Sparsity for parametric PDEs with log-gamma random inputs and applications

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Abstract

We propose a novel method for establishing the sparsity of the coefficients of the Laguerre generalized polynomial chaos expansion of solutions to parametric elliptic PDEs with log-gamma inputs on $\mathbb{R}_+^\infty$. The established sparsity is quantified by ℓ_p-summability and weighted ℓ_2-summability of the coefficients. Building on these sparsity results, we derive convergence rates for semi-discrete approximations in the parametric variables. These rates apply to sparse-grid polynomial interpolations, extended least-squares approximations and the associated semi-discrete quadrature rules. Moreover, a counterpart of our method for parametric elliptic PDEs with log-normal inputs yields a significant improvement in the sufficient condition for ℓ_p-summability when the component functions in the log-normal representation of the parametric diffusion coefficients have global support, compared with results obtained in prior works.

Keywords and Phrases: Uncertainty Quantification; Parametric PDEs with log-gamma random inputs; Laguerre generalized polynomial chaos expansion; Sparsity; Sparse-grid polynomial interpolation; Least squares approximation.

Mathematics Subject Classifications (2020): 60H35, 65C30, 65D32, 65N35, 41A25.

1 Introduction

In Computational Uncertainty Quantification, efficient approximation of infinite-dimensional parametric PDEs with random inputs has seen significant progress recently. The field is vast,

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making a full catalog impractical. For a thorough survey and bibliography, see [1, 8, 12, 13, 16]. The primary driver of convergence rates for numerical integration and interpolation of a parametric solution (as a mapping from the parametric domain to a Bochner space) is the sparsity of its generalized polynomial chaos (GPC) expansion coefficients. This sparsity is quantified by ℓ_p-summability or weighted ℓ_2-summability. The authors of [14] used real-variable bootstrapping arguments for investigation the sparsity of the coefficients of the Hermite GPC expansion of solutions to parametric PDEs with log-normal inputs based on the Hermite's second-order differential equation. Real-variable bootstrapping arguments were also used in [3, 4, 5] to study the sparsity of the coefficients of the GPC expansion of solutions to parametric elliptic PDEs with log-normal and affine inputs. The papers [3, 4, 5] significantly improved the results on sparsity in the case when the supports of the component functions in the presentation of the parametric diffusion coefficient have a finite number of overlaps. However, the real-variable bootstrapping arguments suffer certain technical difficulties in generalizing and extending to some important classes of parametric PDEs (cf. [3, 12]). An efficient and widely applicable approach to establish the sparsity is the complex method which is based on analytic continuation and complex-variable arguments to bound parametric derivatives and GPC expansion coefficients. This approach has been developed in [2, 7, 9, 18, 19] for "the compact case", where the random parameters of the random inputs range in compact subsets of $\mathbb{R}$. Unlike in these references, the work [12] investigated this approach for parametric PDEs with Gaussian field inputs when the parameter domain $\mathbb{R}^\infty$ is not compact and the associated probability measure is the standard tensor-product Gaussian measure, and more general for analytic Hilbert-valued functions on $\mathbb{R}^\infty$. The results on sparsity and relevant interpolation and quadrature approximation algorithms in [12] can be applied to a wide range of parametric and stochastic PDEs with Gaussian field inputs.

In the present paper, we develop a new approach in to study the sparsity for parametric PDEs with log-gamma random inputs when the non-compact parameter domain is $\mathbb{R}_+^\infty$. The gamma probability distribution has important applications in various fields, including econometrics, Bayesian statistics, biology and medicine (genomics, phylogenomics, neuroscience, oncology...). Unfortunately, the techniques and arguments of methods employed in [2, 7, 9, 12, 18, 19] which are based particularly on the Rodrigues formula for the associated orthonormal polynomials, are not suitable for the log-gamma context. We propose a distinct method for establishing the sparsity of the coefficients of the Laguerre GPC expansion of solutions to parametric PDEs with log-gamma inputs on $\mathbb{R}_+^\infty$. This entails significant modifications of mathematical arguments as compared to those in [7, 9, 18, 19, 12].

Consider an important model PDE, the divergence-form diffusion elliptic equation

$$-\operatorname{div}(a\nabla u) \;=\; f \quad \text{in} \quad D, \quad u|_{\partial D} \;=\; 0, \tag{1.1}$$

for a bounded Lipschitz domain $D \subset \mathbb{R}^d$, a right-hand side f and a spatially variable scalar diffusion coefficient a. Denote by $V := H_0^1(D)$ the energy space and $V' = H^{-1}(D)$ the dual space of V. Assume that $f \in V'$ and $a \in L_\infty(D)$ (in what follows this preliminary assumption always holds without mention). If a satisfies the ellipticity assumption

$$0 < \operatorname*{essinf}_{\boldsymbol{x}\in D} a(\boldsymbol{x}) \le a \le \operatorname*{esssup}_{\boldsymbol{x}\in D} a(\boldsymbol{x}) < \infty,$$

by the well-known Lax-Milgram lemma, there exists a unique weak solution $u \in V$ to the equation (1.1) satisfying the variational equation

$$\int_D a(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \;=\; \langle f, v\rangle, \quad \forall v \in V.$$

For the equation (1.1), we consider the diffusion coefficients having a parametric form $a = a(\boldsymbol{y})$, where $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$ is a sequence of real-valued parameters ranging in $\mathbb{R}_+^\infty$ or $\mathbb{R}^\infty$. Denote by $u(\boldsymbol{y})$ the weak solution to the parametric diffusion divergence-form elliptic equation

$$-\operatorname{div}(a(\boldsymbol{y})\nabla u(\boldsymbol{y})) \ = \ f \quad \text{in} \quad D, \quad u(\boldsymbol{y})|_{\partial D} \ = \ 0, \tag{1.2}$$

where the parametric diffusion coefficient a is of the form

$$a(\boldsymbol{y}) = \exp(b(\boldsymbol{y})), \quad \text{with} \ \ b(\boldsymbol{y}) = \sum_{j=1}^{\infty} y_j\psi_j, \ \ \boldsymbol{y} \in \mathbb{U}^\infty, \ \ \text{and} \ \ \psi_j \in L_\infty(D), \tag{1.3}$$

and

- (Γ) y_j are i.i.d. gamma random variables on $\mathbb{U} = \mathbb{R}_+$, or
- (G) y_j are i.i.d. standard Gaussian random variables on $\mathbb{U} = \mathbb{R}$.

The problem of analyticity and sparsity has been studied in [12, Section 3.6] for the parametric equation with with log-normal random inputs (1.2)–(1.3)(G). To the knowledge of the authors, so far this problem for the parametric equation with log-gamma random inputs (1.2)–(1.3)(Γ) has not been considered in any prior works. In the present paper, we focus our attention to this problem for equation (1.2)–(1.3)(Γ) and briefly revisit it for equation (1.2)–(1.3)(G).

We shortly describe the main contribution of this paper.

- (i) We propose a novel method for establishing the sparsity of the coefficients of the Laguerre GPC expansion of solutions $u(\boldsymbol{y})$ to the parametric equation with log-gamma random inputs (1.2)–(1.3)(Γ). Unlike approaches in previous works which rely particularly on the Rodrigues formula, our method is based specially on the Laguerre's second order differential equation for the Laguerre polynomials. The established sparsity is quantified by ℓ_p-summability and weighted ℓ_2-summability of these coefficients (for detailed comments, see Remark 2.1).
- (ii) From the sparsity results, as applications, we establish convergence rates for the semi-discrete parametric-variable approximations of $u(\boldsymbol{y})$ by sparse-grid polynomial interpolations and extended least squares approximations, as well as for the associated semi-discrete quadratures.
- (iii) A counterpart of our method for the parametric equation with log-normal random inputs (1.2)–(1.3)(G) yields a significant improvement in the sufficient condition for ℓ_p-summability when the component functions in the log-normal representation of the parametric diffusion coefficients have global support, compared with results obtained in the earlier works [4, 12, 14] (for detailed comments, see Remark 4.1).

The remaining part of the present paper is organized as follows. In Section 2, we propose a method for establishing the sparsity of the coefficients of the Laguerre GPC expansion of solutions $u(\boldsymbol{y})$ to parametric equation with log-gamma random inputs (1.2)–(1.3)(Γ). In Section 3, we prove convergence rates of sparse-grid polynomial interpolations, extended least squares approximation as well as the generated semi-discrete quadratures for $u(\boldsymbol{y})$. In Section 4, we extend the sparsity results in Section 2 to the parametric PDEs with log-normal random inputs (1.2)–(1.3)(G).

Notation As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, $\mathbb{R}_+$ the real non-negative numbers, $\mathbb{C}$ the complex numbers, $\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}$. Denote by $\mathbb{R}^\infty$ and $\mathbb{R}^\infty_+$ the sets of all sequences $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$ with $y_j \in \mathbb{R}$ and $y_j \in \mathbb{R}_+$, respectively. Denote by $|\boldsymbol{y}|_0$ the number of nonzero components y_j of $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$. Denote by $\mathbb{F}$ the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j\in\mathbb{N}}$ such that their support $\operatorname{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. For $\boldsymbol{s}, \boldsymbol{k} \in \mathbb{F}$ and $k \in \mathbb{R}$, $|\boldsymbol{s}|_1 := \sum_{j\in\mathbb{N}} s_j$, $|\boldsymbol{s}|_\infty := \max\{s_j, j \in \mathbb{N}\}$, $\boldsymbol{s}^k := \prod_{j\in\operatorname{supp}(\boldsymbol{s})} s_j^k$, $\boldsymbol{s}^{\boldsymbol{k}} := \prod_{j\in\operatorname{supp}(\boldsymbol{s})} s_j^{k_j}$, $\boldsymbol{s}! := \prod_{j\in\mathbb{N}} s_j!$. If $\boldsymbol{\alpha} = (\alpha_j)_{j\in\mathcal{J}}$ is a set of positive numbers with any index set $\mathcal{J}$, then we use the notation $\boldsymbol{\alpha}^{-1} := (a_j^{-1})_{j\in\mathcal{J}}$. Denote $\boldsymbol{\alpha}^{\boldsymbol{\beta}} := \left(\alpha_j^{\beta_j}\right)_{j\in\mathcal{J}}$ for the sets $\boldsymbol{\alpha} = (a_j)_{j\in\mathcal{J}}$ and $\boldsymbol{\beta} = (\beta_j)_{j\in\mathcal{J}}$. We use letters C and K to denote general positive constants which may take different values, and $C_{\alpha,\beta\ldots}$ and $K_{\alpha,\beta,\ldots}$ constants depending on $\alpha, \beta, \ldots$. Denote by $|G|$ the cardinality of the set G.

2 Sparsity for parametric PDEs

2.1 The Laguerre GPC expansion

For a fixed number $a > 0$, let the gamma probability measure (associated with gamma distribution) λ_a on $\mathbb{R}_+$ be defined via the density function

$$l_a(y) := \frac{1}{\Gamma(a)} e^{-y} y^{a-1}, \tag{2.1}$$

where Γ is the gamma function. Let $\left(L_k^{(a-1)}\right)_{k\in\mathbb{N}_0}$ be the sequence of Laguerre polynomials on $\mathbb{R}_+$ normalized with respect to λ_a, i.e., $\int_{\mathbb{R}_+} |L_k^{(a-1)}(y)|^2 \mathrm{d}\lambda_a(y) = 1,\ k \in \mathbb{N}_0$. For simplicity, we adopt the following abbreviations: $L_k := L_k^{(a-1)}$ and $\lambda := \lambda_a$.

We next recall a concept of probability measure $\boldsymbol{\lambda}(\boldsymbol{y})$ on $\mathbb{R}^\infty_+$ as the infinite tensor product of the measures $\lambda(y_j)$:

$$\boldsymbol{\lambda}(\boldsymbol{y}) := \bigotimes_{j\in\mathbb{N}} \lambda(y_j), \quad \boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in \mathbb{R}^\infty_+.$$

Let X be a separable Hilbert space. For $0 < p < \infty$, denote by $L_p(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ the Bochner space of strongly $\boldsymbol{\lambda}$-measurable mappings v from $\mathbb{R}^\infty_+$ to X, equipped with the norm

$$\|v\|_{L_p(\mathbb{R}^\infty_+, X;\boldsymbol{\lambda})} := \left(\int_{\mathbb{R}^\infty_+} \|v(\boldsymbol{y})\|_X^p \,\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \right)^{1/p}.$$

A function $v \in L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ can be represented by the Laguerre GPC expansion

$$v(\boldsymbol{y}) = \sum_{\boldsymbol{s}\in\mathbb{F}} v_{\boldsymbol{s}}\, L_{\boldsymbol{s}}(\boldsymbol{y}), \quad v_{\boldsymbol{s}} \in X, \tag{2.2}$$

with convergence in $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$, where

$$L_{\boldsymbol{s}}(\boldsymbol{y}) = \bigotimes_{j\in\mathbb{N}} L_{s_j}(y_j), \quad v_{\boldsymbol{s}} := \int_{\mathbb{R}^\infty_+} v(\boldsymbol{y})\, L_{\boldsymbol{s}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}), \quad \boldsymbol{s} \in \mathbb{F}.$$

Here $\mathbb{F}$ is the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j\in\mathbb{N}}$ such that their support $\operatorname{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. Notice that $(L_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ is an orthonormal basis of $L_2(\mathbb{R}^\infty_+, \mathbb{C}; \boldsymbol{\lambda})$. Moreover, a strongly $\boldsymbol{\lambda}$-measurable function v on $\mathbb{R}^\infty_+$ belongs to $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ if and only if v is represented by the series (2.2) converging in $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ and it holds the Parseval's identity

$$\|v\|^2_{L_2(\mathbb{R}^\infty_+, X;\boldsymbol{\lambda})} = \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|^2_X. \tag{2.3}$$

The the Bochner space $L_p(\mathbb{R}^\infty, X; \boldsymbol{\gamma})$ and the associated infinite tensor product standard Gaussian measure $\boldsymbol{\gamma}(\boldsymbol{y})$ on $\mathbb{R}^\infty$ can be defined in an analogous fashion.

2.2 Sparsity for the Laguerre GPC expansion

We recall a result on regularity with respect to parametric variables proven in [12, Lemma 3.9], which gives bounds for partial derivatives of the weak solution $u(\boldsymbol{y})$ to equation (1.2)–(1.3).

Lemma 2.1 *Assume that there exist a positive sequence $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ and a positive number κ satisfying*

$$\left\| \sum_{j\in\mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} \le \kappa < \frac{\pi}{2}\,. \tag{2.4}$$

Then we have the following. Let $\boldsymbol{y} \in \mathbb{R}^\infty$ with $b(\boldsymbol{y}) \in L_\infty(D)$ and $\boldsymbol{s} \in \mathbb{F}$ such that $\operatorname{supp}(\boldsymbol{s}) \subseteq \operatorname{supp}(\boldsymbol{\rho})$. *Then we have*

$$\|\partial^{\boldsymbol{s}} u(\boldsymbol{y})\|_V \le C_0 \frac{\boldsymbol{s}!}{\boldsymbol{\rho}^{\boldsymbol{s}}} \exp\left(\|b(\boldsymbol{y})\|_{L_\infty(D)}\right), \quad \boldsymbol{y} \in \mathbb{R}^\infty,$$

where $C_0 = e^\kappa (\cos\kappa)^{-1} \|f\|_{V'}$.

In what follows, throughout Sections 2 and 3, we denote by u or $u(\boldsymbol{y})$ the weak solution to parametric equation with log-gamma random inputs (1.2)–(1.3)(Γ).

If $b(\boldsymbol{y}) \in L_\infty(D)$ with $\boldsymbol{y} \in \mathbb{R}^\infty_+$, we have the estimate

$$\|u(\boldsymbol{y})\|_V \le \|f\|_{V'} \|a(\boldsymbol{y})^{-1}\|_{L_\infty(D)} \le \exp\left(\|b(\boldsymbol{y})\|_{L_\infty(D)}\right) \|f\|_{V'}. \tag{2.5}$$

It follows from the a-priori estimate (2.5) that for $f \in V'$ the parametric elliptic diffusion problem equation (1.2)–(1.3)(L) admits a unique solution for parameters $\boldsymbol{y}$ in the set

$$U_0 := \left\{\boldsymbol{y} \in \mathbb{R}^\infty_+ : b(\boldsymbol{y}) \in L_\infty(D)\right\}. \tag{2.6}$$

Lemma 2.2 *Assume that for every $j \in \mathbb{N}$, $\psi_j \in L_\infty(D)$, and there exists a positive sequence $(\rho_j)_{j\in\mathbb{N}}$ such that $\left(\exp(-\rho_j)\right)_{j\in\mathbb{N}} \in \ell_1(\mathbb{N})$ and the series $\sum_{j\in\mathbb{N}} \rho_j |\psi_j|$ converges in $L_\infty(D)$. Then the set U_0 has full measure, i.e., $\boldsymbol{\lambda}(U_0) = 1$, and contains all $\boldsymbol{y} \in \mathbb{R}^\infty_+$ with $|\boldsymbol{y}|_0 < \infty$.*

Proof. The proof of this lemma is similar to the proof of [4, Theorem 2.2]. We omit it. □

It is known that, the Laguerre polynomials $\left(L_k\right)_{k\in\mathbb{N}_0}$ are nontrivial solutions to the Laguerre's differential equation of second order

$$\mathcal{D}u = ku, \quad \text{where} \quad \mathcal{D} := -y\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (a - y)\frac{\mathrm{d}}{\mathrm{d}y} \tag{2.7}$$

(see, e.g., [17, 5.1.3]). This means that

$$\mathcal{D}L_k = kL_k, \quad k \in \mathbb{N}_0.$$

Observe that, it holds the equality

$$\mathcal{D}u = -e^y y^{1-a} \frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y}y^a \frac{\mathrm{d}u}{\mathrm{d}y}\right), \tag{2.8}$$

which can be shown by taking differentiation of the function $e^{-y}y^a\frac{\mathrm{d}u}{\mathrm{d}y}$.

For a given finite set $J \subset \mathbb{N}$ and $r \in \mathbb{N}$, we define the product differential operator

$$\mathcal{D}_J^r := (\mathcal{D}_J)^r, \quad \mathcal{D}_J := (-1)^{|J|} \prod_{j\in J} e^{y_j} y_j^{1-a} \frac{\mathrm{d}}{\mathrm{d}y_j}\left(e^{-y_j}y_j^a \frac{\mathrm{d}}{\mathrm{d}y_j}\right). \tag{2.9}$$

Notice that there exist polynomials $p_j(t)$, $j = 1, ..., 2r$, of degree at most r such that

$$\mathcal{D}^r = \left(-y\frac{\mathrm{d}^2}{\mathrm{d}y^2} - (a-y)\frac{\mathrm{d}}{\mathrm{d}y}\right)^r = \sum_{j=1}^{2r} p_j(y)\frac{\mathrm{d}^j}{\mathrm{d}y^j},$$

and the coefficients of $p_j(y)$ depend on r and a only. Hence, there exists a constant $C_{a,r}$ such that for $y \geq 0$,

$$|p_j(y)| \leq C_{a,r}(1+y)^r, \quad j = 1, \dots, 2r. \tag{2.10}$$

For a finite set $J \subset \mathbb{N}$, $r \in \mathbb{N}$ and $\boldsymbol{s} \in \mathbb{F}$, we define

$$\nu_{J,\boldsymbol{s}} := \prod_{j\in J} s_j. \tag{2.11}$$

Lemma 2.3 *If* $v \in L_2(\mathbb{R}_+^\infty, V; \boldsymbol{\lambda})$ *and* $\mathcal{D}_J^r v \in L_2(\mathbb{R}_+^\infty, V; \boldsymbol{\lambda})$*, then we have that*

$$\mathcal{D}_J^r v = \sum_{\boldsymbol{s}\in\mathbb{F}} \nu_{J,\boldsymbol{s}}^r v_{\boldsymbol{s}} L_{\boldsymbol{s}}. \tag{2.12}$$

Proof. We first prove that if $v \in L_2(\mathbb{R}_+, V; \lambda)$ and $\mathcal{D}v \in L_2(\mathbb{R}_+, V; \lambda)$, then we have that

$$\mathcal{D}v = \sum_{s\in\mathbb{N}_0} s v_s L_s. \tag{2.13}$$

Indeed, by (2.7), (2.8) and integration by parts twice, we have that,

$$\begin{aligned} s v_s &:= \int_{\mathbb{R}_+} v(y)\ (sL_s(y))\,\mathrm{d}\lambda(y) = \int_{\mathbb{R}_+} v(y)\mathcal{D}L_s(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}_+} v(y)\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y}y^a\frac{\mathrm{d}L_s(y)}{\mathrm{d}y}\right)\mathrm{d}y = \int_{\mathbb{R}_+} \frac{\mathrm{d}v(y)}{\mathrm{d}y}e^{-y}y^a\frac{\mathrm{d}L_s(y)}{\mathrm{d}y}\,\mathrm{d}y \\ &= -\int_{\mathbb{R}_+} \frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y}y^a\frac{\mathrm{d}v(y)}{\mathrm{d}y}\right)L_s(y)\,\mathrm{d}y = \int_{\mathbb{R}_+} e^{-y}y^{a-1}\mathcal{D}v(y)L_s(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}_+} \mathcal{D}v(y)L_s(y)\,\mathrm{d}\lambda(y), \end{aligned}$$

which implies (2.13). By using the tensor product argument, (2.13), definition (2.9) one can easily deduce (2.12). ∎

For $\theta, \lambda \geq 0$, we define the set $\boldsymbol{p}(\theta,\lambda) := (p_{\boldsymbol{s}}(\theta,\lambda))_{\boldsymbol{s}\in\mathbb{F}}$ by

$$p_{\boldsymbol{s}}(\theta,\lambda) := \prod_{j\in\mathbb{N}} (1+\lambda s_j)^\theta, \quad \boldsymbol{s}\in\mathbb{F}.$$

We often use the following inequality for fixed θ and λ

$$(1+\lambda k)^\theta \leq C_{\theta,\lambda} k^\theta, \quad k\in\mathbb{N}, \tag{2.14}$$

with some constant $C_{\theta,\lambda}$ depending on θ,λ only. For fixed $0<p<\infty$ and $\theta\geq 0$, let $r_{p,\theta}\in\mathbb{N}$ be chosen such that

$$p(r_{p,\theta}-\theta) > 1, \quad \text{and denote} \quad C_{p,\theta} := \sum_{k\in\mathbb{N}} k^{-p(r_{p,\theta}-\theta)}. \tag{2.15}$$

For a finite subset J of $\mathbb{N}$ and $r\in\mathbb{N}$, we define

$$A_r(J) := \left(\int_{\mathbb{R}^\infty_+} \prod_{j\in J} (1+y_j)^{2r} \exp\left(2\|b(\boldsymbol{y})\|_{L_\infty(D)}\right) \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \right)^{1/2}. \tag{2.16}$$

Theorem 2.4 *Let $0<p<\infty$, $\boldsymbol{\rho}=(\rho_j)_{j\in\mathbb{N}}$ be a positive sequence satisfying condition (2.4), and $\boldsymbol{\rho}^{-1}\in\ell_p(\mathbb{N})$. Assume that for any $r\in\mathbb{N}$, there exist a constant K_r such that*

$$A_r(J) \leq K_r^{|J|} \tag{2.17}$$

for any finite set $J\subset\mathbb{N}$. Then $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$.

Moreover, if in addition, $p<2$, for any fixed $\theta,\lambda\geq 0$, we can construct a set $\boldsymbol{\sigma}=(\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ with positive $\sigma_{\boldsymbol{s}}$, and a constant M such that

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|u_{\boldsymbol{s}}\|_V)^2 \right)^{1/2} \leq M^{1/2} < \infty, \quad \textit{with} \quad \left\|\boldsymbol{p}(\theta,\lambda)\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{F})} \leq M^{1/q} < \infty, \tag{2.18}$$

where $q := 2p/(2-p)$. The set $\boldsymbol{\sigma}$ and constant M depend on $a,p,\theta,\lambda,\kappa$ only.

Proof. We have by Lemma 2.3 and the Parseval's identity (2.3)

$$\int_{\mathbb{R}^\infty_+} \|\mathcal{D}_J^r u(\boldsymbol{y})\|_V^2 \,\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) = \sum_{\boldsymbol{s}\in\mathbb{F}} \nu_{J,\boldsymbol{s}}^{2r} \|u_{\boldsymbol{s}}\|_V^2 \,. \tag{2.19}$$

From (2.10), we get for $\boldsymbol{y}\in\mathbb{R}^\infty_+$,

$$\begin{aligned} \|\mathcal{D}_J^r u(\boldsymbol{y})\|_V &= \left\| \prod_{j\in J} \left(-y_j \frac{\mathrm{d}^2}{\mathrm{d}y_j^2} - (a-y_j)\frac{\mathrm{d}}{\mathrm{d}y_j} \right)^r u(\boldsymbol{y}) \right\|_V \\ &\leq C_{a,r}^{|J|} \prod_{j\in J} (1+y_j)^r \sum_{\mathrm{supp}(\boldsymbol{k})=J,\ |\boldsymbol{k}|_\infty\leq 2r} \left\|\partial^{\boldsymbol{k}} u(\boldsymbol{y})\right\|_V, \end{aligned}$$

where $C_{a,r}$ is the constant as in (2.10). Applying Lemma 2.1 gives

$$\|\mathcal{D}_J^r u(\boldsymbol{y})\|_V \le C_{a,r}^{|J|} \prod_{j\in J}(1+y_j)^r \sum_{\operatorname{supp}(\boldsymbol{k})=J,\ |\boldsymbol{k}|_\infty\le 2r} C_0 \frac{\boldsymbol{k}!}{\boldsymbol{\rho}^{\boldsymbol{k}}} \exp\big(\|b(\boldsymbol{y})\|_{L_\infty(D)}\big).$$

Hence,

$$\int_{\mathbb{R}_+^\infty} \|\mathcal{D}_J^r u(\boldsymbol{y})\|_V^2 \,\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \le A_r(J)^2 C_0^2 C_{a,r}^{2|J|} \left(\sum_{\operatorname{supp}(\boldsymbol{k})=J,\ |\boldsymbol{k}|_\infty\le 2r} \frac{\boldsymbol{k}!}{\boldsymbol{\rho}^{\boldsymbol{k}}} \right)^2.$$

Moreover, $\boldsymbol{k}! \le ((2r)!)^{|J|}$ when $\operatorname{supp}(\boldsymbol{k}) = J$, $|\boldsymbol{k}|_\infty \le 2r$. Therefore, for $\boldsymbol{s} \in \mathbb{F}$ with $\operatorname{supp}(\boldsymbol{s}) = J$, we have $\nu_{J,\boldsymbol{s}}^{2r} = \boldsymbol{s}^{2r}$ and by (2.19) and (2.17),

$$\begin{aligned}
\|u_{\boldsymbol{s}}\|_V &\le \boldsymbol{s}^{-r} \left(\int_{\mathbb{R}_+^\infty} \|\mathcal{D}_J^r u(\boldsymbol{y})\|_V^2 \,\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \right)^{1/2} \\
&\le \boldsymbol{s}^{-r} A_r(J) C_0 C_{a,r}^{|J|} ((2r)!)^{|J|} \sum_{\operatorname{supp}(\boldsymbol{k})=\operatorname{supp}(\boldsymbol{s}),\ |\boldsymbol{k}|_\infty\le 2r} \boldsymbol{\rho}^{-\boldsymbol{k}} \qquad (2.20)\\
&\le C_0 C_1^{|\operatorname{supp}(\boldsymbol{s})|} \boldsymbol{s}^{-r} \sum_{\operatorname{supp}(\boldsymbol{k})=\operatorname{supp}(\boldsymbol{s}),\ |\boldsymbol{k}|_\infty\le 2r} \boldsymbol{\rho}^{-\boldsymbol{k}} =: \beta_{\boldsymbol{s}},
\end{aligned}$$

where $C_1 := K_r C_{a,r} (2r)!$ and K_r is as in (2.17).

Let $\boldsymbol{\beta} = (\beta_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$. In the following, for any $\theta', \lambda \ge 0$ we will prove that there exists $r \in \mathbb{N}$ depending on θ' and p only and a constant M depending on $a, p, \theta', \lambda, \kappa$ only such that

$$\big\|\boldsymbol{p}(\theta',\lambda)\boldsymbol{\beta}\big\|_{\ell_p(\mathbb{F})} \ \le \ M^{1/p}. \qquad (2.21)$$

By (2.14) we have

$$p_{\boldsymbol{s}}(\theta',\lambda)\beta_{\boldsymbol{s}} \le C_0 \boldsymbol{s}^{-r+\theta'} (C_{\theta',\lambda} C_1)^{|\operatorname{supp}(\boldsymbol{s})|} \sum_{\operatorname{supp}(\boldsymbol{k})=\operatorname{supp}(\boldsymbol{s}),\ |\boldsymbol{k}|_\infty\le 2r} \boldsymbol{\rho}^{-\boldsymbol{k}},$$

where $C_{\theta',\lambda}$ is as in (2.14). Hence,

$$\sum_{\operatorname{supp}(\boldsymbol{s})=J} \big(p_{\boldsymbol{s}}(\theta',\lambda)\beta_{\boldsymbol{s}}\big)^p \le C_0 (C_{\theta',\lambda} C_1)^{p|J|} \sum_{\operatorname{supp}(\boldsymbol{s})=J} \boldsymbol{s}^{-p(r-\theta')} \left(\sum_{\operatorname{supp}(\boldsymbol{k})=J,\ |\boldsymbol{k}|_\infty\le 2r} \boldsymbol{\rho}^{-\boldsymbol{k}} \right)^p. \qquad (2.22)$$

Let $r = r_{p,\theta'}$ be chosen and $C_{p,\theta'}$ be defined as in (2.15). Then we have

$$\sum_{\operatorname{supp}(\boldsymbol{s})=J} \boldsymbol{s}^{-p(r-\theta')} = \prod_{j\in J} \sum_{s_j\in\mathbb{N}} s_j^{-p(r-\theta')} \le C_{p,\theta'}^{|J|}. \qquad (2.23)$$

Letting $|J| := m$ and $J := \{j_1, \dots, j_m\}$, we get

$$\left(\sum_{\operatorname{supp}(\boldsymbol{k})=J,\ |\boldsymbol{k}|_\infty\le 2r} \boldsymbol{\rho}^{-\boldsymbol{k}} \right)^p = \prod_{i=1}^m \left(\sum_{\ell=1}^{2r} (\rho_{j_i})^\ell \right)^p.$$

Hence,

$$\sum_{\operatorname{supp}(\boldsymbol{s})=J} \big(p_{\boldsymbol{s}}(\theta',\lambda)\beta_{\boldsymbol{s}}\big)^p \le C_0 (C_{\theta',\lambda} C_1 C_{p,\theta'})^m \prod_{i=1}^m \left(\sum_{\ell=1}^{2r} (\rho_{j_i})^\ell \right)^p = C_0 \prod_{i=1}^m K \left(\sum_{\ell=1}^{2r} (\rho_{j_i})^\ell \right)^p,$$

where

$$K = C_{\theta',\lambda} K_r C_{a,r} C_{p,\theta'} (2r)!.$$

It follows that

$$\begin{aligned}
\sum_{\boldsymbol{s}\in\mathbb{F}} \big(p_{\boldsymbol{s}}(\theta',\lambda)\beta_{\boldsymbol{s}}\big)^p &\le C_0 \sum_{m=1}^{\infty} \sum_{j_1,\dots,j_m=1}^{\infty} \prod_{i=1}^{m} K \left(\sum_{\ell=1}^{2r} (\rho_{j_i})^\ell\right)^p \\
&= C_0 \prod_{k=1}^{\infty} \left(1 + K \left(\sum_{\ell=1}^{2r} (\rho_{j_i})^\ell\right)^p\right) \\
&\le C_0 \exp\left(K \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{2r} (\rho_{j_i})^\ell\right)^p\right),
\end{aligned}$$

which is finite since $\boldsymbol{\rho}^{-1} \in \ell_p(\mathbb{N})$. This proves (2.21). Hence, from (2.20) we deduce that $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ by choosing $\theta' = 0$.

We now prove (2.18). If $0 < p < 2$, we define the set $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ by

$$\sigma_{\boldsymbol{s}} := \beta_{\boldsymbol{s}}^{p/2-1}, \ \boldsymbol{s} \in \mathbb{F}.$$

Then we have

$$\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|u_{\boldsymbol{s}}\|_V)^2 \ \le \ \sum_{\boldsymbol{s}\in\mathbb{F}} \left(\beta_{\boldsymbol{s}}^{p/2-1}\beta_{\boldsymbol{s}}\right)^2 \ = \ \|\boldsymbol{\beta}\|_{\ell_p(\mathbb{F})}^p \ \le \ M,$$

and by choosing $\theta' = \theta q/p$,

$$\left\|\boldsymbol{p}(\theta,\lambda)\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{F})}^q \ = \ \sum_{\boldsymbol{s}\in\mathbb{F}} \left(\beta_{\boldsymbol{s}}^{1-p/2}\right)^{2p/(2-p)} p_{\boldsymbol{s}}(\theta,\lambda)^q \ = \ \left\|\boldsymbol{p}(\theta',\lambda)\boldsymbol{\beta}\right\|_{\ell_p(\mathbb{F})}^p \ \le \ M,$$

which proves (2.18). $\square$

Lemma 2.5 *Let $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ be defined by $b_j := \|\psi_j\|_{L_\infty(D)}$. Assume $\boldsymbol{b} \in \ell_1(\mathbb{N})$ and*

$$\|\boldsymbol{b}\|_{\ell_\infty(\mathbb{N})} = b_0 < \frac{1}{2}.$$

Then

$$A_r(J) \le \exp\left(\frac{a}{1-2b_0}\|\boldsymbol{b}\|_{\ell_1(\mathbb{N})}\right) K_{a,r,\boldsymbol{b}}^{|J|}, \tag{2.24}$$

where

$$K_{a,r,\boldsymbol{b}} := \left(\int_{\mathbb{R}_+} \frac{(1+y)^{2r}y^{a-1}}{\Gamma(a)} \exp\big(y(2b_0-1)\big)\mathrm{d}y\right)^{1/2} < \infty. \tag{2.25}$$

Proof. For any finite set $J \subset \mathbb{N}$ we have

$$\begin{aligned}
A_r(J)^2 &:= \int_{\mathbb{R}_+^\infty} \prod_{j\in J} (1+y_j)^{2r} \exp\Big(2\|b(\boldsymbol{y})\|_{L_\infty(D)}\Big)\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \\
&\le \int_{\mathbb{R}_+^\infty} \prod_{j\in J} (1+y_j)^{2r} \exp\Big(2\sum_{j\in\mathbb{N}} b_j y_j\Big) \prod_{j\in\mathbb{N}} \frac{1}{\Gamma(a)} y_j^{a-1}\exp(-y_j)\mathrm{d}y_j \\
&\le \prod_{j\in J} \int_{\mathbb{R}_+} \frac{(1+y_j)^{2r}y^{a-1}}{\Gamma(a)} \exp\big(y_j(2b_0-1)\big)\mathrm{d}y_j \prod_{j\notin J} \int_{\mathbb{R}_+} \frac{y^{a-1}}{\Gamma(a)} \exp\big(y_j(2b_j-1)\big)\mathrm{d}y_j.
\end{aligned}$$

By changing variable $y_j(1-2b_j)=\xi_j$ and by we get

$$A_r(J)^2 \le \left(K_{a,r,\boldsymbol{b}}^{|J|}\right)^2 \prod_{j\notin J} \frac{1}{(1-2b_j)^a}.$$

We have

$$\begin{aligned}\log\left(\prod_{j\notin J}\frac{1}{(1-2b_j)^a}\right) &= a\sum_{j\notin J}\log\frac{1}{1-2b_j} = a\sum_{j\notin J}\log\left(1+\frac{2b_j}{1-2b_j}\right)\\ &\le a\sum_{j\notin J}\frac{2b_j}{1-2b_0} \le \frac{2a}{1-2b_0}\|\boldsymbol{b}\|_{\ell_1(\mathbb{N})}.\end{aligned}$$

The bound (2.24) has been proven. ☐

Theorem 2.6 *Let* $0<p\le 1$ *and* $\boldsymbol{b}=(b_j)_{j\in\mathbb{N}}$ *be defined by* $b_j := \|\psi_j\|_{L_\infty(D)}$. *Let* $r=r_{p,0}\in\mathbb{N}$ *be a fixed number satisfying* (2.15). *Assume that* $\boldsymbol{b}\in\ell_p(\mathbb{N})$,

$$\|\boldsymbol{b}\|_{\ell_\infty(\mathbb{N})} = b_0 < \frac{1}{2},$$

and

$$\|\boldsymbol{b}\|_{\ell_1(\mathbb{N})} < K^{-1}, \tag{2.26}$$

where

$$K := e(2r)!C_{a,r}C_{p,0}K_{a,r,\boldsymbol{b}},$$

and the constants $C_{a,r}$, $C_{p,0}$, $K_{a,r,\boldsymbol{b}}$, *are defined as in* (2.10), (2.15), (2.25), *respectively. Then* $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}}\in\ell_p(\mathbb{F})$.

Moreover, for any fixed $\theta,\lambda\ge 0$, *if* $q:=2p/(2-p)$, $\theta'=2\theta/(2-p)$, $r=r_{p,\theta'}$ *is a fixed number* (2.15) *and* K *in* (2.26) *is replaced by*

$$K := e(2r)!C_{a,r}C_{p,\theta'}C_{\theta',\lambda}K_{a,r,\boldsymbol{b}} \tag{2.27}$$

with $C_{\theta',\lambda}$ *being defined as in* (2.14), *then we can construct a set* $\boldsymbol{\sigma}=(\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$, *and a constant* M *such that*

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}}(\sigma_{\boldsymbol{s}}\|u_{\boldsymbol{s}}\|_V)^2\right)^{1/2} \le M^{1/2} < \infty, \quad \textit{with} \quad \left\|\boldsymbol{p}(\theta,\lambda)\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{F})} \le M^{1/q} < \infty. \tag{2.28}$$

The set $\boldsymbol{\sigma}$ *and constant* M *depend on* $a,\boldsymbol{b},p,\theta,\lambda$ *only.*

Proof. We define the sequence $\boldsymbol{\rho_k}=(\rho_{\boldsymbol{k},j})_{j\in\mathbb{N}}$ depending on $\boldsymbol{k}\in\mathbb{F}$, by

$$\rho_{\boldsymbol{k},j} := \begin{cases}\frac{k_j}{b_j|\boldsymbol{k}|_1} & \text{if } j\in\operatorname{supp}(\boldsymbol{k}),\\ 0 & \text{if } j\notin\operatorname{supp}(\boldsymbol{k}).\end{cases}$$

Notice that

$$\sup_{\boldsymbol{k}\in\mathbb{F}}\left\|\sum_{j\in\mathbb{N}}\rho_{\boldsymbol{k},j}|\psi_j|\right\|_{L_\infty(D)} \le 1.$$

Applying Lemma 2.1 gives for $\boldsymbol{y} \in \mathbb{R}_+^\infty$ and $\boldsymbol{k} \in \mathbb{F}$,

$$\left\|\partial^{\boldsymbol{k}} u(\boldsymbol{y})\right\|_V \le C_0 \frac{\boldsymbol{k}!}{\boldsymbol{\rho}_{\boldsymbol{k}}^{\boldsymbol{k}}} \exp\left(\|b(\boldsymbol{y})\|_{L_\infty(D)}\right) = C_0 \frac{\boldsymbol{k}! \boldsymbol{b}^{\boldsymbol{k}} |\boldsymbol{k}|^{|\boldsymbol{k}|}}{\boldsymbol{k}^{\boldsymbol{k}}} \exp\left(\|b(\boldsymbol{y})\|_{L_\infty(D)}\right).$$

Similarly to the proof of the previous theorem we can derive

$$\int_{\mathbb{R}_+^\infty} \|\mathcal{D}_J^r u(\boldsymbol{y})\|_V^2 \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \le A_r(J)^2 C_0^2 C_{a,r}^{2|J|} \left(\sum_{\operatorname{supp}(\boldsymbol{k})=J, \ |\boldsymbol{k}|_\infty \le 2r} \frac{\boldsymbol{k}! \boldsymbol{b}^{\boldsymbol{k}} |\boldsymbol{k}|^{|\boldsymbol{k}|}}{\boldsymbol{k}^{\boldsymbol{k}}} \right)^2,$$

where $C_{a,r}$ is the constant as in (2.10) and $A_r(J)$ as in (2.16). By Lemma 2.5 and the inequalities $\frac{|\boldsymbol{k}|^{|\boldsymbol{k}|}}{\boldsymbol{k}^{\boldsymbol{k}}} \le \frac{|\boldsymbol{k}|! e^{|\boldsymbol{k}|}}{\boldsymbol{k}!}$ and $\boldsymbol{k}! \le ((2r)!)^{|J|}$ with $\operatorname{supp}(\boldsymbol{k}) = J$ and $|\boldsymbol{k}|_\infty \le 2r$, we have

$$\int_{\mathbb{R}_+^\infty} \|\mathcal{D}_J^r u(\boldsymbol{y})\|_V^2 \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) \le C_1^2 K_{a,r,\boldsymbol{b}}^{2|J|} C_{a,r}^{2|J|} ((2r)!)^{2|J|} \left(\sum_{\operatorname{supp}(\boldsymbol{k})=J, \ |\boldsymbol{k}|_\infty \le 2r} \frac{(e\boldsymbol{b})^{\boldsymbol{k}} |\boldsymbol{k}|!}{\boldsymbol{k}!} \right)^2,$$

where

$$C_1 := C_0 \exp\left(\frac{a}{1 - 2b_0} \|\boldsymbol{b}\|_{\ell_1(\mathbb{N})} \right).$$

This together with and (2.19) implies for $\boldsymbol{s} \in \mathbb{F}$ with $\operatorname{supp}(\boldsymbol{s}) = J$,

$$\|u_{\boldsymbol{s}}\|_V \le C_1 C_2^{|J|} \boldsymbol{s}^{-r} \sum_{\operatorname{supp}(\boldsymbol{k})=J, \ |\boldsymbol{k}|_\infty \le 2r} \frac{(e\boldsymbol{b})^{\boldsymbol{k}} |\boldsymbol{k}|!}{\boldsymbol{k}!}. \tag{2.29}$$

where

$$C_2 := (2r)! K_{a,r,\boldsymbol{b}} C_{a,r}.$$

Denote by $\beta_{\boldsymbol{s}}$ the right-hand side of (2.29) and let $\boldsymbol{\beta} = (\beta_{\boldsymbol{s}})_{\boldsymbol{s} \in \mathbb{F}}$. For any $\theta', \lambda \ge 0$ we will prove that there exists a constant M depending on $a, \boldsymbol{b}, p, \theta, \lambda$ only such that

$$\|\boldsymbol{p}(\theta, \lambda) \boldsymbol{\beta}\|_{\ell_p(\mathbb{F})} \ \le \ M^{1/p}. \tag{2.30}$$

By using the inequality (2.14), we get

$$\sum_{\operatorname{supp}(\boldsymbol{s})=J} \left(p_{\boldsymbol{s}}(\theta', \lambda) \beta_{\boldsymbol{s}}\right)^p \le C_1 (C_{\theta',\lambda} C_2)^{p|J|} \sum_{\operatorname{supp}(\boldsymbol{s})=J} \boldsymbol{s}^{-p(r-\theta')} \left(\sum_{\operatorname{supp}(\boldsymbol{k})=J, \ |\boldsymbol{k}|_\infty \le 2r} \frac{(e\boldsymbol{b})^{\boldsymbol{k}} |\boldsymbol{k}|!}{\boldsymbol{k}!} \right)^p,$$

where $C_{\theta',\lambda}$ is as in (2.14). Let $r = r_{p,\theta'}$ be chosen and $C_{p,\theta'}$ be defined as in (2.15). By (2.23) we get

$$\begin{aligned} \sum_{\operatorname{supp}(\boldsymbol{s})=J} \left(p_{\boldsymbol{s}}(\theta', \lambda) \beta_{\boldsymbol{s}}\right)^p &\le C_1 C_3^{p|J|} \sum_{\operatorname{supp}(\boldsymbol{k})=J, \ |\boldsymbol{k}|_\infty \le 2r} \left(\frac{(e\boldsymbol{b})^{\boldsymbol{k}} |\boldsymbol{k}|!}{\boldsymbol{k}!} \right)^p \\ &= C_1 \sum_{\operatorname{supp}(\boldsymbol{k})=J, \ |\boldsymbol{k}|_\infty \le 2r} \left(\frac{(eC_3\boldsymbol{b})^{\boldsymbol{k}} |\boldsymbol{k}|!}{\boldsymbol{k}!} \right)^p, \end{aligned}$$

where $C_3 := e^{-1} K$ and K is as in (2.27). Then we obtain

$$\sum_{\boldsymbol{s} \in \mathbb{F}} p_{\boldsymbol{s}}(\theta', \lambda)^p \beta_{\boldsymbol{s}}^p \le C_1 \sum_{\boldsymbol{k} \in \mathbb{F}\colon |\boldsymbol{k}|_\infty \le 2r} \left(\frac{\bar{\boldsymbol{b}}^{\boldsymbol{k}} |\boldsymbol{k}|!}{\boldsymbol{k}!} \right)^p, \tag{2.31}$$

where $\bar{\boldsymbol{b}} := K\boldsymbol{b}$. We have by (2.30) that $\|\bar{\boldsymbol{b}}\|_{\ell_p(\mathbb{N})} < \infty$ and $\|\bar{\boldsymbol{b}}\|_{\ell_1(\mathbb{N})} < 1$. Hence, from [9, Theorem 7.2] and (2.31) we derive (2.26), and therefore, $\|\boldsymbol{p}(\theta,\lambda)\boldsymbol{\beta}\|_{\ell_p(\mathbb{F})} \leq M^{1/p}$. It follows from (2.29) that $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ by choosing $\theta' = 0$.

The proof of (2.28) is similar to that of (2.18) in Theorem 2.4. $\square$

Remark 2.1 We compare our method in establishing the sparsity for Laguerre GPC expansion coefficients with the method for Hermite GPC expansion coefficients presented in [12, Section 3.6]. Both methods are based on the bounds of parametric partial derivatives as in Lemma 2.1. They both also employ the criterion for ℓ_p-summability as in [9, Theorem 7.2].

A notable difference distinguishes the two methods. The method in [12, Section 3.6] relies on an identity between an weighted square sum of the energy norms of Hermite GPC expansion coefficients and weighted square sum of the $L_2(\mathbb{R}^\infty, V; \boldsymbol{\gamma})$-norms of the parametric partial derivatives [4, Theorem 3.3]. This equality is established by combining Parseval's identity and Rodrigues' formula, tool sets that do not extend to the Laguerre setting. By contrast, our method employs the identity (2.12) from Lemma 2.3 which expresses the derivative the derivative $\mathcal{D}^r_J v$ in terms of Laguerre GPC expansion coefficients of v. This representation is derived from the Laguerre's second order differential equation satisfied by the Laguerre polynomials.

3 Semi-discrete approximations

In this section, from the weighted ℓ_2-summability results of Theorems 2.4 and 2.6, we derive convergence rates for the semi-discrete linear parametric-variable approximations of the solution $u(\boldsymbol{y})$ to the parametric equation with with log-gamma random inputs (1.2)–(1.3)(Γ) by sparse-grid polynomial interpolations, extended least-squares sampling algorithms and the associated semi-discrete quadratures.

3.1 Sparse-grid polynomial interpolation

In this section, we construct sparse-grid polynomial interpolations for semi-discrete parametric approximation of the solution $u(\boldsymbol{y})$ to the parametric equation (1.2)–(1.3)(Γ). Observe that under the assumption of Lemma 2.2, Notice that $u(\boldsymbol{y})$ is well-defined for every $\boldsymbol{y} \in U_0$, where U_0 is the set defined as in (2.6). By Lemma 2.2, the set U_0 has full measure, i.e., $\boldsymbol{\lambda}(U_0) = 1$, and contains all $\boldsymbol{y} \in \mathbb{R}^\infty_+$ with $|\boldsymbol{y}|_0 < \infty$, where $|\boldsymbol{y}|_0$ denotes the number of nonzero components y_j of $\boldsymbol{y}$. Moreover, $u(\boldsymbol{y})$ can be treated as a representative of an element in $L_2(\mathbb{R}^\infty_+, V; \boldsymbol{\lambda})$.

For $m \in \mathbb{N}_0$, let $Y_m = (y_{m;k})_{k=1}^m$ be the increasing sequence of the m roots of the Laguerre polynomial L_m, ordered as $0 < y_{m;1} < \ldots < y_{m;m}$. We use also the convention $Y_0 = (y_{0;0})$ with $y_{0;0} = 0$.

For a function v on $\mathbb{R}_+$ taking values in a Hilbert space V and $m \in \mathbb{N}_0$, we define the Lagrange interpolation operator I_m by

$$I_m v := \sum_{k=1}^{m} v(y_{m;k})\ell_{m;k}, \quad \ell_{m;k}(y) := \prod_{1\leq j\leq m,\ j\neq k} \frac{y - y_{m;j}}{y_{m;k} - y_{m;j}}, \tag{3.1}$$

(in particular, $I_0 v = v(y_{0;0})\ell_{0;0}(y) = v(0)$ and $\ell_{0;0}(y) = 1$). Notice that $I_m v$ is a function on $\mathbb{R}_+$ taking values in V and interpolating v at $y_{m;k}$, i.e., $I_m v(y_{m;k}) = v(y_{m;k})$.

Let

$$\lambda_m := \sup_{\left\|v\sqrt{l_a}\right\|_{L_\infty(\mathbb{R}_+)}\le 1} \left\|(I_m v)\sqrt{l_a}\right\|_{L_\infty(\mathbb{R}_+)}$$

be the Lebesgue constant, where l_a is given as in (2.1). It was proven in [15] that

$$\lambda_m \ \approx \ C m^{1/6}, \quad m \in \mathbb{N},$$

for some positive constant C independent of m.

For a function v on $\mathbb{R}_+^\infty$ taking values in a Hilbert space V and $\boldsymbol{s} \in \mathbb{F}$, we introduce the tensor product operator $\Delta_{\boldsymbol{s}}$, $\boldsymbol{s} \in \mathbb{F}$, by

$$\Delta_{\boldsymbol{s}} v := \bigotimes_{j\in\mathbb{N}} \Delta_{s_j} v, \quad \Delta_m := \ I_m - I_{m-1}, \quad I_{-1} = 0 \ \ (m \in \mathbb{N}_0),$$

where the operator Δ_{s_j} is successively applied to the univariate function $\bigotimes_{i<j} \Delta_{s_i} v$ by considering it as a function of variable y_j with the other variables held fixed. We define for $\boldsymbol{s} \in \mathbb{F}$,

$$\ell_{\boldsymbol{s};\boldsymbol{k}} := \bigotimes_{j\in\mathbb{N}} \ell_{s_j;k_j}, \quad \pi_{\boldsymbol{s}} := \prod_{j\in\mathbb{N}} \pi_{s_j}.$$

For $\boldsymbol{s} \in \mathbb{F}$ and $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{s}$, let $E_{\boldsymbol{s}}$ be the subset in $\mathbb{F}$ of all $\boldsymbol{e}$ such that e_j is either 1 or 0 if $s_j > 0$, and e_j is 0 if $s_j = 0$, and let $\boldsymbol{y}_{\boldsymbol{s};\boldsymbol{k}} := (y_{s_j;k_j})_{j\in\mathbb{N}} \in \mathbb{R}_+^\infty$. Here, the inequities $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{s}$ mean by convention that $1 \le k_j \le s_j$ for $j \in \operatorname{supp}(\boldsymbol{s})$ and $\operatorname{supp}(\boldsymbol{k}) = \operatorname{supp}(\boldsymbol{s})$. Recall that $|\boldsymbol{s}|_1 := \sum_{j\in\mathbb{N}} s_j$ for $\boldsymbol{s} \in \mathbb{F}$. It is easy to check that the interpolation operator $\Delta_{\boldsymbol{s}}$ can be represented in the form

$$\Delta_{\boldsymbol{s}} v \ = \ \sum_{\boldsymbol{e}\in E_{\boldsymbol{s}}} (-1)^{|\boldsymbol{e}|_1} \sum_{\mathbf{1}\le\boldsymbol{k}\le\boldsymbol{s}-\boldsymbol{e}} v(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}) \ell_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}. \tag{3.2}$$

Let $0 < q < \infty$ and $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ be a set of positive numbers. Let the set $\Lambda(\xi)$ for $\xi > 1$ be defined by

$$\Lambda(\xi) := \ \big\{\boldsymbol{s} \in \mathbb{F} : \ \sigma_{\boldsymbol{s}} \le \xi^{1/q}\big\}. \tag{3.3}$$

We introduce the sparse-grid polynomial interpolation operator $I_{\Lambda(\xi)}$ by

$$I_{\Lambda(\xi)} := \sum_{\boldsymbol{s}\in\Lambda(\xi)} \Delta_{\boldsymbol{s}}.$$

By the formula (3.2) we can represent $I_{\Lambda(\xi)}$ in the form

$$I_{\Lambda(\xi)} v \ = \sum_{(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k})\in G(\xi)} (-1)^{|\boldsymbol{e}|_1} v(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}) \ell_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}, \tag{3.4}$$

where

$$G(\xi) := \ \{(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k}) \in \mathbb{F}\times\mathbb{F}\times\mathbb{F} : \ \boldsymbol{s} \in \Lambda(\xi), \ \boldsymbol{e} \in E_{\boldsymbol{s}}, \ \mathbf{1} \le \boldsymbol{k} \le \boldsymbol{s}-\boldsymbol{e}\}.$$

Note $I_{\Lambda(\xi)} v$ is determined by the values of v at the points $\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}$, $(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k}) \in G(\xi)$, and the number of these points is $|G(\xi)|$. Moreover, $|\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}|_0 < \infty$, and, consequently $\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}} \in U_0$ for every $(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k}) \in G(\xi)$. Hence, for the solution u to the parametric elliptic PDE (1.2)–(1.3)(Γ), the function $I_{\Lambda(\xi)} u$ is well-defined.

Theorem 3.1 *Let the assumptions of either Theorem 2.4 or Theorem 2.6 for $0 < p < 1$. Let $\Lambda(\xi)$ be the set defined in (3.3) for the set $\boldsymbol{\sigma}$ in these theorems satisfying (2.18) or (3.5), respectively. Then there exists a constant C such that for each $n > 1$, we can construct a sequence of points $(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}})_{(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k})\in G(\xi_n)}$ so that $|G(\xi_n)| \le n$ and*

$$\|u - I_{\Lambda(\xi_n)}u\|_{L_2(\mathbb{R}^\infty_+,V;\boldsymbol{\lambda})} \le Cn^{-(1/p-1)}.$$

Proof. This theorem can be proven similarly to [10, Corollary 3.1] for functions in the Bochner space $L_2(\mathbb{R}^\infty, X; \boldsymbol{\gamma})$. ∎

3.2 Sparse-grid quadrature for numerical integration

If v is a function defined on $\mathbb{R}$ taking values in a Hilbert space X, the function $I_m v$ in (3.1) generates the quadrature formula which is defined by

$$Q_m v := \int_{\mathbb{R}_+} I_m v(y)\,\mathrm{d}\lambda(y) \ = \ \sum_{k=0}^{m} \omega_{m;k}\, v(y_{m;k}), \quad \omega_{m;k} := \int_{\mathbb{R}_+} \ell_{m;k}(y)\,\mathrm{d}\lambda(y).$$

For a function v defined on $\mathbb{R}^\infty_+$ taking value in X, we introduce the operator $\Delta^{\mathrm{Q}}_{\boldsymbol{s}}$ defined for $\boldsymbol{s} \in \mathbb{F}$ by

$$\Delta^{\mathrm{Q}}_{\boldsymbol{s}} v := \bigotimes_{j\in\mathbb{N}} \Delta^{\mathrm{Q}}_{s_j} v, \quad \Delta^{\mathrm{Q}}_m := \ Q_m - Q_{m-1}, \ \ Q_{-1} := 0 \ \ (m \in \mathbb{N}),$$

where the univariate operator $\Delta^{\mathrm{Q}}_{s_j}$ is applied to the univariate function $\bigotimes_{i<j} \Delta^{\mathrm{Q}}_{s_i} v$ by considering it as a function of variable y_i with the other variables held fixed. For a finite set $\Lambda \subset \mathbb{F}$, we introduce the quadrature operator Q_Λ by

$$Q_\Lambda v := \sum_{\boldsymbol{s}\in\Lambda} \Delta^{\mathrm{Q}}_{\boldsymbol{s}} v.$$

Further, if $\phi \in V'$ is a bounded linear functional on V, denote by $\langle \phi, v\rangle$ the value of ϕ in v.

By the formula (3.4) we can represent the operator $Q_{\Lambda(\xi)}$ in the form

$$Q_{\Lambda(\xi)} v \ = \sum_{(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k})\in G(\xi)} (-1)^{|\boldsymbol{e}|_1} \omega_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}} v(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}}),$$

where $\omega_{\boldsymbol{s};\boldsymbol{k}} := \prod_{j\in\mathrm{supp}(\boldsymbol{s})} \omega_{s_j;k_j}$ for $\boldsymbol{s} \in \mathbb{F}$ and $\mathbf{1} \le \boldsymbol{k} \le \boldsymbol{s}$.

Theorem 3.2 *Under the assumption of Theorem 3.1, there exists a constant C such that for each $n > 1$, we can construct a sequence of points $(\boldsymbol{y}_{\boldsymbol{s}-\boldsymbol{e};\boldsymbol{k}})_{(\boldsymbol{s},\boldsymbol{e},\boldsymbol{k})\in G(\xi_n)}$ so that $|G(\xi_n)| \le n$ and*

$$\left\| \int_{\mathbb{R}^\infty_+} u(\boldsymbol{y})\,\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - Q_{\Lambda(\xi_n)} u \right\|_V \ \le \ Cn^{-(1/p-1)},$$

and, if additionally, $\phi \in V'$ is a bounded linear functional on V,

$$\left| \int_{\mathbb{R}^\infty_+} \langle \phi, u(\boldsymbol{y})\rangle \,\mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - \langle \phi, Q_{\Lambda(\xi_n)} u\rangle \right| \ \le \ C\,\|\phi\|_{V'}\, n^{-(1/p-1)}.$$

Proof. This theorem can be proven similarly to [10, Corollary 4.1] for functions in the Bochner space $L_2(\mathbb{R}^\infty, X; \boldsymbol{\gamma})$. ∎

3.3 Extended least squares approximation

We analyze the convergence rate of the extended least-squares sampling algorithms for the solution $u(\boldsymbol{y})$ to equation (1.2)–(1.3)(Γ). Let the assumptions of either Theorem 2.4 or Theorem 2.6 hold for $0 < p < 2$. These theorems state that one can construct a set $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$, and a constant M such that

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|u_{\boldsymbol{s}}\|_V)^2\right)^{1/2} \leq M^{1/2} < \infty, \quad \text{with} \quad \left\|\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{F})} \leq M^{1/q} < \infty, \tag{3.5}$$

where the set $\boldsymbol{\sigma}$ and constant M depend on $a, \boldsymbol{b}, p, \theta, \lambda$ only. Provided this weighted ℓ_2-summability, the series (2.2) converges unconditionally in $L_2(\mathbb{R}^\infty_+, V; \boldsymbol{\lambda})$ to u. This unconditional convergence can be established in the same way as the proof of [11, Lemma 3.1]. Hence, putting $U := \mathbb{R}^\infty_+$, $\sigma_j := \sigma_{\boldsymbol{s}_j}$, $\varphi_j := L_{\boldsymbol{s}_j}$ and $u_j := u_{\boldsymbol{s}_j}$, by (3.5) we can reorder the countable set $\mathbb{F}$ as $\mathbb{F} = (\boldsymbol{s}_j)_{j\in\mathbb{N}}$ so that the sequence $(\sigma_{\boldsymbol{s}_j})_{j\in\mathbb{N}}$ is non-decreasing, and the weak solution u is represented by the series

$$u = \sum_{j\in\mathbb{N}} u_j \, \varphi_j,$$

with

$$\left(\sum_{j\in\mathbb{N}} (\sigma_j \|u_j\|_V)^2\right)^{1/2} \leq M^{1/2}, \quad \text{with} \quad \left\|\left(\sigma_j^{-1}\right)_{j\in\mathbb{N}}\right\|_{\ell_q(\mathbb{N})} \leq M^{1/q}. \tag{3.6}$$

Notice that $u(\boldsymbol{y})$ is well-defined for every $\boldsymbol{y} \in U_0$, where U_0 is the set defined as in (2.6) for $U = \mathbb{R}^\infty_+$. By Lemma 2.2, the set U_0 has full measure, i.e., $\boldsymbol{\lambda}(U_0) = 1$.

Let us construct an extension of a least squares approximation in the space $L_2(\mathbb{R}^\infty_+, \mathbb{C}; \boldsymbol{\lambda})$ to a space $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ (for detail in the general setting, see [6]). For $n, m \in \mathbb{N}$ with $n \geq m$, let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n \in U_0$ be points, $\omega_1, \ldots, \omega_n \geq 0$ be weights, and $V_m = \operatorname{span}\{\varphi_j\}_{j=1}^m$ the subspace spanned by the functions φ_j, $j = 1, ..., m$. The weighted least squares approximation $S_n^{\mathbb{C}} f = S_n^{\mathbb{C}}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n, \omega_1, \ldots, \omega_n, V_m) f$ of a function $f\colon \mathbb{R}^\infty_+ \to \mathbb{C}$ is given by

$$S_n^{\mathbb{C}} f = \operatorname*{arg\,min}_{g\in V_m} \sum_{i=1}^n \omega_i |f(\boldsymbol{y}_i) - g(\boldsymbol{y}_i)|^2. \tag{3.7}$$

For every $n \in \mathbb{N}$, let

$$S_n^{\mathbb{C}} f := \sum_{i=1}^n f(\boldsymbol{y}_i) h_i,$$

be the least squares sampling algorithm constructed by (3.7) for these sample points and weights, where $h_1, ..., h_n \in L_2(\mathbb{R}^\infty_+, \mathbb{C}; \boldsymbol{\lambda})$. We define the extension of this least squares algorithm to the Bochner space $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ by replacing $f \in L_2(\mathbb{R}^\infty_+, \mathbb{C}; \boldsymbol{\lambda})$ with $v \in L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$:

$$S_n^X v := S_n^X(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n, \omega_1, \ldots, \omega_n, V_m) v := \sum_{i=1}^n v(\boldsymbol{y}_i) h_i.$$

As the least squares approximation is a linear operator, worst-case error bounds carry over from the usual Lebesgue space $L_2(\mathbb{R}^\infty_+, \mathbb{C}; \boldsymbol{\lambda})$ to the Bochner space $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$. We define $\tilde{S}_n^X$ by

$$\tilde{S}_n^X v := S_n^X(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n, \omega_1, \ldots, \omega_n, V_m) v := \sum_{i=1}^n v(\boldsymbol{y}_i) h_i, \quad \text{for} \quad m := \left\lceil \frac{n}{43200} \right\rceil.$$

Theorem 3.3 *Let the assumptions of either Theorem 2.4 or Theorem 2.6 hold for* $0 < p < 2$. *Then there exist a constant* C *such that for all* $n \in \mathbb{N}$,

(i)
$$\left\| u - \tilde{S}_n^V u \right\|_{L_2(\mathbb{R}_+^\infty, V; \boldsymbol{\lambda})} \leq CMn^{-(1/p-1/2)};$$

(ii) *moreover, for the quadrature*
$$\tilde{Q}_n^V u = \sum_{i=1}^n w_i u(\boldsymbol{y}_i), \quad w_i := \int_{\mathbb{R}_+^\infty} \tilde{h}_i(\boldsymbol{y}) \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}),$$

it holds
$$\left\| \int_{\mathbb{R}_+^\infty} u(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - \tilde{Q}_n^V u \right\|_V \leq CMn^{-(1/p-1/2)}, \tag{3.8}$$

and, if additionally, $\phi \in V'$ *be a bounded linear functional on* V,
$$\left| \int_{\mathbb{R}_+^\infty} \langle \phi, u(\boldsymbol{y}) \rangle \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - \left\langle \phi, \tilde{Q}_n^V u \right\rangle \right| \leq C \|\phi\|_{V'} n^{-(1/p-1/2)}. \tag{3.9}$$

Proof. Claim (i) is directly derived from [6, Corollary 2.1] and the weighted ℓ_2-summability (3.6). Let us prove claim (ii). From the equality
$$\int_{\mathbb{R}_+^\infty} u(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - \tilde{Q}_n^V u = \int_{\mathbb{R}_+^\infty} \left(u(\boldsymbol{y}) - \tilde{S}_n^V u(\boldsymbol{y}) \right) \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}),$$

and the claim (i) it follows that
$$\begin{aligned} \left\| \int_{\mathbb{R}_+^\infty} u(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - \tilde{Q}_n^V u \right\|_V &\leq \left\| u - \tilde{S}_n^V u \right\|_{L_1(\mathbb{R}_+^\infty, V; \boldsymbol{\lambda})} \\ &\leq \left\| u - \tilde{S}_n^V u \right\|_{L_2(\mathbb{R}_+^\infty, V; \boldsymbol{\lambda})} \leq CMn^{-(1/p-1/2)}, \end{aligned}$$

which proves (3.8). In a similar way, we can establish (3.9) by using the equality
$$\int_{\mathbb{R}_+^\infty} \langle \phi, u(\boldsymbol{y}) \rangle \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}) - \left\langle \phi, \tilde{Q}_n^V u \right\rangle = \int_{\mathbb{R}_+^\infty} \left\langle \phi, u(\boldsymbol{y}) - \tilde{S}_n^V u(\boldsymbol{y}) \right\rangle \, \mathrm{d}\boldsymbol{\lambda}(\boldsymbol{y}).$$

□

Remark 3.1 The convergence rate of the approximation of u by least squares sampling algorithms (Theorem 3.3) is markedly superior to that of the approximation of u by sparse-grid polynomial interpolations (Theorem 3.1) by a factor of $n^{-1/2}$. The former matches the optimal rate of best n-term polynomial approximation of u, which can be shown via Stechkin's lemma (see, e.g., [8, Section 3]).

4 Parametric PDEs with Gaussian log-normal inputs

In this section, we present a counterpart of our method for parametric elliptic PDEs with Gaussian log-normal inputs. Throughout the section, we denote by u or $u(\boldsymbol{y})$ the weak solution to equation (1.2)–(1.3)(G).

Let γ be the standard Gaussian probability measure on $\mathbb{R}$ with the density

$$g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Let $(H_k)_{k\in\mathbb{N}_0}$ be the sequence of Hermite polynomials on $\mathbb{R}$ normalized with respect to the measure γ, i.e.,

$$\int_{\mathbb{R}} |H_k(y)|^2 \mathrm{d}\gamma(y) = \int_{\mathbb{R}} |H_k(y)|^2 g(y)\mathrm{d}y = 1, \quad k \in \mathbb{N}_0.$$

The the Bochner space $L_2(\mathbb{R}^\infty, X; \boldsymbol{\gamma})$ and the associated infinite tensor product standard Gaussian measure $\boldsymbol{\gamma}(\boldsymbol{y})$ on $\mathbb{R}^\infty$ can be defined in the same fashion as the Bochner space $L_2(\mathbb{R}^\infty_+, X; \boldsymbol{\lambda})$ and the measure $\boldsymbol{\lambda}(\boldsymbol{y})$.

A function $v \in L_2(\mathbb{R}^\infty, X; \boldsymbol{\gamma})$ can be represented by the Gaussian GPC expansion

$$v(\boldsymbol{y}) = \sum_{\boldsymbol{s}\in\mathbb{F}} v_{\boldsymbol{s}}\, H_{\boldsymbol{s}}(\boldsymbol{y}), \quad v_{\boldsymbol{s}} \in X,$$

with convergence in $L_2(\mathbb{R}^\infty, X; \boldsymbol{\gamma})$, where

$$H_{\boldsymbol{s}}(\boldsymbol{y}) = \bigotimes_{j\in\mathbb{N}} H_{s_j}(y_j), \quad v_{\boldsymbol{s}} := \int_{\mathbb{R}^\infty} v(\boldsymbol{y})\, H_{\boldsymbol{s}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{\gamma}(\boldsymbol{y}), \quad \boldsymbol{s} \in \mathbb{F}.$$

It is known that, the Hermite polynomials $\left(H_k\right)_{k\in\mathbb{N}_0}$ are nontrivial solution to Hermite's (linear second-order) differential equation

$$\mathcal{D}u = ku, \quad \text{where} \quad \mathcal{D} := -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y\frac{\mathrm{d}}{\mathrm{d}y}$$

(see, e.g., [17, 5.5.2]). This means that

$$\mathcal{D}H_k = H_k, \quad k \in \mathbb{N}_0.$$

Analogously to the Laguerre polynomials, it holds the equality

$$\mathcal{D}u = -e^{y^2/2}\frac{\mathrm{d}}{\mathrm{d}y}\left(e^{-y^2/2}\frac{\mathrm{d}u}{\mathrm{d}y}\right).$$

For a given finite set $J \subset \mathbb{N}$ and $r \in \mathbb{N}$, we define the product differential operator

$$\mathcal{D}_J^r := \left(\mathcal{D}_J\right)^r, \quad \mathcal{D}_J := (-1)^{|J|} \prod_{j\in J} e^{y_j^2/2}\frac{\mathrm{d}}{\mathrm{d}y_j}\left(e^{-y_j^2/2}\frac{\mathrm{d}}{\mathrm{d}y_j}\right).$$

Notice that there exist polynomials $q_j(y)$, $j = 1, ..., 2r$, of degree at most r such that

$$\mathcal{D}^r = \sum_{j=1}^{2r} q_j(y)\frac{\mathrm{d}^j}{\mathrm{d}y^j},$$

and the coefficients of $q_j(y)$ depend on r only. Hence, there exists a constant C_r such that

$$|q_j(y)| \le C_r(1+|y|)^r, \quad j = 1, \dots, 2r. \tag{4.1}$$

Hence, in a way similar to the proof of Lemma 2.3 we can prove

Lemma 4.1 *If $v \in L_2(\mathbb{R}^\infty, V; \boldsymbol{\gamma})$ and $\mathcal{D}_J^r v \in L_2(\mathbb{R}^\infty, V; \boldsymbol{\gamma})$, then we have that*

$$\mathcal{D}_J^r v = \sum_{\boldsymbol{s} \in \mathbb{F}} \nu_{J,\boldsymbol{s}}^r v_{\boldsymbol{s}} H_{\boldsymbol{s}},$$

where $\nu_{J,\boldsymbol{s}}$ is defined in (2.11).

We recall the following result proven in [4, Theorem 2.2].

Lemma 4.2 *Assume that*

$$\exists \ (\rho_j)_{j\in\mathbb{N}},\ \rho_j > 0\text{:}\ \sum_{j\in\mathbb{N}} \rho_j |\psi_j| \ \textit{converges in}\ L_\infty(D)\ \textit{and} \quad \sum_{j\in\mathbb{N}} \exp\big(-\rho_j^2\big) < \infty.$$

Then for any $k \in \mathbb{N}$

$$\int_{\mathbb{R}^\infty} \exp\big(k\|b(\boldsymbol{y})\|_{L_\infty(D)}\big) \mathrm{d}\boldsymbol{\gamma}(\boldsymbol{y}) < \infty.$$

We put

$$\tilde{A}_r(J) := \left(\int_{\mathbb{R}^\infty} \prod_{j\in J} (1+y_j)^{2r} \exp\Big(2\|b(\boldsymbol{y})\|_{L_\infty(D)}\Big) \mathrm{d}\boldsymbol{\gamma}(\boldsymbol{y}) \right)^{1/2}.$$

Under the assumption of Lemma 4.2, by Hölder's inequality we deduce that

$$\tilde{A}_r(J)^4 \ \le \ \int_{\mathbb{R}^\infty} \exp\big(4\|b(\boldsymbol{y})\|_{L_\infty(D)}\big) \mathrm{d}\boldsymbol{\gamma}(\boldsymbol{y}) \left(\int_{\mathbb{R}} (1+y)^{4r} \mathrm{d}\gamma(y) \right)^{|J|}. \tag{4.2}$$

The following result can be established analogously, with appropriate modifications to the proof of Theorem 2.4 by using, in particular, inequality (4.2).

Theorem 4.3 *Let $0 < p < \infty$, $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ be a positive sequence satisfying condition* (2.4), *and $\boldsymbol{\rho}^{-1} \in \ell_p(\mathbb{N})$. Then $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$.*

Moreover, if in addition, $p < 2$, for any fixed $\theta, \lambda \ge 0$, we can construct a set $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ with positive $\sigma_{\boldsymbol{s}}$, and a constant M such that

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|u_{\boldsymbol{s}}\|_V)^2 \right)^{1/2} \le M^{1/2} < \infty, \quad \textit{with} \quad \big\|\boldsymbol{p}(\theta,\lambda)\boldsymbol{\sigma}^{-1}\big\|_{\ell_q(\mathbb{F})} \le M^{1/q} < \infty,$$

where $q := 2p/(2-p)$. The set $\boldsymbol{\sigma}$ and constant M depend on $a, p, \theta, \lambda, \kappa$ only.

We have the following estimate for $\tilde{A}_r(J)$ in the case of arbitrary supports of ψ_j.

Lemma 4.4 *Let* $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ *be defined by* $b_j := \|\psi_j\|_{L_\infty(D)}$. *Assume* $\boldsymbol{b} \in \ell_1(\mathbb{N})$ *and*

$$\|\boldsymbol{b}\|_{\ell_\infty(\mathbb{N})} = b_0 < \infty.$$

Then

$$\tilde{A}_r(J) \le K_{r,\boldsymbol{b}}^{|J|} \exp\left(\|\boldsymbol{b}\|_{\ell_2(\mathbb{N})}^2 + \frac{\sqrt{2}}{\pi}\|\boldsymbol{b}\|_{\ell_1(\mathbb{N})}\right),$$

where

$$K_{r,\boldsymbol{b}} := \left(\int_{\mathbb{R}} \frac{(1+y)^{2r}}{\sqrt{2\pi}} \exp\left(2b_0|y| - \frac{|y|^2}{2}\right)\mathrm{d}y\right)^{1/2} < \infty. \tag{4.3}$$

Proof. Indeed, by using the inequality

$$\int_{\mathbb{R}} \exp(b|y|)\mathrm{d}\gamma(y) \le \exp\left(\frac{b^2}{2} + \frac{\sqrt{2}b}{\pi}\right),$$

(see, e.g., [4, (38)]), we derive for any finite set $J \subset \mathbb{N}$,

$$\begin{aligned}
\tilde{A}_r(J)^2 &\le \int_{\mathbb{R}^\infty} \prod_{j\in J}(1+y_j)^{2r} \exp\left(2\sum_{j\in\mathbb{N}} b_j y_j\right) \prod_{j\in\mathbb{N}} \mathrm{d}\gamma(y_j) \\
&\le \prod_{j\in J} \int_{\mathbb{R}} \frac{(1+y_j)^{2r}}{\sqrt{2\pi}} \exp\left(2b_0|y_j| - \frac{|y_j|^2}{2}\right)\mathrm{d}y_j \prod_{j\notin J} \int_{\mathbb{R}} \exp\left(2b_j|y_j|\right)\mathrm{d}\gamma(y_j). \\
&\le K_{r,\boldsymbol{b}}^{2|J|} \prod_{j\notin J} \exp\left(2b_j^2 + \frac{2\sqrt{2}b_j}{\pi}\right) \le K_{r,\boldsymbol{b}}^{2|J|} \exp\left(2\|\boldsymbol{b}\|_{\ell_2(\mathbb{N})}^2 + \frac{2\sqrt{2}}{\pi}\|\boldsymbol{b}\|_{\ell_1(\mathbb{N})}\right).
\end{aligned}$$

□

In a manner analogous to the proof of Theorem 2.6, with certain modifications we can obtain the following result.

Theorem 4.5 *Let* $0 < p \le 1$ *and* $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ *be defined by* $b_j := \|\psi_j\|_{L_\infty(D)}$. *Let* $r = r_{p,0} \in \mathbb{N}$ *be a fixed number satisfying* (2.15). *Assume that* $\boldsymbol{b} \in \ell_p(\mathbb{N})$ *and*

$$\|\boldsymbol{b}\|_{\ell_1(\mathbb{N})} < K^{-1},$$

where

$$K := e(2r)!C_r C_{p,0} K_{r,\boldsymbol{b}}, \tag{4.4}$$

and the constants C_r, $C_{p,0}$, $K_{r,\boldsymbol{b}}$ *are defined as in* (4.1), (2.15), (4.3) *respectively. Then* $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$.

Moreover, for any fixed $\theta, \lambda \ge 0$, *if* $q := 2p/(2-p)$, $\theta' = 2\theta/(2-p)$, $r = r_{p,\theta'}$ *in* (2.15) *and* K *in* (4.4) *is replaced by*

$$K := e(2r)!C_r C_{p,\theta'} C_{\theta',\lambda} K_{r,\boldsymbol{b}}$$

with $C_{p,\theta'}$ *being defined as in* (2.14), *then we can construct a set* $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$, *and a constant* M *such that*

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}}\|u_{\boldsymbol{s}}\|_V)^2\right)^{1/2} \le M^{1/2} < \infty, \quad \text{with} \quad \left\|\boldsymbol{p}(\theta,\lambda)\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{F})} \le M^{1/q} < \infty,$$

where the set $\boldsymbol{\sigma}$ *and constant* M *depend on* $\boldsymbol{b}, p, \theta, \lambda$ *only.*

Remark 4.1 Theorem 4.3 presents a sufficient condition for the ℓ_p-summability $(\|u_{\boldsymbol{s}}\|_V)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$ for $0 < p < \infty$. Sufficient conditions for this ℓ_p-summability considered in [14, Proposition 4.4] for $0 < p \leq 1$, and in [4, Theorem 1.2] for $0 < p < 2$. For $0 < p < 2$, the ℓ_p-summability result of [4, Theorem 1.2] under the condition $\boldsymbol{\rho}^{-1} \in \ell_q(\mathbb{N})$ with $q := 2p/(2-p)$, is better than the ℓ_p-summability result of Theorem 4.3.

Consider the component function ψ_j with arbitrary supports in the parametric equation with log-normal random inputs (1.2)–(1.3)(G), including the case of globally supported functions such as the Fourier series. In this context, the sufficient condition $\big(\|\psi_j\|_{L_\infty(D)}\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ in Theorem 4.5 for the ℓ_p-summability $\big(\|u_{\boldsymbol{s}}\|_V\big)_{\boldsymbol{s}\in\mathbb{F}} \in \ell_p(\mathbb{F})$, significantly improves the sufficient conditions $\big(\|j\psi_j\|_{L_\infty(D)}\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ in [14], and $\big(\|j^\alpha\psi_j\|_{L_\infty(D)}\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ with some $\alpha > 1/2$, in [4, Corollary 6.3] and [12, Section 3.6.2].

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