}_l, \tag{26}$$

where $\mathbf{m} = (m_1, \dots, m_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$.

The first equation relates to the centers of the IV-WVs, while the second pertains to their widths. Therefore, solving the first equation, given by:

$$A_E \cdot \mathbf{m} = \frac{\mathbf{v}_l + \mathbf{v}_r}{2} \implies \mathbf{m} = A_E^+ \cdot \frac{\mathbf{v}_l + \mathbf{v}_r}{2} + [c, \dots, c]^T$$

is sufficient for ranking the intervals based on their centers. □

Similarly, we also consider the proposed multiplicative model for incomplete FPRs.

#### 4.5.3. Multiplicative model for incomplete FPRs

The multiplicative model for incomplete FPR can be also derived from **MAI**. If $(\mathbf{R}, E)$ is an incomplete FPR, where $\mathbf{R} = (r_{ij})$, then by setting $x_i = \ln w_i$ we can prove that the optimal WV $(x_1, \dots, x_n)$ should satisfy equation:

$$A_E \cdot (x_1, \dots, x_n)^T - (c_1, \dots, c_n)^T = 0,$$

where $c_k = \sum_{j=1}^n e_{kj}\phi(r_{kj})$. If $A_E$ has rank $n - 1$, the optimal WV can be calculated:

$$\mathbf{w} = (e^{x_1}, \dots, e^{x_n}), \text{ where } \mathbf{x} = (x_1, \dots, x_n) = A_E^+ \cdot \mathbf{c}_{R,E} + (c, \dots, c)^T.$$

| **Model** $\mathbf{MAI}^*$ : the multiplicative model for incomplete FPR $(R, E)$ |
|---|
| 1. Create $A_E = S - E$, using Eq. (23) and calculate its pseudoinverse $A_E^+$; |
| 2. Calculate vector $\mathbf{c}_{R,E} = (c_1, \dots, c_n)$, where $c_k = \sum_{j=1}^n e_{kj}\phi(r_{kj})$; |
| 3. Calculate $\mathbf{x} = A_E^+ \cdot \mathbf{c}_{R,E} = (x_1, \dots, x_n)^T$; |
| 4. Return $\mathbf{w} = (\exp(x_1), \dots, \exp(x_n))^T$. |

If $R = (r_{ij})_{n\times n}$ is a full FPR, then the weight vector $\mathbf{w}^* = (w_1^*, \dots, w_n^*)$, where

$$w_k^* = \left(\prod_{j=1}^n \frac{r_{kj}}{1 - r_{kj}}\right)^{1/n}, \text{ for} k = 1, \dots, n, \tag{27}$$

minimizes the function $\mathcal{M}^*_{R,E}$, moreover $\prod_{k=1}^n w_k^* = 1$.

## 5. Ranking the top tennis players of all-time

Many studies analyze sports outcomes, including tennis predictions: Kovalchik [40] forecasted event results, Lisi and Zanella [41] used logistic regression for win probabilities, and Elo [42]'s method outperformed others, as confirmed by Vaughan Williams et al. [43]'s 2018 Wimbledon analysis. Gu and Saaty [44] combined objective and subjective data to predict tournaments, validated on the 2015 US Open. Ramón et al. [45] ranked players via data envelopment analysis. Bozóki et al. [23] and Temesi et al. [21] ranked players using incomplete PCMs from win-loss ratios with eigenvalue and logarithmic least squares methods.

A key contribution is a technique that transforms this data into an incomplete IV-FPR using Bernoulli distribution and Bayesian statistics.

### 5.1. Data acquisition

This section outlines the raw data acquisition methods from WTA and ATP websites and the ranking evaluation methodology. All datasets described in Sections 5.1 and 5.2 are also available in the supplementary material.

**Table 2**
Head-to-head results among top WTA players from 1973 to early July 2025, with total career wins, losses, and win-loss ratio presented in the final three columns.

| Players | ① | ② | ③ | ④ | ⑤ | ⑥ | ⑦ | ⑧ | ⑨ | ⑩ | ⑪ | ⑫ | ⑬ | ⑭ | ⑮ | ⑯ | ⑰ | ⑱ | ⑲ | ⑳ | ㉑ | ㉒ | ㉓ | ㉔ | ㉕ | ㉖ | ㉗ | ㉘ | ㉙ | W | L | W/L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ① Goolagong | | 12 | 12 | 4 | | | | | | | | | | | | | | | | | | | | | | | | | | 704 | 165 | 4.27 |
| ② Evert | 26 | | 37 | 9 | 6 | 1 | 2 | | | | | | | | | | | | | | | | | | | | | | | 1304 | 144 | 9.06 |
| ③ Navratilova | 15 | 43 | | 20 | 9 | 12 | 7 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | 1442 | 219 | 6.58 |
| ④ Austin | 4 | 8 | 13 | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | 335 | 90 | 3.72 |
| ⑤ Graf | | 7 | 9 | 1 | | 28 | 10 | 10 | 8 | 1 | 7 | 3 | 1 | 1 | | | | | | | | | | | | | | | | 900 | 115 | 7.83 |
| ⑥ Sanchez | | 1 | 3 | | 8 | | 3 | 6 | 7 | 1 | 2 | 3 | 4 | 0 | 1 | | | | | | | | | | | | | | | 759 | 295 | 2.57 |
| ⑦ Seles | | 1 | 10 | | 5 | 20 | | 9 | 3 | 3 | 5 | 1 | 1 | 1 | 4 | | | 1 | | | | | | | | | | | | 595 | 122 | 4.88 |
| ⑧ Capriati | | | 1 | | 1 | 4 | 5 | | 3 | 4 | 4 | 0 | 7 | 3 | 2 | | | 1 | | | | | | | | | | | | 430 | 176 | 2.44 |
| ⑨ Davenport | | | 0 | | 6 | 5 | 10 | 9 | | 12 | 14 | 14 | 4 | 8 | 5 | 2 | 4 | 1 | 1 | | 1 | 1 | | | | | | | | 753 | 194 | 3.88 |
| ⑩ Mauresmo | | | | | 0 | 2 | 2 | 7 | 4 | | 7 | 3 | 2 | 7 | 6 | 4 | 6 | 3 | 6 | 0 | | 0 | | | | | | | | 545 | 227 | 2.40 |
| ⑪ Hingis | | | | | 2 | 18 | 15 | 5 | 11 | 8 | | 11 | 6 | 4 | 2 | 2 | 0 | 1 | 1 | 0 | | 2 | | | | | | | | 548 | 135 | 4.06 |
| ⑫ WilliamsV | | | | | 2 | 6 | 9 | 4 | 13 | 5 | 10 | | 12 | 6 | 7 | 3 | 7 | 3 | 9 | 6 | 3 | 7 | 3 | 1 | 0 | 4 | 1 | | | 818 | 278 | 2.94 |
| ⑬ WilliamsS | | | | | 1 | 3 | 4 | 10 | 10 | 10 | 7 | 19 | | 7 | 8 | 6 | 10 | 20 | 9 | 18 | 6 | 10 | 10 | 2 | 2 | 3 | 1 | | | 858 | 156 | 5.50 |
| ⑭ Clijsters | | | | | 0 | 4 | 0 | 3 | 9 | 8 | 5 | 7 | 2 | | 13 | 8 | 8 | 5 | 6 | 4 | 0 | 3 | 1 | | | 0 | | | | 523 | 132 | 3.96 |
| ⑮ Henin | | | | | | 0 | 3 | 5 | 7 | 8 | 2 | 2 | 6 | 12 | | 5 | 10 | 7 | 5 | | 2 | 1 | | | | | | | | 525 | 115 | 4.57 |
| ⑯ Safina | | | | | | | | | 1 | 3 | 1 | 1 | 1 | 2 | 1 | | 3 | 3 | 1 | 4 | | 1 | 1 | | | | | | | 360 | 173 | 2.08 |
| ⑰ Jankovic | | | | | | | | | 2 | 1 | 2 | 7 | 4 | 1 | 0 | 4 | | 1 | 3 | 4 | 2 | 5 | 1 | 1 | | 2 | | | | 644 | 370 | 1.74 |
| ⑱ Sharapova | | | | | | | 0 | 0 | 5 | 1 | 2 | 5 | 2 | 4 | 3 | 4 | 8 | | 10 | 8 | 4 | 7 | 7 | 2 | 1 | 3 | 0 | | | 645 | 171 | 3.77 |
| ⑲ Ivanovic | | | | | | | | | 0 | 2 | 1 | 3 | 1 | 0 | 0 | 3 | 9 | 4 | | 3 | 5 | 5 | 3 | 0 | | 0 | | | | 480 | 225 | 2.13 |
| ⑳ Azarenka | | | | | | | | | | 2 | 1 | 3 | 5 | 3 | | 2 | 7 | 7 | 5 | | 10 | 7 | 2 | 4 | 1 | 2 | 1 | 1 | 1 | 638 | 270 | 2.36 |
| ㉑ Kerber | | | | | | | | | 0 | | | 6 | 3 | 1 | 0 | | 4 | 5 | 2 | 1 | | 8 | 6 | 7 | 2 | 3 | 5 | 0 | 1 | 683 | 378 | 1.81 |
| ㉒ Wozniacki | | | | | | | | | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 6 | 4 | 2 | 4 | 8 | | 5 | 6 | 3 | 3 | 2 | 0 | 1 | 655 | 280 | 2.34 |
| ㉓ Halep | | | | | | | | | | | | 4 | 2 | 0 | | 0 | 7 | 2 | 2 | 3 | 6 | 2 | | 8 | 3 | 3 | 4 | 2 | 3 | 580 | 243 | 2.39 |
| ㉔ Pliskova | | | | | | | | | | | | 2 | 2 | | | | 1 | 0 | 5 | 5 | 5 | 4 | 4 | | 2 | 9 | 5 | 0 | 2 | 641 | 372 | 1.72 |
| ㉕ Barty | | | | | | | | | | | | 2 | 0 | | | | | 2 | | 3 | 4 | 0 | 1 | 6 | | 3 | 2 | 2 | 4 | 305 | 102 | 2.99 |
| ㉖ Muguruza | | | | | | | | | | | | 2 | 3 | 1 | | | 3 | 1 | 1 | 3 | 5 | 3 | 4 | 2 | 1 | | 0 | 1 | 2 | 449 | 238 | 1.89 |
| ㉗ Osaka | | | | | | | | | | | | 1 | 3 | | | | | 1 | | 3 | 2 | 1 | 1 | 2 | 2 | 1 | | 1 | 1 | 305 | 175 | 1.74 |
| ㉘ Swiatek | | | | | | | | | | | | | | | | | | | | 5 | 3 | 2 | 2 | 3 | 0 | 0 | 2 | | 8 | 377 | 86 | 4.38 |
| ㉙ Sabalenka | | | | | | | | | | | | | | | | | | | | 5 | 0 | 2 | 2 | 3 | 4 | 1 | 0 | 5 | | 458 | 196 | 2.34 |

### 5.1.1. *WTA data*

Created in 1973 by Billie Jean King, the WTA is a global organization of women's professional tennis, bringing together over 2500 players from nearly 100 countries. WTA head-to-head data offers match statistics, player performances, and updates, filtering options based on time, surface, events, or rounds. Player rankings are based on tournament outcomes, sparking ongoing debates about the criteria for ranking top players over time.

This section estimates the all-time rankings of 29 former and current top-ranked female players. This analysis uses WTA data, similar to that found in Temesi et al. [21], Vo and Nguyen [30], from a dataset updated as of July 04, 2025. This dataset, retrieved from https://www.wtatennis.com/head-to-head/ and detailed in Table 2, includes player's names, head-to-head win/loss stats and career win-loss scores.

Consequently, we expect that the top-ranked players, Iga Swiatek and Aryna Sabalenka, will rise in the all-time ranking list.

This study also corrects several errors found in Temesi et al. [21], especially regarding A. Barty, who retired in March 2022. Her actual win-loss records are: vs. C. Wozniacki 0–3, vs. A. Kerber 4–2, and vs. V. Azarenka 3–1. However, Temesi et al. [21] (from August 2022) incorrectly listed them as 4–3, 3–2, and 1–1.

Table 2 displays a $29 \times 29$ incomplete matrix, where each entry $W_{ij}$ represents the number of matches player $P_i$ won against player $P_j$. Here, $P_i$ refers to the $i$th player, corresponding to the circled number ($i = 1, \ldots, 29$). For example, $P_5$ (S. Graf) defeated $P_6$ (A. Vicario) 28 times, while $P_6$ won 8 times against $P_5$, resulting in a win-loss ratio of 28 to 8.

**Table 3**
Head-to-head win-loss records between ATP top players from 1973 to June 16, 2025, including their career wins, losses, and win-loss ratios in the last three columns.

| Players | ① | ② | ③ | ④ | ⑤ | ⑥ | ⑦ | ⑧ | ⑨ | ⑩ | ⑪ | ⑫ | ⑬ | ⑭ | ⑮ | ⑯ | ⑰ | ⑱ | ⑲ | ⑳ | ㉑ | ㉒ | ㉓ | ㉔ | ㉕ | ㉖ | ㉗ | ㉘ | ㉙ | W | L | W/L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ① Nastase | | 4 | 15 | 7 | 3 | 0 | 0 | | 0 | | | | | | | | | | | | | | | | | | | | | 908 | 334 | 2.72 |
| ② Newcombe | 1 | | 2 | 3 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | 568 | 227 | 2.50 |
| ③ Connors | 12 | 4 | | 8 | 14 | 13 | 0 | 6 | 0 | 0 | 0 | 0 | | | | | | | | | | | | | | | | | | 1274 | 283 | 4.50 |
| ④ Borg | 10 | 2 | 15 | | 7 | 6 | 1 | | | | | | | | | | | | | | | | | | | | | | | 654 | 140 | 4.67 |
| ⑤ McEnroe | 6 | 1 | 20 | 7 | | 15 | 7 | 7 | 2 | 1 | 0 | 2 | | | | | | | | | | | | | | | | | | 883 | 198 | 4.46 |
| ⑥ Lendl | 1 | | 22 | 2 | 21 | | 15 | 13 | 11 | 4 | 3 | 6 | 4 | | | | 0 | | | | | | | | | | | | | 1068 | 242 | 4.41 |
| ⑦ Wilander | 1 | | 5 | 0 | 6 | 7 | | 11 | 3 | | 1 | 2 | 2 | | | 1 | 2 | | | | | | | | | | | | | 571 | 222 | 2.57 |
| ⑧ Edberg | | | 6 | | 6 | 14 | 9 | | 10 | 4 | 6 | 3 | 10 | 1 | 1 | 1 | 3 | | | | | | | | | | | | | 801 | 270 | 2.97 |
| ⑨ Becker | 1 | | 6 | | 8 | 10 | 7 | 25 | | 6 | 7 | 4 | 2 | 3 | 2 | 4 | 2 | | 0 | 1 | | | | | | | | | | 713 | 214 | 3.33 |
| ⑩ Courier | | | 3 | | 2 | 0 | | 6 | 1 | | 4 | 7 | 7 | 0 | 2 | 1 | 0 | 1 | 1 | | | | | | | | | | | 506 | 237 | 2.14 |
| ⑪ Sampras | | | 2 | | 3 | 5 | 2 | 8 | 12 | 16 | | 20 | 9 | 2 | 3 | 11 | 12 | 2 | 3 | 4 | | 1 | 0 | | | | | | | 762 | 222 | 3.43 |
| ⑫ Agassi | | | 2 | | 2 | 2 | 5 | 6 | 10 | 5 | 14 | | 5 | 1 | 3 | 8 | 10 | 7 | 3 | 4 | 2 | 5 | 3 | 0 | | | | | | 870 | 274 | 3.18 |
| ⑬ Muster | | | | | | 1 | 0 | 0 | 1 | 5 | 2 | 4 | | 3 | 4 | 4 | 0 | 0 | 0 | | | | | | | | | | | 625 | 273 | 2.29 |
| ⑭ Rios | | | | | | | | 0 | 2 | 3 | 0 | 2 | 1 | | 5 | 2 | 1 | 2 | 1 | 2 | 1 | 0 | 0 | | | | | | | 391 | 192 | 2.04 |
| ⑮ Moya | | | | | | | | 0 | 2 | 1 | 1 | 1 | 4 | 2 | | 3 | 3 | 3 | 4 | 5 | 6 | 1 | 0 | 2 | 2 | 0 | | | | 575 | 319 | 1.80 |
| ⑯ Kafelnikov | | | | | | | 1 | 2 | 1 | 5 | 2 | 4 | 1 | 6 | 3 | | 3 | 5 | 2 | 1 | 2 | | 4 | | | | | | | 609 | 306 | 1.99 |
| ⑰ Rafter | | | | | | 1 | 1 | 0 | 1 | 3 | 4 | 5 | 3 | 2 | 1 | 2 | | 4 | 1 | 1 | 1 | | 3 | | | | | | | 358 | 191 | 1.87 |
| ⑱ Kuerten | | | | | | | | | | 0 | 1 | 4 | 3 | 2 | 4 | 7 | 4 | | 4 | 1 | 2 | 1 | 2 | | | | | | | 358 | 195 | 1.84 |
| ⑲ Safin | | | | | | | | | 1 | 1 | 4 | 3 | 1 | 3 | 3 | 2 | 0 | 3 | | 7 | 6 | 3 | 2 | 0 | 2 | 1 | | | | 422 | 267 | 1.58 |
| ⑳ Hewitt | | | | | | | | | 0 | | 5 | 4 | | 3 | 7 | 7 | 3 | 3 | 7 | | 6 | 7 | 9 | 4 | 1 | 0 | | | | 616 | 262 | 2.35 |
| ㉑ Ferrero | | | | | | | | | | | | 3 | | 3 | 8 | 1 | 2 | 3 | 6 | 4 | | 0 | 3 | 2 | 1 | 0 | | | | 479 | 262 | 1.83 |
| ㉒ Roddick | | | | | | | | | | | 2 | 1 | | 2 | 4 | | | 1 | 4 | 7 | 5 | | 3 | 3 | 5 | 3 | | | | 612 | 213 | 2.87 |
| ㉓ Federer | | | | | | | | | | | 1 | 8 | | 2 | 7 | 2 | 0 | 1 | 10 | 18 | 10 | 21 | | 16 | 23 | 14 | 3 | | | 1251 | 275 | 4.55 |
| ㉔ Nadal | | | | | | | | | | | | 2 | | | 6 | | | | 2 | 7 | 7 | 7 | 24 | | 29 | 17 | 5 | 2 | 3 | 1080 | 228 | 4.74 |
| ㉕ Djokovic | | | | | | | | | | | | | | | 2 | | | | 0 | 6 | 2 | 4 | 27 | 31 | | 25 | 10 | | 4 | 1145 | 230 | 4.98 |
| ㉖ Murray | | | | | | | | | | | | | | | 2 | | | | 0 | 1 | 3 | 8 | 11 | 7 | 11 | | 0 | 1 | 1 | 739 | 262 | 2.82 |
| ㉗ Medvedev | | | | | | | | | | | | | | | | | | | | | | | 0 | 1 | 5 | 3 | | 2 | 7 | 397 | 168 | 2.36 |
| ㉘ Alcaraz | | | | | | | | | | | | | | | | | | | | | | | | 1 | | 1 | 6 | | 8 | 246 | 61 | 4.03 |
| ㉙ Sinner | | | | | | | | | | | | | | | | | | | | | | | | 0 | 5 | 1 | 8 | 4 | | 281 | 82 | 3.43 |

### 5.1.2. ATP data

Similar to the WTA, the ATP introduced its computerized ranking system on 23 August 1973. The system ranks professional male players according to their results over the previous 52 weeks. Points are awarded based on tournament progress, with higher-tier events such as Grand Slams and Masters 1000 awarding significantly more points. Since 1979, the rankings have been updated weekly. Between 1973 and 2025, 29 different players have reached the world No. 1 ranking, 19 of whom have finished the year as year-end No. 1. The system's strong emphasis on recent performance and tournament prestige directly influences seeding and qualification for major events.

Table 3 presents the win-loss records for 29 top-ranked ATP players retrieved from https://www.atptour.com/en/players/atp-head-2-head.

### 5.1.3. Expert-based ranking and assessment method

To assess the accuracy of the proposed ranking methods, we will utilize expert rankings, such as Jon Crim's list [46], which evaluates top tennis players based on comprehensive career statistics. This dataset includes titles categorized by surface (grass, clay, hard) and event type (Olympic, Grand Slam, etc.), with separate rankings for men and women, each featuring a top 10 and honorable mentions (7 for women, 6 for men).

**Table 4**
The undisputed greatest tennis players of all time according to Jon Crim [46].

| Female players | Male players |
|---|---|
| 1. **Serena Williams** | 1. **Novak Djokovic** |
| 2. **Steffi Graf** | 2. **Rafael Nadal** |
| 3. *Margaret Court* | 3. **Roger Federer** |
| 4. **Martina Navratilova** | 4. *Rod Laver* |
| 5. **Chris Evert** | 5. **Pete Sampras** |
| 6. *Billie Jean King* | 6. **Björn Borg** |
| 7. **Monica Seles** | 7. **Ivan Lendl** |
| 8. **Justine Henin** | 8. **Jimmy Connors** |
| 9. **Evonne Goolagong** | 9. **John McEnroe** |
| 10. **Venus Williams** | 10. **Andre Agassi** |
| 11. *Helen Wills Moody* | 11. *Ken Rosewell* |
| 12. *Susanne Lenglen* | 12. *Fred Perry* |
| 13. **Martina Hingis** | 13. **Stefan Edberg** |
| 14. **Maria Sharapova** | 14. **Boris Becker** |
| 15. **Lindsay Davenport** | 15. **Mats Wilander** |
| 16. *Maureen C. Brinker* | 16. *Arthur Ashe* |
| 17. *Margaret O. DuPont* | |

**Table 5**
Peak Elo ratings of 29 male tennis players who reached world No. 1 in the Open Era. Source: https://www.ultimatetennisstatistics.com/peakEloRatings as of Dec 31, 2024.

| Nr | Players | Peak Elo | | … | | | |
|---|---|---|---|---|---|---|---|
| | | Value | Rank | 15 | Carlos Moya | 2190 | 29 |
| 1 | Ilie Nastase | 2363 | 15 | 16 | Yevgeny Kafelnikov | 2216 | 28 |
| 2 | John Newcombe | 2327 | 18 | 17 | Patrick Rafter | 2255 | 24 |
| 3 | Jimmy Connors | 2521 | 6 | 18 | Gustavo Kuerten | 2281 | 22 |
| 4 | Bjorn Borg | 2622 | 2 | 19 | Marat Safin | 2218 | 27 |
| 5 | John McEnroe | 2583 | 3 | 20 | Lleyton Hewitt | 2295 | 20 |
| 6 | Ivan Lendl | 2518 | 7 | 21 | Juan-Carlos Ferrero | 2225 | 26 |
| 7 | Mats Wilander | 2371 | 13 | 22 | Andy Roddick | 2274 | 23 |
| 8 | Stefan Edberg | 2370 | 14 | 23 | Roger Federer | 2550 | 5 |
| 9 | Boris Becker | 2419 | 9 | 24 | Rafael Nadal | 2552 | 4 |
| 10 | Jim Courier | 2330 | 17 | 25 | Novak Djokovic | 2629 | 1 |
| 11 | Pete Sampras | 2407 | 11 | 26 | Andy Murray | 2500 | 8 |
| 12 | Andre Agassi | 2376 | 12 | 27 | Daniil Medvedev | 2316 | 19 |
| 13 | Thomas Muster | 2287 | 21 | 28 | Carlos Alcaraz | 2348 | 16 |
| 14 | Marcelo Rios | 2241 | 25 | 29 | Jannik Sinner | 2414 | 10 |

Table 4 displays Jon Crim's lists of the greatest tennis players of all time. Pre-Open Era players, italicized in the lists, are excluded from our analysis due to missing WTA and ATP head-to-head data. Consequently, only 11 women and 12 men will be analyzed in the following sections.

The Elo rating system, originally developed for chess Elo [42], has become a widely accepted alternative for ranking performance in sports such as tennis. At its core, the system is self-correcting: a player's rating increases after a victory and decreases after a defeat. The magnitude of this adjustment is determined by the difference between the actual and expected outcomes. When a highly rated favorite wins as anticipated, the rating change is only marginal. In contrast, an upset, where a lower-rated underdog defeats a stronger opponent, produces a substantially larger rating shift. Because these ratings provide a consistent measure of relative skill, they enable historians and fans to compare players' dominance across different eras of tennis history.

Peak Elo rating denotes the highest Elo rating achieved by a player at any point in their career. In sports and gaming, it serves as a widely accepted measure of a player's absolute peak—the point of greatest dominance over their contemporaries. Because the Elo system is cumulative and interdependent, accurate calculation of Peak Elo ratings requires a complete dataset containing the results of all official matches from the professional tour. Table 5 presents the Peak Elo ratings for the 29 male tennis stars considered in this study, published on 31 December 2024 on the Ultimate Tennis Statistics website https://www.ultimatetennisstatistics.com/peakEloRatings.

In the following sections, we will employ Spearman's rank correlation to assess the similarity between the ranking lists produced by our algorithms and those by Jon Crim for both WTA and ATP data. Recall that Spearman's rank correlation uses the Pearson correlation formula on the ranks of variables to evaluate the strength and direction of monotonic relationships. Specifically, if $pos_1(a_i)$ and $pos_2(a_i)$ denote the positions of options $a_i$ in the two ranking lists $\mathbf{L}_1$ and $\mathbf{L}_2$, then the Spearman's correlation coefficient $\rho(\mathbf{L}_1, \mathbf{L}_2)$ is defined as:

$$\rho(\mathbf{L}_1, \mathbf{L}_2) = 1 - \frac{6\sum_{i=1}^{n}(pos_1(a_i) - pos_2(a_i))^2}{n(n^2-1)}. \tag{28}$$

**Table 6**
Expert-based ranking comparison using win-loss data from 29 WTA players. Second column: the positions of top tennis players in the ranking list based on career win-loss ratios. Third column: the rankings of selected players returned by LLSM-K30 in Temesi et al. [21].

| Jon Crim's | WL | [21] | $Matrix_0(2, 0.2)$ | | | | | | $Matrix_2(0.2)$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ranking | | | | M2* | A3* | M3* | AAI | MAI | A2* | M2* | A3* | M3* | AAI | MAI |
| 1. Williams S. | 1 | 1 | 1 | 2 | 1 | 5 | 1 | 1 | 1 | 3 | 1 | 4 | 2 | 2 |
| 2. Graf | 2 | 2 | 2 | 1 | 2 | 4 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 1 |
| 3. Navratilova | 4 | 3 | 6 | 4 | 6 | 1 | 3 | 4 | 6 | 4 | 8 | 3 | 4 | 4 |
| 4. Evert | 8 | 7 | 8 | 6 | 7 | 3 | 5 | 5 | 5 | 2 | 4 | 1 | 3 | 3 |
| 5. Seles | 10 | 9 | 11 | 11 | 5 | 6 | 9 | 8 | 10 | 7 | 9 | 6 | 7 | 6 |
| 6. Henin | 3 | 5 | 4 | 3 | 4 | 7 | 4 | 3 | 3 | 5 | 3 | 7 | 5 | 5 |
| 7. Goolagong | 11 | 11 | 5 | 10 | 11 | 2 | 11 | 11 | 11 | 10 | 11 | 5 | 10 | 8 |
| 8. Williams V. | 6 | 6 | 3 | 5 | 3 | 10 | 6 | 6 | 4 | 8 | 7 | 10 | 9 | 10 |
| 9. Hingis | 5 | 4 | 9 | 8 | 8 | 9 | 10 | 10 | 9 | 6 | 6 | 8 | 8 | 9 |
| 10. Sharapova | 9 | 10 | 10 | 9 | 9 | 11 | 8 | 9 | 7 | 11 | 5 | 11 | 11 | 11 |
| 11. Davenport | 7 | 8 | 7 | 7 | 10 | 8 | 7 | 7 | 8 | 9 | 10 | 9 | 6 | 7 |
| Spearman's $\rho$ | 0.53 | 0.64 | 0.50 | 0.60 | 0.70 | 0.70 | 0.72 | 0.74 | 0.57 | 0.83 | 0.54 | 0.85 | 0.79 | 0.87 |

### 5.2. From raw data to incomplete IV-FPR

In tennis, there are no draws, the criterion for $P_i$ being ranked higher than $P_j$ is $W_{ij} > W_{ji}$. Nevertheless, the infrequent encounters among elite players, particularly those from distinct eras, can result in records like $1/0$, $0/1$, or $1/1$, which might not offer a genuine comparison of their abilities. Vo and Nguyen [30] suggested using a credible interval—rather than a single value—to represent the probability of $P_i$ beating $P_j$ for each pair of players $(P_i, P_j)$. The experiment results showed that credible intervals derived by MAP (Maximum a Posteriori) performed better than credible intervals derived by MLE (Maximum Likelihood Estimator).

The MAP approach for interval generation is briefly outlined as follows: In $n$ matches between players $A$ and $B$, with $w$ wins for $A$ and $l = n - w$ for $B$, outcomes $\mathbf{X} = (X_1, \ldots, X_n)$ follow a Bernoulli process with success probability $p \in [0, 1]$ ($P(X_i = 1) = p$). Using the MAP estimator with a Beta-Bernoulli model, we infer an interval for $p$ that maximizes the posterior probability based on the win-loss ratio.

- **Conjugate Prior:** Assume $p$ follows a Beta distribution: $p \sim \mathcal{B}(a, b)$.
- **Posterior:** After observing $w$ victories and $l$ defeats, the posterior distribution of $p$ becomes $\mathcal{B}(a + w, b + l)$.
- **Conclusion:** $p \in [\beta_{\alpha/2}, \beta_{1-\alpha/2}]$, where $1 - \alpha$ is a predefined credibility level.

Here, $\beta_x$ is the $x$-quantile of $\mathcal{B}(a + w, b + l)$ (i.e., $P(p \leq \beta_x) = x$), Interval $[\beta_{\alpha/2}, \beta_{1-\alpha/2}]$ is called the $1 - \alpha$ credible interval for a Beta-distributed random variable. Values $a$ and $b$ represent prior beliefs about the winning chances of two players, which can be established by one of the following methods:

1. $Matrix_0(k, \alpha)$ **- without prior knowledge:** This method assumes that both players have an equal chance of winning before the match. Thus, we set $a = b = k$, where $k$ and $\alpha$ are preset parameters.
2. **Using career win-loss records:** This method assumes that a higher career win-loss ratio means a better chance of winning. We suggest two ways to use career win-loss records in the ranking process:
   (a) $Matrix_1(\alpha)$: Vo and Nguyen [30] proposed to add a new player to the list called “The Other”. If a player has $w$ career wins and $l$ losses, her record against “The Other” is shown as $w/l : 1$.
   (b) $Matrix_2(\alpha)$: Set the value of $a$ and $b$ by the career win-loss ratio, i.e. the last column of Table 2.

As an example, consider the pair of players S. Graf and A. Sanchez. Starting with the conjugate prior $\mathcal{B}(2, 2)$ and a win-loss record of $28/8$, Graf's winning probability $p$ has a posterior distribution of $\mathcal{B}(30, 10)$, leading to an 80% credible interval of $[0.66, 0.83]$. Using career win-loss records, Graf's conjugate prior is $\mathcal{B}(7.83, 2.57)$, resulting in an 80% confidence interval of $[0.69, 0.85]$.

For the pair S. Graf vs A. Mauresmo, if we start with the conjugate prior $\mathcal{B}(2, 2)$ and a win-loss record of $1/0$, Graf's winning probability $p$ has a posterior distribution of $\mathcal{B}(3, 2)$, leading to an 80% credible interval of $[0.32, 0.86]$.

Using career win-loss records, Graf's conjugate prior is $\mathcal{B}(7.83, 2.4)$, gives an interval of [0.62, 0.92]. Therefore, career win-loss records significantly reduced uncertainty for Graf-Mauresmo but not too much for Graf-Sanchez.

### 5.3. Experiment results for WTA dataset

In [30], the ranking of top WTA players was examined using similar data processing techniques. Nonetheless, the final IV-WV was determined using ranking models **A2***, **A3***, **M2***, and **M3***. These models are modifications of the linear programming models **A2**, **A3**, **M2**, and **M3** from Vo and Nguyen [31], respectively, with constraints are restricted to pairs $(i, j)$ where $e_{ij} = 1$. Fig. 4 shows the IV-WVs derived from the **AAI** and **MAI** models, presented both without and with career win-loss scores.

Table 6 compares rankings for 11 players. Columns $Matrix_0$ and $Matrix_2$ refer to the IV-FPR generation methods presented in Section 5.2. The “WL” presents the ranking based on win/loss ratios from Table 2. Spearman's $\rho$ values appear in the last row. The

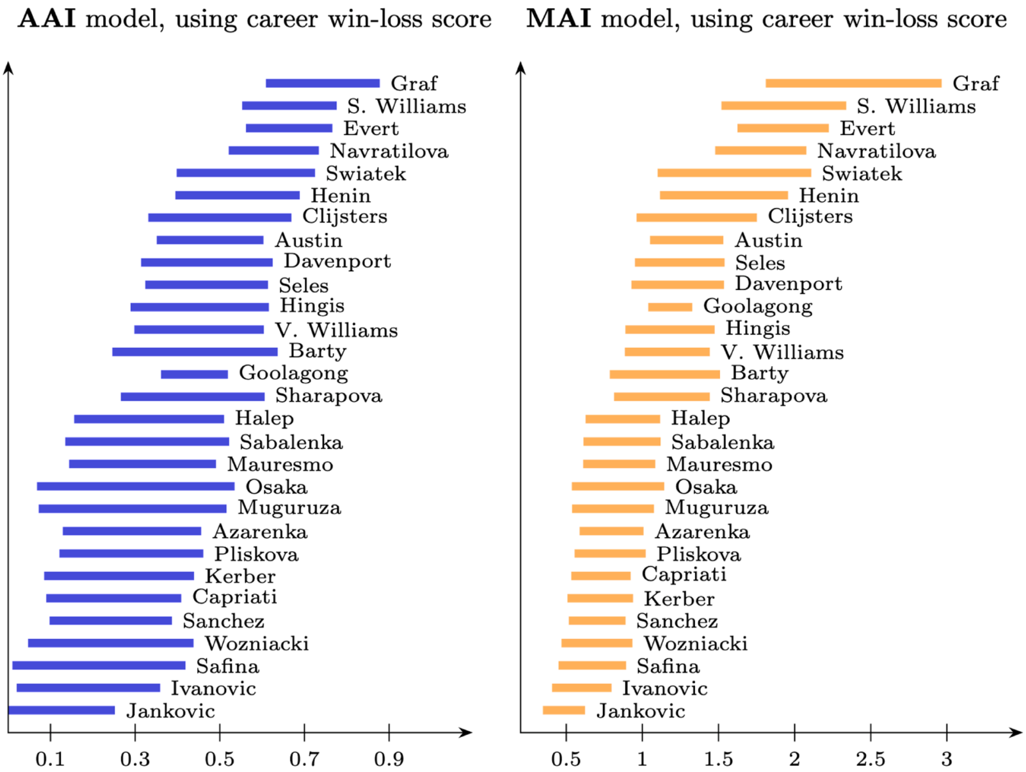



**Fig. 4.** Above: IV-WVs derived by (**AAI**) and (**MAI**) models using WTA win-loss data. Below: Rankings derived from the (**AAI**) model (left) and (**MAI**) model (right) with career win-loss scores. Players are ordered by the center of their interval weights.

results show that using career win-loss records improves ranking accuracy—except for **A3***—and multiplicative models generally outperform additive ones. Models **M3***, **AAI** and **MAI** outperform the method LLSM-K30 in Temesi et al. [21]. The best results come from the **MAI** model with career win-loss data. Fig. 4 (below) presents the illustration of the best ranking returned by the algorithms **AAI** and **MAI**.

A key feature of IV-WVs is that they can produce different rankings. The table below shows how changing the $\lambda$ value in the score function $S_\lambda$ (Eq. (9)) results in different rankings of 3 players, based on interval weights from **AAI**.

| Player | $l_w$ | $r_w$ | $c(w)$ | $d(w)$ | $S_{-0.5}$ | $S_{0.1}$ | $S_{0.8}$ |
|---|---|---|---|---|---|---|---|
| Osaka | 0.215 | 0.603 | 0.409 | 0.388 | −0.3279 | −0.2376 | **−0.1322** |
| Seles | 0.278 | 0.545 | 0.4115 | 0.267 | **−0.2599** | −0.2171 | −0.1672 |
| Sharapova | 0.266 | 0.57 | **0.418** | 0.304 | −0.2601 | **−0.2046** | −0.1399 |

Consequently, IV-WVs represent a class of multiple rankings. However, some player subgroups, e.g., players ranked in the 9–15 and in the 16–22 positions, exhibited uncertain rankings, suggesting that greater precision could be achieved by incorporating additional criteria.

### *5.4. Experiment results for ATP dataset*

This section demonstrates the accuracy of the described method in evaluating the all-time rankings of 29 tennis players who have achieved the No.1 position in the ATP rankings. Similar to the previous section, we compare the ranking lists generated by **AAI** and **MAI** across three distinct data generation approaches $Matrix_0$, $Matrix_1$, $Matrix_2$, as described in Section 5.2.

Table 7 presents the accuracy of those diferent settings. Similarly, $Matrix_2$ shows to be the best IV-FPR method. Notably, unlike the WTA case, **AAI** derives slightly more effective ranking in combination with career win-loss scores than **MAI**. Fig. 5 presents the illustration of these best ranking lists.

The next experiment examines the importance of preference relation connectivity in the ranking problem. For this, we restrict the win-loss data to include only the 12 top tennis players. Table 8 presents the win-loss data, restricted to the top 12 players. A notable observation is the presence of two distinct player subgroups: the first 3 players and the subsequent 9. Significantly, N. Djokovic had no recorded matches with any player from the second subgroup. However, the overall connectivity between these two groups

**Table 7**
Expert-based ranking evaluation using data from 29 top male tennis players.

| Jon Crim's ranking | $Matrix_0(3, 0.2)$ AAI | $Matrix_0(3, 0.2)$ MAI | $Matrix_1(0.2)$ AAI | $Matrix_1(0.2)$ MAI | $Matrix_2(0.2)$ AAI | $Matrix_2(0.2)$ MAI |
|---|---|---|---|---|---|---|
| 1. Djokovic | 4 | 4 | 4 | 4 | 2 | 2 |
| 2. Nadal | 1 | 1 | 1 | 1 | 1 | 1 |
| 3. Federer | 2 | 2 | 2 | 2 | 3 | 3 |
| 4. Sampras | 3 | 3 | 3 | 3 | 6 | 6 |
| 5. Borg | 8 | 8 | 5 | 6 | 4 | 5 |
| 6. Lendl | 6 | 6 | 6 | 7 | 5 | 4 |
| 7. Connors | 12 | 12 | 12 | 12 | 9 | 9 |
| 8. McEnroe | 11 | 11 | 11 | 11 | 7 | 7 |
| 9. Agassi | 7 | 7 | 8 | 8 | 10 | 10 |
| 10. Edberg | 9 | 10 | 10 | 10 | 11 | 11 |
| 11. Becker | 5 | 5 | 7 | 5 | 8 | 8 |
| 12. Wilander | 10 | 9 | 9 | 9 | 12 | 12 |
| Spearman's $\rho$ | 0.65 | 0.636 | 0.748 | 0.671 | 0.916 | 0.909 |

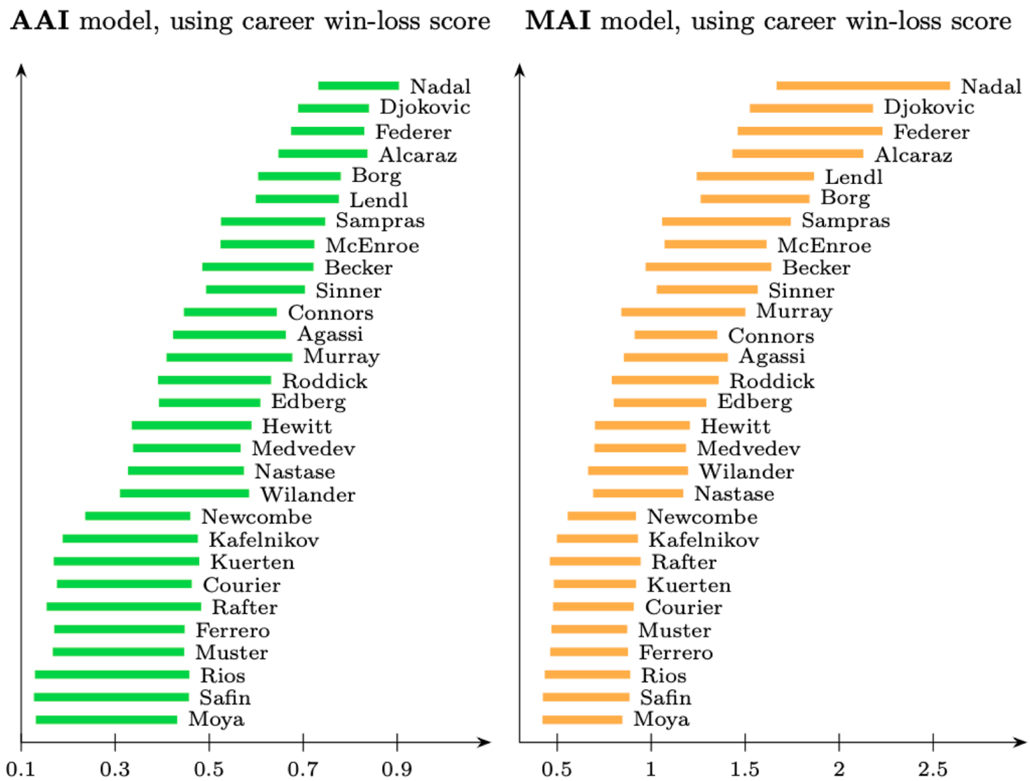



**Fig. 5.** Ranking calculated using (**AAI**) model (left) and (**MAI**) model (right) using the ATP's win-loss matrix statistics enriched by the career's win-loss scores. The players are sorted by the centers of corresponding intervals.

remains preserved due to matches played among R. Nadal, R. Federer, P. Sampras, and A. Agassi. This experiment's results are shown in Table 8.

Note that, due to the visualization of interval-valued weights, the last 10 ATP top players in both rankings do not have a clear preference order.

The last experiment investigates the agreement between the rankings generated by the methodology proposed in this paper (Fig. 5) and the ranking derived from the Peak Elo rating (Table 5).

Spearman rank correlation for the ranking lists generated by **AAI** and **MAI** (shown in Fig. 5) are as follows:

**Table 8**
Win-loss table restricted to the top 12 players in Crim's list and expert-based ranking evaluation using their win-loss data only.

| | | | | | | | | | | | | | $Matrix_1(0.2)$ | | $Matrix_2(0.2)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Players | ① | ② | ③ | ④ | ⑤ | ⑥ | ⑦ | ⑧ | ⑨ | ⑩ | ⑪ | ⑫ | **AAI** | **MAI** | **AAI** | **MAI** |
| ① Djokovic | | 31 | 27 | | | | | | | | | | 2 | 2 | 1 | 1 |
| ② Nadal | 29 | | 24 | | | | | | 2 | | | | 1 | 1 | 2 | 2 |
| ③ Federer | 23 | 16 | | 1 | | | | | 8 | | | | 3 | 3 | 3 | 3 |
| ④ Sampras | | | 0 | | | 5 | 2 | 3 | 20 | 8 | 12 | 2 | 4 | 4 | 6 | 7 |
| ⑤ Borg | | | | | | 6 | 15 | 7 | | | | 1 | 5 | 5 | 4 | 4 |
| ⑥ Lendl | | | | 3 | 2 | | 22 | 21 | 6 | 13 | 11 | 15 | 7 | 7 | 5 | 5 |
| ⑦ Connors | | | | 0 | 8 | 13 | | 14 | 0 | 6 | 0 | 0 | 12 | 12 | 10 | 9 |
| ⑧ McEnroe | | | | 0 | 7 | 15 | 20 | | 2 | 7 | 2 | 7 | 9 | 9 | 7 | 6 |
| ⑨ Agassi | | 0 | 3 | 14 | | 2 | 2 | 2 | | 6 | 10 | 5 | 8 | 8 | 9 | 10 |
| ⑩ Edberg | | | | 6 | | 14 | 6 | 6 | 3 | | 10 | 9 | 11 | 11 | 11 | 11 |
| ⑪ Becker | | | | 7 | | 10 | 6 | 8 | 4 | 25 | | 7 | 6 | 6 | 8 | 8 |
| ⑫ Wilander | | | | 1 | 0 | 7 | 5 | 6 | 2 | 11 | 3 | | 10 | 10 | 12 | 12 |
| Spearman's $\rho$ | | | | | | | | | | | | | 0.79 | 0.79 | 0.909 | 0.895 |

| Ranking List from Fig. 5 | Spearman's $\rho$ | $p$-value |
|---|---|---|
| **AAI** | 0.870 | $8.458 \times 10^{-10}$ |
| **MAI** | 0.868 | $1.074 \times 10^{-9}$ |

It is worth noting that both the **AAI** and **MAI** models produce rankings that show a high correlation with Peak Elo ratings, comparable to the levels reported in Tables 7 and 8. The main discrepancy concerns the position of the current world No. 1, Carlos Alcaraz.

In the Peak Elo ranking (as of 31 December 2024), he is placed 16th, whereas in the rankings generated in this study (based on data from 4 July 2025, see Fig. 5), he ranks 4th. During this period, Alcaraz won five titles—including Roland Garros and the Queen's Club Championships—and achieved an 18-match winning streak. We estimate that updating his Peak Elo position to 10th would increase the Spearman's $\rho$ to approximately 0.9.

## 6. Conclusion

This paper introduces ranking models that utilize incomplete IV-FPRs, constructed from incomplete pairwise comparisons, to determine an IV-WV as output. To assess the consistency of an IV-FPR, we used the Frobenius norm to set up an optimization problem for the additive consistency model **AAI**, while also incorporating a logarithmic transformation for the multiplicative consistency model **MAI**. One of the paper's new contributions is the use of masking matrices to describe the presence or absence of pairwise comparison values in the models's input. We present the pseudoinverse matrix method to find optimal IV-WVs for the models. Furthermore, it specifies the conditions for the models to have optimal solutions.

In applications, the paper applied the methodology to two empirical datasets to rank the top female tennis players (WTA) and male tennis players (ATP). We detail all steps from collecting and processing raw data to its conversion into an incomplete IV-FPR input and the subsequent determination of IV-WVs for assessing player rankings. Notably, the paper utilized Bernoulli distribution combined with Bayesian statistics to construct IV-FPRs. Besides, we compared the rankings from different models, and also compared them with expert rankings. We also plan to develop novel rank correlation metrics that compare a strict definitive ranking list with an IV-WV. However, in reality, the selection of an interval comparison approach is also contingent upon the assessment criteria.

Future work will focus on developing more advanced ranking models, e.g., by extending the eigenvector concept to IV-FPRs, and exploring their practical applications. We also plan to investigate ranking methods based on interval comparison techniques, as well as efficient algorithms for computing the Moore–Penrose pseudoinverse of the block matrices $B_E$ defined in Eq. (15).

### Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author(s) used Gemini in order to translate and polish the language. After using this tool/service, the author(s) reviewed and edited the content as needed and take(s) full responsibility for the content of the published article.

## CRediT authorship contribution statement

**Bich Khue Vo:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization; **Hung Son Nguyen:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Project administration, Methodology, Formal analysis, Conceptualization.

## Data availability

The data used is publicly available on websites.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Bich Khue Vo reports administrative support was provided by University of Finance–Marketing. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Supplementary material

Supplementary material associated with this article can be found in the online version at 10.1016/j.ijar.2026.109719.

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