

# ELLIPTIC PDES ON LOG-GAUSSIAN SHAPES: SPARSITY AND FINITE ELEMENT DISCRETIZATION

DINH DŨNG, HELMUT HARBRECHT, VAN KIEN NGUYEN, AND CHRISTOPH SCHWAB

ABSTRACT. In this article, we consider the solution to elliptic diffusion problems on a class of random domains obtained by log-Gaussian random homothety of the unit disk respectively an annulus. We model the problem under consideration and verify the existence and uniqueness of the random solution by path-wise pullback to the nominal unit disk respectively annulus. We moreover prove the analytic regularity of the solution with respect to the random input parameter. Finally, we consider the numerical approximation of the random diffusion problem by means of a continuous, piecewise linear Lagrangian finite element method with numerical quadrature in the nominal domain, and by sparse grid interpolation and quadrature of Gauss-Hermite Smolyak and Quasi-Monte Carlo type in the parameter domain. The theoretical findings are complemented by numerical results.

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## 1. INTRODUCTION

Domain uncertainties appear in many applications from engineering and sciences since the shape of the object under consideration may not be perfectly known. For example, one might think of manufacturing imperfections in the shape of products fabricated by line production, or shapes which stem from inverse problems such as tomography. Similarly, the boundary of the object might not be well-defined in the sense that it is the result of a smooth transition between different materials or states. Here, one may think of applications in cell biology for example.

Boundary value problems on random domain have been considered by several authors, see [7, 8, 20, 22, 24, 27, 29] for example. However, in all these articles, the domain has been modeled by bounded random variables to ensure the random domain under consideration does not degenerate. In contrast, we model the random domain by using log-Gaussian distributed random variables. To this end, we restrict ourselves to homothetic and star-shaped random domains where the logarithm of radial function is a Fourier series with Gaussian coefficients. This setup guarantees that almost every realization of the random domain is homothetic to a nominal annulus resp. to the unit disc, and hence well-defined.

Since the realizations of the random domain can exhibit large variations, we apply the domain mapping approach meaning that the given boundary value problem on the random domain is pulled back to the unit disc resp. to a nominal annular domain. Thus, the boundary value problem posed on the random domain becomes a boundary value problem posed on a deterministic domain but with log-Gaussian random diffusion matrix and right-hand side. This allows to extend the complex-variable methods proposed in [10, Section 2.2] and, specifically, in [15, Section 3.6], using holomorphic extension of the solution to the last equation to establish bounds for its parametric partial derivatives in the energy norm, for the analysis of Sparse Grid and Quasi-Monte Carlo integration.

**1.1. Contents.** This paper is organized as follows. We present the model problem and the geometry parametrization in Section 2. The model problem is formulated in Section 2.1 and upon homothetic pullback on the reference domain in Section 2.2. Geometry parametrization and precise analytic dependency on the input parameters is investigated in Section 2.3. Next, in Section 3, we consider the situation that the input parameters are random. Geometry uncertainty is modelled by tensor-product Gaussian measures on the parameter sequences. The particular log-Gaussian domain model is formulated in Section (3.1). Holomorphy of the parameters-to-solution map is verified in 3.2, where the precise bounds of the particular partial derivatives are derived in Section 3.3. The semidiscretization with respect to the random parameter is the topic of Section 4. Here, we consider the sparse grid interpolation of the random solution in Section 4.1 and the quasi-Monte Carlo quadrature based on Halton points for directly computing the random solution's expectation in Section 4.2. The finite element discretization and implementation of the problem under consideration is the topic of Section 5. The finite element discretization in the spatial variable of a generic parameter-dependent boundary value problem is analysed in Section 5.1, while numerical quadrature is studied in Section 5.2. This theory is then applied to the problem under consideration in Section 5.3. Numerical experiments which verify our theoretical findings are carried out in Section 6. Finally, the conclusion is drawn in Section 7.

**1.2. Notation.** We denote the natural numbers by  $\mathbb{N} = \{1, 2, \dots\}$ , where  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . As usual,  $\mathbb{Z}$  are the integers,  $\mathbb{R}$  the real numbers,  $\mathbb{R}_+$  the real non-negative numbers, and  $\mathbb{C}$  the complex numbers. For a complex number  $z \in \mathbb{C}$ ,  $\Re(z)$  and  $\Im(z)$  denote the real and imaginary parts of  $z$ , respectively.

Denote by  $\mathbb{R}^{\mathbb{N}}$  and  $\mathbb{R}^{\mathbb{Z}}$  the sets of all sequences  $\mathbf{y} = (y_j)_{j \in \mathbb{N}}$  and  $\mathbf{y} = (y_j)_{j \in \mathbb{Z}}$ , respectively, with  $y_j \in \mathbb{R}$ . For a nonnegative sequence  $\boldsymbol{\rho} = (\rho_j)_{j \geq 1} \in [0, \infty)^{\mathbb{N}}$ , we denote its support  $\text{supp}(\boldsymbol{\rho}) = \{j \in \mathbb{N} : \rho_j > 0\}$ . With  $|\mathbf{y}|_0$  we count the number of nonzero components  $y_j$  of  $\mathbf{y} = (y_j)_{j \in \mathbb{N}}$ . The set  $\mathbb{F}$  consists of all sequences of non-negative integers  $\boldsymbol{\nu} = (\nu_j)_{j \in \mathbb{N}}$  such that  $\text{supp}(\boldsymbol{\nu}) := \{j \in \mathbb{N} : \nu_j > 0\}$  is a finite set. For  $\boldsymbol{\nu}, \mathbf{k} \in \mathbb{F}$  and  $k \in \mathbb{N}$ ,  $|\boldsymbol{\nu}|_1 := \sum_{j \in \mathbb{N}} \nu_j$ ,  $|\boldsymbol{\nu}|_\infty := \max\{\nu_j, j \in \mathbb{N}\}$ , and

$$\boldsymbol{\nu}^k := \prod_{j \in \text{supp}(\boldsymbol{\nu})} \nu_j^k, \quad \boldsymbol{\nu}^{\mathbf{k}} := \prod_{j \in \text{supp}(\boldsymbol{\nu})} \nu_j^{k_j}, \quad \boldsymbol{\nu}! := \prod_{j \in \mathbb{N}} \nu_j!$$

If  $\boldsymbol{\alpha} = (\alpha_j)_{j \in \mathcal{J}}$  is a set of positive numbers with any index set  $\mathcal{J} \subseteq \mathbb{N}$ , we use the notation  $\boldsymbol{\alpha}^{-1} := (\alpha_j^{-1})_{j \in \mathcal{J}}$ . Given the sets  $\boldsymbol{\alpha} = (\alpha_j)_{j \in \mathcal{J}}$  and  $\boldsymbol{\beta} = (\beta_j)_{j \in \mathcal{J}}$ , we define  $\boldsymbol{\alpha}^{\boldsymbol{\beta}} := (\alpha_j^{\beta_j})_{j \in \mathcal{J}}$ .

We use the letters  $C$  and  $K$  to denote general positive constants which may take different values, and  $C_{\alpha, \beta, \dots}$  and  $K_{\alpha, \beta, \dots}$  constants depending on  $\alpha, \beta, \dots$ . For the quantities  $A_n(f, \mathbf{k}, \mathbf{y}, \dots)$  and  $B_n(f, \mathbf{k}, \mathbf{y}, \dots)$  depending on  $n, f, \mathbf{k}, \mathbf{y}, \dots$ , we write  $A_n(f, \mathbf{k}, \mathbf{y}, \dots) \lesssim B_n(f, \mathbf{k}, \mathbf{y}, \dots)$  if there exists a constant  $C > 0$  independent of  $n, f, \mathbf{k}, \mathbf{y}, \dots$  such that  $A_n(f, \mathbf{k}, \mathbf{y}, \dots) \leq CB_n(f, \mathbf{k}, \mathbf{y}, \dots)$ . Finally, we denote by  $|G|$  the cardinality of the set  $G$ .

## 2. ELLIPTIC DIFFUSION PROBLEMS ON VARIABLE DOMAINS

**2.1. A class of homothetic parametric domains.** We shall first model the class of random domains under consideration. The model we use is in accordance with [20], where however uniformly distributed random variables on a compact domain have been considered for the input parameters. Since log-Gaussian random domains can become unbounded, we restrict ourselves here to homothetic domains in order to ensure that the domains under consideration are always well-defined.

To this end, we consider the parametric Poisson equation

$$(1) \quad -\Delta u(a) = f \text{ in } D_\kappa(a), \quad u(a) = 0 \text{ on } \partial D_\kappa(a),$$

where  $f \in L^2(\mathbb{R}^2)$  and  $\kappa \in [0, 1)$  and  $a$  describe the shape of the domain  $D_\kappa(a) \subset \mathbb{R}^2$  in polar coordinates according to the homothety  $r \mapsto a(\theta)r$ , i.e.,

$$(2) \quad D_\kappa(a) := \text{int} \{ \mathbf{x} = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 : 0 \leq \kappa a(\theta) \leq r < a(\theta) \}.$$

Here, “int” denotes the interior of a set. In (2), we take  $a \in W_\infty^1(\mathbb{S}^1)$ , understood to be positive, and  $2\pi$ -periodic, where  $\mathbb{S}^1$  is the unit circle and  $W_\infty^1(\mathbb{S}^1)$  is the space of  $2\pi$ -periodic, real-valued, Lipschitz-continuous functions on  $\mathbb{S}^1$ , equipped with the norm

$$\|a\|_{W_\infty^1(\mathbb{S}^1)} := \|a\|_{L^\infty(\mathbb{S}^1)} + \|a'\|_{L^\infty(\mathbb{S}^1)}.$$

When  $\|a\|_{W_\infty^1(\mathbb{S}^1)}$  is finite and  $a(\theta) > 0$ ,  $\theta \in \mathbb{S}^1$ , the domain  $D_0(a)$  is a bounded domain. In particular, the domain  $D_\kappa(a)$  is Lipschitz since the boundary admits a global representation with respect to the polar angle as graph of the Lipschitz smooth radial function. Thus, there exists a unique weak solution  $u(a) \in$

$H_0^1(D_\kappa(a))$  to (1), satisfying the variational formulation

$$(3) \quad \int_{D_\kappa(a)} \nabla u(a) \cdot \nabla v = \langle f, v \rangle, \quad v \in H_0^1(D_\kappa(a)).$$

We remark that for  $0 < \kappa < 1$ ,  $\partial D_\kappa(a)$  has two components, one of them with arbitrary large curvature as  $\kappa \downarrow 0$ .

**2.2. Pullback of the solution to a reference domain.** In order to study the parameter dependent problem (1) and (2), we transform it to a parameter independent, fixed *reference domain*  $D_{\text{ref},\kappa}$ . Throughout this article, we consider for  $0 \leq \kappa < 1$  the reference domain

$$(4) \quad D_{\text{ref},\kappa} := \text{int}\{\boldsymbol{\xi} \in \mathbb{R}^2 : \kappa \leq \|\boldsymbol{\xi}\|_2 < 1\}.$$

For  $\kappa = 0$ ,  $D_{\text{ref},\kappa}$  is the open unit disc, and  $D_{\text{ref},\kappa}$  and  $D_\kappa(a)$  are star-shaped with respect to the origin. For  $0 < \kappa < 1$ ,  $D_{\text{ref},\kappa}$  is an annulus.

Given  $a$  and  $\kappa \in [0, 1)$ , let  $F(a)$  be the homothetic transform which maps  $D_{\text{ref},\kappa}$  onto  $D_\kappa(a)$  that is defined by

$$(5) \quad F(a)(\boldsymbol{\xi}) := F(a)(r \cos \theta, r \sin \theta) := (a(\theta)r \cos \theta, a(\theta)r \sin \theta)$$

for  $\boldsymbol{\xi} = (r \cos \theta, r \sin \theta) \in D_{\text{ref},\kappa}$ . The Jacobian matrix of  $F(a)(\boldsymbol{\xi})$  is given by

$$\frac{dF(a)}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) = a(\theta) \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & h(\theta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

with

$$(6) \quad h(\theta) := \frac{a'(\theta)}{a(\theta)},$$

where  $a'(\theta)$  denotes the derivative with respect to variable  $\theta$ . The determinant of the Jacobian is

$$(7) \quad J(a)(\boldsymbol{\xi}) = \det \left( \frac{dF(a)}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \right) = a(\theta)^2 > 0,$$

which implies that the map  $F(a)(\boldsymbol{\xi})$  is one-to-one.

We denote the pullback solution of (1) by

$$(8) \quad \hat{u}(a) = u(a) \circ F(a) \in V, \quad \text{where } V := H_0^1(D_{\text{ref},\kappa}).$$

Utilizing  $F(a)$  as a change of variable in (3),  $\hat{u}$  solves

$$(9) \quad \hat{u}(a) \in V : \int_{D_{\text{ref},\kappa}} M(a) \nabla \hat{u}(a) \cdot \nabla v = \langle f_{\text{ref}}(a), v \rangle \quad \forall v \in V.$$

This is the variational formulation of the equation

$$(10) \quad -\text{div}(M(a) \nabla \hat{u}(a)) = f_{\text{ref}}(a) \text{ in } D_{\text{ref},\kappa}, \quad \hat{u}(a) = 0 \text{ on } \partial D_{\text{ref},\kappa}$$

(for  $0 < \kappa < 1$ ,  $\partial D_{\text{ref},\kappa} = \kappa \mathbb{S}^1 \cup \mathbb{S}^1$  and for  $\kappa = 0$ ,  $\partial D_{\text{ref},\kappa} = \mathbb{S}^1$ ) with the diffusion matrix

$$(11) \quad \begin{aligned} M(a)(\boldsymbol{\xi}) &:= J(a)(\boldsymbol{\xi}) \left( \frac{dF(a)}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \right)^{-1} \left( \frac{dF(a)}{d\boldsymbol{\xi}}(\boldsymbol{\xi}) \right)^{-\top} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 + h(\theta)^2 & -h(\theta) \\ -h(\theta) & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$

and the right-hand side

$$(12) \quad f_{\text{ref}}(a)(\boldsymbol{\xi}) := J(a)(\boldsymbol{\xi})(f \circ F(a)(\boldsymbol{\xi})).$$

Note that the equation (1) on the random domain  $D_\kappa(a)$  and the equation (10) on the deterministic domain  $D_{\text{ref},\kappa}$  are in different Cartesian coordinate systems: (1) is in  $\boldsymbol{x}$ -coordinates, while (10) is in  $\boldsymbol{\xi}$ -coordinates. The variables  $\boldsymbol{x}$  and  $\boldsymbol{\xi}$  are connected by the equation  $\boldsymbol{x} = F(a)(\boldsymbol{\xi})$  of the change of variables. For simplicity,  $\boldsymbol{x}$  and  $\boldsymbol{\xi}$  are omitted in the sequel in some equations and expressions.

**2.3. Parametric analyticity of the pullback solution.** Throughout the rest of this article, we assume

$$(13) \quad f \text{ is a fixed, real analytic function on } \mathbb{R}^2 \text{ and } a \in W_\infty^1(\mathbb{S}^1; \mathbb{C}).$$

We define the complex extension of the weak solution as

$$(14) \quad \hat{u}(a) \in V_{\mathbb{C}} : \quad B(\hat{u}(a), v; a) = L(v; a) \quad \forall v \in V_{\mathbb{C}},$$

where  $V_{\mathbb{C}} = H_0^1(D_{\text{ref},\kappa}; \mathbb{C})$  and with  $\bar{v}$  denoting the complex conjugate for  $v \in V_{\mathbb{C}}$ .

$$(15) \quad B(\hat{u}(a), v; a) = \int_{D_{\text{ref},\kappa}} M(a) \nabla \hat{u}(a) \cdot \nabla \bar{v} \quad \text{and} \quad L(v; a) = \langle f_{\text{ref}}(a), \bar{v} \rangle.$$

Due to the expressions of  $M(a)$  in (11) and of  $J(a)$  in (7), and the assumption that  $f$  is real analytic, the mapping  $a \mapsto L(\cdot; a)$  is holomorphic from  $W_\infty^1(\mathbb{S}^1; \mathbb{C})$  to  $V_{\mathbb{C}}$ , and  $a \mapsto B(\cdot, \cdot; a)$  is linear, hence holomorphic, from  $L^\infty(D_{\text{ref},\kappa}; \mathbb{C})$  to  $\mathfrak{B}(V_{\mathbb{C}}, V_{\mathbb{C}})$ , the *bilinear* forms  $V_{\mathbb{C}} \times V_{\mathbb{C}} \rightarrow \mathbb{C}$ . It is also holomorphic from  $W_\infty^1(\mathbb{S}^1; \mathbb{C}) \subset L^\infty(\mathbb{S}^1; \mathbb{C})$  to  $\mathfrak{B}(V_{\mathbb{C}}, V_{\mathbb{C}})$ .

Let the real symmetric matrix  $R(a)(\boldsymbol{\xi})$  be defined by

$$R(a)(\boldsymbol{\xi}) := \Re(M(a)(\boldsymbol{\xi})),$$

and let  $\lambda_{\min}(a)(\boldsymbol{\xi})$  resp.  $\lambda_{\max}(a)(\boldsymbol{\xi})$  denote its smallest resp. largest eigenvalue. Denote

$$\lambda_{\min}(a) := \min_{\boldsymbol{\xi} \in D_{\text{ref},\kappa}} \lambda_{\min}(a)(\boldsymbol{\xi}), \quad \lambda_{\max}(a) := \max_{\boldsymbol{\xi} \in D_{\text{ref},\kappa}} \lambda_{\max}(a)(\boldsymbol{\xi}).$$

Then, we have

$$(16) \quad \Re(B(v, v; a)) = \int_{D_{\text{ref},\kappa}} R(a) \nabla v \cdot \nabla \bar{v} \geq \lambda_{\min}(a) \|v\|_{V_{\mathbb{C}}}^2$$

for any  $v \in V_{\mathbb{C}}$ . Therefore, the coercivity condition

$$(17) \quad |B(v, v; a)| \geq \lambda_{\min}(a) \|v\|_{V_{\mathbb{C}}}^2$$

holds. If  $\lambda_{\min}(a) > 0$ , by the Lax-Milgram lemma, we obtain

$$(18) \quad \|\hat{u}(a)\|_{V_{\mathbb{C}}} \leq \lambda_{\min}(a)^{-1} \|f_{\text{ref}}(a)\|_{L^2(D_{\text{ref},\kappa}; \mathbb{C})}.$$

Next, we derive a bound on  $\lambda_{\min}(a)^{-1}$ .

**Lemma 2.1.** *Assume that  $a \in W_\infty^1(\mathbb{S}^1; \mathbb{C})$  and let  $h(\theta)$  be defined in (6). Assume also that*

$$\|\mathfrak{J}(h(\theta))\|_{L^\infty(\mathbb{S}^1)} < 1.$$

*Then there holds*

$$(19) \quad \lambda_{\min}(a)^{-1} \leq \frac{2 + \|\Re(h(\theta))\|_{L^\infty(\mathbb{S}^1)}}{1 - \|\mathfrak{J}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}}$$

and

$$(20) \quad \lambda_{\max}(a) \leq 2 + \|\mathfrak{R}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2.$$

*Proof.* The eigenvalues of  $R(a)(\xi)$  are the roots of the quadratic equation

$$\lambda^2 - B\lambda + C$$

and, with  $\mathfrak{R}(h)^2$  denoting  $\mathfrak{R}(h^2)$ , that

$$C := 1 - \mathfrak{I}(h(\theta))^2, \quad B := 2 + \mathfrak{R}(h(\theta))^2 - \mathfrak{I}(h(\theta))^2.$$

Therefore,

$$\begin{aligned} \lambda_{\min}(a)(\xi) &= \frac{2 + \mathfrak{R}(h(\theta))^2 - \mathfrak{I}(h(\theta))^2 - \sqrt{(\mathfrak{R}(h(\theta))^2 - \mathfrak{I}(h(\theta))^2)^2 + 4\mathfrak{R}(h(\theta))^2}}{2} \\ &= \frac{2 - 2\mathfrak{I}(h(\theta))}{2 + \mathfrak{R}(h(\theta))^2 - \mathfrak{I}(h(\theta))^2 + \sqrt{(\mathfrak{R}(h(\theta))^2 - \mathfrak{I}(h(\theta))^2)^2 + 4\mathfrak{R}(h(\theta))^2}}. \end{aligned}$$

Then we have the estimate

$$\lambda_{\min}(a)(\xi)^{-1} \leq \frac{2 + \mathfrak{R}(h(\theta))^2 + \sqrt{\mathfrak{R}(h(\theta))^4 + 4\mathfrak{R}(h(\theta))^2}}{2 - 2\mathfrak{I}(h(\theta))} \leq \frac{2 + \mathfrak{R}(h(\theta))^2}{1 - \mathfrak{I}(h(\theta))^2},$$

which implies

$$\lambda_{\min}(a)^{-1} \leq \frac{2 + \|\mathfrak{R}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2}{1 - \|\mathfrak{I}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2}.$$

The bound (20) is derived in complete analogy.  $\square$

From (18) and (19), we obtain

$$(21) \quad \|\hat{u}(a)\|_{V_{\mathbb{C}}} \leq \frac{2 + \|\mathfrak{R}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2}{1 - \|\mathfrak{I}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2} \|f_{\text{ref}}(a)\|_{L^2(D_{\text{ref},\kappa};\mathbb{C})}.$$

We also have the estimate

$$(22) \quad \begin{aligned} \|f_{\text{ref}}(a)\|_{L^2(D_{\text{ref},\kappa};\mathbb{C})} &= \|J(a)(f \circ F(a))\|_{L^2(D_{\text{ref},\kappa};\mathbb{C})} \\ &\leq \sqrt{\|J(a)\|_{L^\infty(D_{\text{ref},\kappa};\mathbb{C})}} \|f\|_{L^2(D_\kappa(a))} \\ &\leq \|a(\theta)\|_{L^\infty(\mathbb{S}^1;\mathbb{C})} \|f\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Thus, we finally arrive at

$$(23) \quad \|\hat{u}(a)\|_{V_{\mathbb{C}}} \leq \frac{2 + \|\mathfrak{R}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2}{1 - \|\mathfrak{I}(h(\theta))\|_{L^\infty(\mathbb{S}^1)}^2} \|a(\theta)\|_{L^\infty(\mathbb{S}^1;\mathbb{C})} \|f\|_{L^2(\mathbb{R}^2)}.$$

Let  $\mathcal{A} \subset W_\infty^1(\mathbb{S}^1;\mathbb{C})$  be the set of all  $a \in W_\infty^1(\mathbb{S}^1;\mathbb{C})$  which are lower bounded away from zero and such that

$$(24) \quad |\mathfrak{R}(h(\theta))| < \infty, \quad |\mathfrak{I}(h(\theta))| < 1, \quad h(\theta) := \frac{a'(\theta)}{a(\theta)}, \quad \theta \in \mathbb{S}^1.$$

From [10, Theorem 2.1 and Corollary 2.4] we obtain, using (13), the following.

**Lemma 2.2.** *For any  $0 \leq \kappa < 1$ , the pullback solution  $\hat{u}(a) \in V_{\mathbb{C}}$  exists for every  $a \in \mathcal{A}$ . Moreover, the map*

$$a \mapsto \hat{u}(a)$$

*is holomorphic over a neighborhood of  $\mathcal{A}$ .*

## 3. RANDOM VARIATIONS OF THE DOMAIN

**3.1. Log-Gaussian shape parametrization.** In the following, we consider the Poisson problem (1) with  $D_\kappa(a)$ ,  $0 \leq \kappa < 1$  as defined in (2) and  $a$  depending on  $\mathbf{y} = (y_k)_{k \in \mathbb{N}}$  in the log-Gaussian form

$$(25) \quad a(\mathbf{y})(\theta) = \exp \left( \sum_{k \in \mathbb{N}} y_k \psi_k(\theta) \right), \quad \theta \in \mathbb{S}^1, \quad \mathbf{y} = (y_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

Here  $y_k$  are i.i.d. standard Gaussian random variables on  $\mathbb{R}$ , and  $\psi_k \in W_\infty^1(\mathbb{S}^1)$ . With (25), we rewrite (1) in the parametric form

$$-\Delta u(\mathbf{y}) = f \text{ in } D_\kappa(\mathbf{y}), \quad u(\mathbf{y}) = 0 \text{ on } \partial D_\kappa(\mathbf{y}),$$

where

$$D_\kappa(\mathbf{y}) := \text{int} \{ \mathbf{x} = (r \cos \theta, r \sin \theta) : 0 \leq \kappa a(\mathbf{y})(\theta) \leq r < a(\mathbf{y})(\theta) \}.$$

I.e., for any choice of  $\mathbf{y}$ ,  $D_\kappa(\mathbf{y})$  is obtained by a homothetic radial scaling  $r \mapsto a(\theta)r$  of  $D_{\text{ref}, \kappa}$ , which is either an annulus when  $0 < \kappa < 1$  or the unit circle for  $\kappa = 0$ . In the latter case,  $D_\kappa(\mathbf{y})$  is contractible and star-shaped with respect to the origin for any choice of  $\mathbf{y}$ . Randomness of  $D_\kappa(\mathbf{y})$  will be modelled by considering  $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$  endowed with a Gaussian measure  $\gamma$  (GM for short).

Let  $\gamma(y)$  be the standard Gaussian probability measure on  $\mathbb{R}$  with the density

$$(26) \quad \rho(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

We recall (e.g. [3]) the concept of standard Gaussian probability measure  $\gamma(\mathbf{y})$  on  $\mathbb{R}^{\mathbb{N}}$  as countable tensor product of the Gaussian measures  $\gamma(y_j)$ :

$$(27) \quad \gamma(\mathbf{y}) := \bigotimes_{j \in \mathbb{N}} \gamma(y_j), \quad \mathbf{y} = (y_j)_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

If  $X$  is a separable Banach space, the standard Gaussian probability measure  $\gamma$  in (27) on  $\mathbb{R}^{\mathbb{N}}$  induces the Bochner space  $L^2(\mathbb{R}^{\mathbb{N}}, X; \gamma)$  of strongly  $\gamma$ -measurable mappings  $v$  from  $\mathbb{R}^{\mathbb{N}}$  to  $X$ , equipped with the norm

$$(28) \quad \|v\|_{L^2(\mathbb{R}^{\mathbb{N}}, X; \gamma)} := \left( \int_{\mathbb{R}^{\mathbb{N}}} \|v(\cdot, \mathbf{y})\|_X^2 d\gamma(\mathbf{y}) \right)^{1/2}.$$

Elements  $v \in L^2(\mathbb{R}^{\mathbb{N}}, X; \gamma)$  are referred to as *Gaussian random fields (GRFs) taking values in  $X$* .

**Example 3.1.** A typical instance of (25) that we consider also in our numerical experiments, is that in (25)  $b(\mathbf{y}) = \log(a(\mathbf{y}))$  is a stationary Gaussian random field on  $\mathbb{S}^1$ . Then, for fixed  $\mathbf{y}$ ,  $\theta \mapsto b(\mathbf{y})(\theta)$  is a Fourier series, i.e.  $a(\mathbf{y})$  is of the form

$$(29) \quad a(\mathbf{y})(\theta) = \exp(b(\mathbf{y})(\theta)),$$

where  $\mathbf{y} = (y_k)_{k \in \mathbb{Z}}$  is a sequence of i.i.d. standard Gaussian normal variables on a probability space  $(\Omega, \Sigma, \mathbb{P})$  and  $b(\mathbf{y})(\theta)$  is a GRF taking values in  $C(\mathbb{S}^1)$ . We assume the GRF  $b(\mathbf{y})(\theta)$  to be given in the form of a Karhunen-Loève expansion

$$b(\mathbf{y})(\theta) = \sum_{k \in \mathbb{Z}} y_k(\omega) \sigma_k b_k(\theta), \quad \theta \in \mathbb{S}^1,$$

where, due to stationarity,

$$b_0 = \frac{1}{2\pi} \quad \text{and} \quad b_k(\theta) \simeq \cos(k\theta), \quad b_{-k} \simeq \sin(k\theta) \quad \text{for } k \in \mathbb{N},$$

are scaled to be orthonormal with respect to the  $L^2(\mathbb{S}^1)$  innerproduct  $(\cdot, \cdot)_{\mathbb{S}^1}$  given by  $(\varphi, \psi)_{\mathbb{S}^1} = \int_0^{2\pi} \varphi(\theta)\psi(\theta) d\theta$ , i.e.  $(b_k, b_j)_{\mathbb{S}^1} = \delta_{kj}$  for  $j, k \in \mathbb{N}$ . Comparing with (25), we set  $\psi_k(\theta) = \sigma_k b_k(\theta)$ , so that

$$b(\mathbf{y})(\theta) = y_0 \psi_0 + \sum_{0 \neq k \in \mathbb{Z}} y_k \psi_k(\theta).$$

The sequence  $(\sigma_k^2)_{k \in \mathbb{Z}} \subset [0, \infty)$  corresponds to the eigenvalues of the covariance operator of the GRF  $b(\mathbf{y})(\theta)$ . It is a compact and self-adjoint operator on  $L^2(\mathbb{S}^1)$ . We assume the eigenvalue sequence  $\{\sigma_{\pm k}^2\}_{k \in \mathbb{Z}}$  to be enumerated in decreasing order according to  $\sigma_0 \geq \sigma_{\pm 1} \geq \dots \geq \sigma_{\pm k} \geq \dots \geq 0$  and accumulating only at zero. The connection to random Fourier series on  $\mathbb{S}^1$  is via

$$(30) \quad b(\mathbf{y})(\theta) = y_0 \lambda_0 + \sum_{k=1}^{\infty} y_k \lambda_k \cos(k\theta) + y_{-k} \lambda_{-k} \sin(k\theta),$$

so that  $\lambda_k = \mathcal{O}(\sigma_k)$ .

**Remark 3.2.** The derivative with respect to  $\theta$  of the expansion (29) is given by  $a'(\mathbf{y})(\theta) = (a(\mathbf{y})b'(\mathbf{y}))(\theta)$ , so that in (6) we obtain for (29) that  $h(\theta) = b'(\theta)$ .

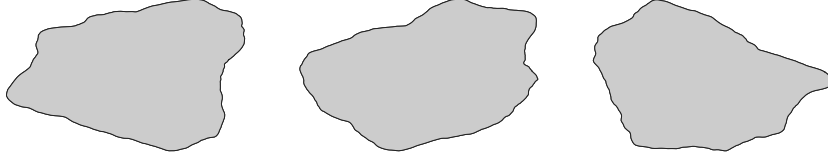


FIGURE 1. Three realizations of the random domain with log-Gaussian random boundary in case of  $\kappa = 0$  and for the sequence  $\lambda_k = (|k| + 1)^{-2}$  for all  $k \in \mathbb{Z}$ .

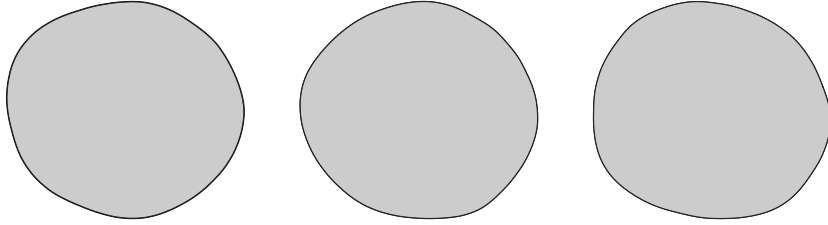


FIGURE 2. Three realizations of the random domain with log-Gaussian random boundary in case of  $\kappa = 0$  and for the sequence  $\lambda_k = (|k| + 1)^{-3}$  for all  $k \in \mathbb{Z}$ .

A visualization of three random samples in case  $\kappa = 0$  and the sequence  $\lambda_k = (|k| + 1)^{-2}$  is shown in Figure 1, while a visualization of three random samples in case of  $\lambda_k = (|k| + 1)^{-3}$  is shown in Figure 2.

Given  $a = a(\mathbf{y})$ , we denote by  $\hat{u}(\mathbf{y}) := \hat{u}(a(\mathbf{y}))$  the weak pullback solution to variational formulation (9). It satisfies the equation

$$(31) \quad \int_{D_{\text{ref},\kappa}} M(\mathbf{y}) \nabla \hat{u}(\mathbf{y}) \cdot \nabla v = \langle f_{\text{ref}}(\mathbf{y}), v \rangle, \quad v \in V,$$

where

$$(32) \quad M(\mathbf{y}) := M(a(\mathbf{y})) \quad \text{and} \quad f_{\text{ref}}(\mathbf{y}) := f_{\text{ref}}(a(\mathbf{y}))$$

are the diffusion matrix and the right-hand side. Note that equation (31) is the variational formulation of the equation

$$-\operatorname{div}(M(\mathbf{y}) \nabla \hat{u}(\mathbf{y})) = f_{\text{ref}}(\mathbf{y}) \text{ in } D_{\text{ref},\kappa}, \quad \hat{u}(\mathbf{y}) = 0 \text{ on } \partial D_{\text{ref},\kappa}.$$

A complex extension of  $\hat{u}(\mathbf{y})$  is  $\mathbf{z} \mapsto \hat{u}(\mathbf{z}) \in V_{\mathbb{C}}$  which satisfies the equation

$$(33) \quad \int_{D_{\text{ref},\kappa}} M(\mathbf{z}) \nabla \hat{u}(\mathbf{z}) \cdot \nabla \bar{v} = \langle f_{\text{ref}}(\mathbf{y}), \bar{v} \rangle, \quad v \in V_{\mathbb{C}},$$

which is the variational formulation of the equation

$$(34) \quad -\operatorname{div}(M(\mathbf{z}) \nabla \hat{u}(\mathbf{z})) = f_{\text{ref}}(\mathbf{z}) \text{ in } D_{\text{ref},\kappa}, \quad \hat{u}(\mathbf{z}) = 0 \text{ on } \partial D_{\text{ref},\kappa},$$

where

$$M(\mathbf{z}) := M(a(\mathbf{z})), \quad f_{\text{ref}}(\mathbf{z}) := f_{\text{ref}}(a(\mathbf{z})),$$

and

$$a(\mathbf{z})(\theta) = \exp\left(\sum_{k \in \mathbb{N}} z_k \psi_k(\theta)\right), \quad \theta \in \mathbb{S}^1, \quad \mathbf{z} = (z_k)_{k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}.$$

We set

$$(35) \quad b(\mathbf{z})(\theta) := \log(a(\mathbf{z})) = \sum_{k \in \mathbb{N}} z_k \psi_k(\theta),$$

where  $\log(\cdot)$  denotes the principal branch of the logarithm of a complex argument. Then we have

$$h(a(\mathbf{z}))(\theta) = b'(\mathbf{z}) = \sum_{k \in \mathbb{N}} z_k \psi'_k(\theta), \quad \theta \in \mathbb{S}^1,$$

and by (23)

$$(36) \quad \begin{aligned} \|\hat{u}(\mathbf{z})\|_{V_{\mathbb{C}}} &\leq \frac{2 + \|\Re(b'(\mathbf{z}))\|_{L^\infty(\mathbb{S}^1)}^2}{1 - \|\Im(b'(\mathbf{z}))\|_{L^\infty(\mathbb{S}^1)}^2} \|a(\mathbf{z})\|_{L^\infty(\mathbb{S}^1; \mathbb{C})} \|f\|_{L^2(\mathbb{R}^2)} \\ &\leq \frac{2 + \|\Re(b'(\mathbf{z}))\|_{L^\infty(\mathbb{S}^1)}^2}{1 - \|\Im(b'(\mathbf{z}))\|_{L^\infty(\mathbb{S}^1)}^2} \exp(\|\Re(b(\mathbf{z}))\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

provided  $\|\Im(b'(\mathbf{z}))\|_{L^\infty(\mathbb{S}^1)} < 1$ .

**3.2. Holomorphy of the parametric pullback solution.** In (25), denote

$$(37) \quad U_0 = \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} : a(\mathbf{y}) \in W_\infty^1(\mathbb{S}^1)\} = \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} : b(\mathbf{y}) \in W_\infty^1(\mathbb{S}^1)\}.$$

In view of [1, Theorem 2.2], we have the following result.

**Lemma 3.3.** *Assume that there exists a sequence  $(\tau_k)_{k \in \mathbb{N}}$  such that  $\exp(-\tau_k^2)_{k \in \mathbb{N}} \in \ell_1(\mathbb{N})$  and the series*

$$\theta \mapsto \sum_{k \in \mathbb{N}} \tau_k |\psi_k(\theta)| \quad \text{and} \quad \theta \mapsto \sum_{k \in \mathbb{N}} \tau_k |\psi'_k(\theta)|$$

converge in  $L^\infty(\mathbb{S}^1)$ . Then  $\gamma(U_0) = 1$ .

In the following we always assume that the sequence  $(\psi_k)_{k \in \mathbb{N}}$  satisfies the conditions in Lemma 3.3. Let  $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$  be a sequence of non-negative numbers and assume that  $\mathbf{u} \subseteq \text{supp}(\boldsymbol{\rho})$  is finite. Define

$$\mathcal{S}_{\mathbf{u}}(\boldsymbol{\rho}) := \bigtimes_{j \in \mathbf{u}} \mathcal{S}_j(\boldsymbol{\rho}),$$

where the strip  $\mathcal{S}_j(\boldsymbol{\rho})$  is given by

$$\mathcal{S}_j(\boldsymbol{\rho}) := \{z_j \in \mathbb{C} : |\Im(z_j)| < \rho_j\}.$$

For  $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ , put

$$\mathcal{S}_{\mathbf{u}}(\mathbf{y}, \boldsymbol{\rho}) := \{(z_j)_{j \in \mathbb{N}} : z_j \in \mathcal{S}_j(\boldsymbol{\rho}) \text{ if } j \in \mathbf{u} \text{ and } z_j = y_j \text{ if } j \notin \mathbf{u}\}.$$

**Lemma 3.4.** *Let the sequence  $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$  satisfy*

$$\left\| \sum_{k \in \mathbb{N}} \rho_k |\psi'_k| \right\|_{L^\infty(\mathbb{S}^1)} < 1.$$

Let  $\mathbf{y}_0 = (y_{0,j})_{j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$  be such that  $b(\mathbf{y}_0)$  belongs to  $W_\infty^1(\mathbb{S}^1)$ , and let  $\mathbf{u} \subseteq \text{supp}(\boldsymbol{\rho})$  be a finite set. Assume (13).

Then the solution  $\hat{u}(\mathbf{y})$  of the parametric variational formulation (31) is holomorphic on  $\mathcal{S}_{\mathbf{u}}(\boldsymbol{\rho})$  as a function of the parameters  $\mathbf{z}_{\mathbf{u}} = (z_j)_{j \in \mathbb{N}} \in \mathcal{S}_{\mathbf{u}}(\mathbf{y}_0, \boldsymbol{\rho})$  taking values in  $V$  with  $z_j = y_{0,j}$  for  $j \notin \mathbf{u}$  held fixed.

*Proof.* Let  $N \in \mathbb{N}$ . We denote

$$\mathcal{S}_{\mathbf{u},N}(\boldsymbol{\rho}) := \{(y_j + i\xi_j)_{j \in \mathbf{u}} \in \mathcal{S}_{\mathbf{u}}(\boldsymbol{\rho}) : |y_j - y_{0,j}| < N\}.$$

For  $\mathbf{z}_{\mathbf{u}} = (y_j + i\xi_j)_{j \in \mathbb{Z}} \in \mathcal{S}_{\mathbf{u}}(\mathbf{y}_0, \boldsymbol{\rho})$  with  $(y_j + i\xi_j)_{j \in \mathbf{u}} \in \mathcal{S}_{\mathbf{u},N}(\boldsymbol{\rho})$ , we have

$$|\Im(b'(\mathbf{z}_{\mathbf{u}})(\theta))| \leq \left\| \sum_{k \in \mathbb{N}} \rho_k |\psi'_k(\theta)| \right\|_{L^\infty(\mathbb{S}^1)} < 1,$$

and

$$\begin{aligned} |\Re(b'(\mathbf{z}_{\mathbf{u}})(\theta))| &= \left| \sum_{k \in \mathbb{N}} y_k \psi'_k(\theta) \right| \\ &\leq \left| \sum_{k \in \mathbb{N}} y_{0,k} \psi'_k(\theta) \right| + \sum_{k \in \mathbf{u}} |y_{0,k} - y_k| \cdot |\psi'_k(\theta)| \\ &\leq \|b'(\mathbf{y}_0)\|_{L^\infty(\mathbb{S}^1)} + N \sum_{k \in \mathbf{u}} |\psi'_k(\theta)| < \infty. \end{aligned}$$

Similarly we have

$$\|\Re(b(\mathbf{z}_{\mathbf{u}}))\|_{L^\infty(\mathbb{S}^1)} < \infty.$$

Hence, by Lemma 2.2, (13) and the analyticity of exponential functions, we conclude that the map  $\mathbf{z}_{\mathbf{u}} \mapsto \hat{u}(\mathbf{z}_{\mathbf{u}})$  is holomorphic on the set  $\mathcal{S}_{\mathbf{u},N}(\boldsymbol{\rho})$ . Since  $N$  is arbitrary, we finally deduce that the map  $\mathbf{z}_{\mathbf{u}} \mapsto \hat{u}(\mathbf{z}_{\mathbf{u}})$  is holomorphic on  $\mathcal{S}_{\mathbf{u}}(\boldsymbol{\rho})$ .  $\square$

**3.3. Derivative estimates.** We shall next study the behaviour of the derivatives  $\partial^\nu \hat{u}(\mathbf{y})$  of the pullback solution. Recall that  $\mathbb{F}$  is the set of all sequences of non-negative integers  $\nu = (\nu_j)_{j \in \mathbb{N}}$  such that their support  $\text{supp}(\nu) := \{j \in \mathbb{N} : \nu_j > 0\}$  is a finite set. The analytic continuation of the parametric solutions  $\{\hat{u}(\mathbf{y}) : \mathbf{y} \in U_0\}$  to  $\mathcal{S}_u(\rho)$  leads to a result on parametric  $V$ -regularity.

**Theorem 3.5.** *Assume that there exist a non-negative sequence  $\rho = (\rho_j)_{j \in \mathbb{N}}$  such that there are constants  $B_\rho, C_\rho$  such that in (25) holds*

$$(38) \quad \left\| \sum_{k \in \mathbb{N}} \rho_k |\psi'_k| \right\|_{L^\infty(\mathbb{S}^1)} \leq B_\rho < 1$$

and

$$(39) \quad \left\| \sum_{k \in \mathbb{N}} \rho_k |\psi_k| \right\|_{L^\infty(\mathbb{S}^1)} \leq C_\rho.$$

Let  $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$  with  $b(\mathbf{y}) \in W_\infty^1(\mathbb{S}^1)$  and  $\nu \in \mathbb{F}$  such that  $\text{supp}(\nu) \subseteq \text{supp}(\rho)$ .

Then

$$\|\partial^\nu \hat{u}(\mathbf{y})\|_V \leq \exp(C_\rho) \frac{\nu! (\|b'(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)} + 2)^2}{\rho^\nu (1 - B_\rho)} \exp(\|b(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)}.$$

*Proof.* Let  $\nu \in \mathbb{F}$  be such that  $\text{supp}(\nu) \subseteq \text{supp}(\rho)$ . Denote  $\mathbf{u} = \text{supp}(\nu)$ . Fixing the variable  $y_j$  when  $j \notin \mathbf{u}$ , the map  $\mathcal{S}_u(\mathbf{y}, \rho) \ni \mathbf{z}_u \mapsto \hat{u}(\mathbf{z}_u)$  is holomorphic on the domain  $\mathcal{S}_u(\mathbf{y}, \rho)$  by Lemma 3.4. Applying Cauchy's integral formula gives

$$\partial^\nu \hat{u}(\mathbf{y}) = \frac{\nu!}{(2\pi i)^{|\mathbf{u}|}} \int_{\mathcal{C}_{\mathbf{y}, \mathbf{u}}(\rho)} \frac{\hat{u}(\mathbf{z}_u)}{\prod_{j \in \mathbf{u}} (z_j - y_j)^{\nu_j + 1}} \prod_{j \in \mathbf{u}} dz_j,$$

where integration is over the cylinders

$$\mathcal{C}_{\mathbf{y}, \mathbf{u}}(\rho) := \bigtimes_{j \in \mathbf{u}} \mathcal{C}_{\mathbf{y}, j}(\rho), \quad \mathcal{C}_{\mathbf{y}, j}(\rho) := \{z_j \in \mathbb{C} : |z_j - y_j| = \rho_j\}.$$

This leads to

$$(40) \quad \|\partial^\nu \hat{u}(\mathbf{y})\|_V \leq \frac{\nu!}{\rho^\nu} \sup_{\mathbf{z}_u \in \mathcal{C}_u(\mathbf{y}, \rho)} \|\hat{u}(\mathbf{z}_u)\|_{V_{\mathbb{C}}}$$

with

$$\mathcal{C}_u(\mathbf{y}, \rho) = \{(z_j)_{j \in \mathbb{N}} \in \mathcal{S}_u(\mathbf{y}, \rho) : (z_j)_{j \in \mathbf{u}} \in \mathcal{C}_{\mathbf{y}, \mathbf{u}}(\rho)\}.$$

Notice that for  $\mathbf{z}_u = (z_j)_{j \in \mathbb{N}} \in \mathcal{C}_u(\mathbf{y}, \rho)$  we can write  $z_j = y_j + \eta_j + i\xi_j \in \mathcal{C}_{\mathbf{y}, j}(\rho)$  with  $|\eta_j| \leq \rho_j$ ,  $|\xi_j| \leq \rho_j$  if  $j \in \mathbf{u}$  and  $\eta_j = \xi_j = 0$  if  $j \notin \mathbf{u}$ . We deduce from (36) that

$$\begin{aligned} \|\hat{u}(\mathbf{z}_u)\|_{V_{\mathbb{C}}} &\leq \frac{2 + \|\Re(b'(\mathbf{z}_u))\|_{L^\infty(\mathbb{S}^1)}}{1 - \|\Im(b'(\mathbf{z}_u))\|_{L^\infty(\mathbb{S}^1)}^2} \exp(\|\Re(b(\mathbf{z}_u))\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)} \\ &\leq \frac{2 + \|b'(\mathbf{y} + \boldsymbol{\eta})\|_{L^\infty(\mathbb{S}^1)}^2}{1 - B_\rho} \exp(\|b(\mathbf{y} + \boldsymbol{\eta})\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)} \\ &\leq \exp(C_\rho) \frac{2 + (\|b'(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)} + 1)^2}{1 - B_\rho} \exp(\|b(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)} \\ &\leq \exp(C_\rho) \frac{(\|b'(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)} + 2)^2}{1 - B_\rho} \exp(\|b(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

By inserting this estimate into (40), we obtain the desired result.  $\square$

This theorem immediately implies the following corollary which concerns the special series expansion (29) which is based on a Fourier series.

**Corollary 3.6.** *Consider  $a(\mathbf{y})(\theta)$  as in (29), with Fourier series  $b(\mathbf{y}) = \log(a(\mathbf{y})(\theta))$ . Assume that there exist a non-negative sequence  $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{Z}}$  and a number  $B_{\boldsymbol{\rho}} > 0$  satisfying*

$$(41) \quad \left\| \sum_{k=1}^{\infty} \rho_k k |\lambda_k \sin(k\theta)| + \rho_{-k} k |\lambda_{-k} \cos(k\theta)| \right\|_{L^\infty(\mathbb{S}^1)} \leq B_{\boldsymbol{\rho}} < 1$$

and

$$C_{\boldsymbol{\rho}} := \left\| \rho_0 \lambda_0 + \sum_{k=1}^{\infty} \rho_k |\lambda_k \cos(k\theta)| + \rho_{-k} |\lambda_{-k} \sin(k\theta)| \right\|_{L^\infty(\mathbb{S}^1)} < \infty.$$

Let  $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{Z}}$  be such that  $|\boldsymbol{\nu}|_1 < \infty$  and  $\text{supp}(\boldsymbol{\nu}) \subseteq \text{supp}(\boldsymbol{\rho})$ . Then for every  $\mathbf{y} \in \mathbb{R}^{\mathbb{Z}}$  with  $b(\mathbf{y}) \in W_{\infty}^1(\mathbb{S}^1)$  holds

$$(42) \quad \|\partial^{\boldsymbol{\nu}} \hat{u}(\mathbf{y})\|_V \leq \exp(C_{\boldsymbol{\rho}}) \frac{\boldsymbol{\nu}! (\|b'(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)} + 2)^2}{\boldsymbol{\rho}^{\boldsymbol{\nu}} (1 - B_{\boldsymbol{\rho}})} \exp(\|b(\mathbf{y})\|_{L^\infty(\mathbb{S}^1)}) \|f\|_{L^2(\mathbb{R}^2)}$$

for the parametric, weak solution  $\hat{u}(\mathbf{y}) \in V$  to (31) with  $a(\mathbf{y})$  as in (29).

#### 4. INTERPOLATION AND QUADRATURE

We address the numerical solution of the parametric PDE (31). In Section 4.1, we address sparse-grid interpolation with respect to the parameter, and sparse Gauss-Hermite Smolyak quadrature with respect to the parameter. In Section 4.2, we address quasi-Monte Carlo quadrature over the parameter domain.

**4.1. Sparse grid interpolation and quadrature.** We apply the analyticity results in Lemma 3.5 to obtain a weighted  $\ell_2$ -summability of the Hermite GPC expansion coefficients of  $u(\mathbf{y})$  which gives necessary conditions for constructing sparse-grid interpolations and establishing a convergence rate that is free from the CoD.

Every function  $v \in L^2(\mathbb{R}^{\mathbb{N}}, X; \gamma)$  can be represented by the Hermite polynomial chaos (PC) expansion

$$(43) \quad v(\mathbf{y}) = \sum_{\boldsymbol{\nu} \in \mathbb{F}} v_{\boldsymbol{\nu}} H_{\boldsymbol{\nu}}(\mathbf{y}), \quad v_{\boldsymbol{\nu}} \in X,$$

where

$$H_{\boldsymbol{\nu}}(\mathbf{y}) = \bigotimes_{j \in \mathbb{N}} H_{\nu_j}(y_j), \quad v_{\boldsymbol{\nu}} := \int_{\mathbb{R}^{\mathbb{N}}} v(\mathbf{y}) H_{\boldsymbol{\nu}}(\mathbf{y}) d\gamma(\mathbf{y}), \quad \boldsymbol{\nu} \in \mathbb{F},$$

with  $(H_k)_{k \in \mathbb{N}_0}$  being the Hermite orthonormal polynomials in  $L^2(\mathbb{R}, \gamma)$ . There holds the Parseval's identity

$$\|v\|_{L^2(\mathbb{R}^{\mathbb{N}}, X; \gamma)}^2 = \sum_{\boldsymbol{\nu} \in \mathbb{F}} \|v_{\boldsymbol{\nu}}\|_X^2.$$

For  $\theta, \lambda \geq 0$ , define the set  $\mathbf{p}(\theta, \lambda) := (p_{\boldsymbol{\nu}}(\theta, \lambda))_{\boldsymbol{\nu} \in \mathbb{F}}$  by

$$p_{\boldsymbol{\nu}}(\theta, \lambda) := \prod_{j \in \mathbb{N}} (1 + \lambda \nu_j)^\theta, \quad \boldsymbol{\nu} \in \mathbb{F}.$$

For integer  $m \in \mathbb{N}$ , we use the following notation:

$$\mathbb{F}_m := \{\boldsymbol{\nu} \in \mathbb{F} : \nu_j \in \mathbb{N}_{0,m}, j \in \mathbb{N}\}, \quad \text{where } \mathbb{N}_{0,m} := \{n \in \mathbb{N}_0 : n = 0, m, m+1, \dots\}.$$

**Lemma 4.1.** *Assume that there exist a non-negative sequence  $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{N}}$  and positive numbers  $B_{\boldsymbol{\rho}}$  and  $C_{\boldsymbol{\rho}}$  satisfying (38) and (39), and  $(\rho_j^{-1})_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$  for some  $p \in (0, 1)$ . Set  $q := 2p/(2-p) \in (0, 2)$ . Let  $m \in \mathbb{N}$  and  $\tau, \lambda \geq 0$  be any fixed numbers. For  $r \in \mathbb{N}$  and the sequence  $\boldsymbol{\varrho} = (\varrho_j)_{j \in \mathbb{N}}$  given by*

$$\varrho_j := \rho_j^{1-p/2} \frac{1}{\sqrt{r!} \|(\rho_j^{-1})_{j \in \mathbb{N}}\|_{\ell_p(\mathbb{N})}^{p/2}},$$

we define the Wiener-Hermite weights  $\sigma_{\boldsymbol{\nu}}$  by

$$(44) \quad \sigma_{\boldsymbol{\nu}}^2 := \sum_{|\boldsymbol{\nu}'|_{\infty} \leq r} \binom{\boldsymbol{\nu}}{\boldsymbol{\nu}'} \boldsymbol{\varrho}^{2\boldsymbol{\nu}'} = \prod_{j \in \mathbb{N}} \left( \sum_{\ell=0}^r \binom{\nu_j}{\ell} \varrho_j^{2\ell} \right), \quad \boldsymbol{\nu} \in \mathbb{F}.$$

Then, for any fixed  $r > 2(\tau + 1)m/q$ , the Hermite coefficients  $\hat{u}_{\boldsymbol{\nu}}$  of the parametric weak solution  $\hat{u}(\mathbf{y})$  of (31) admit the weighted  $\ell_2$ -summability

$$\sum_{\boldsymbol{\nu} \in \mathbb{F}_m} (\sigma_{\boldsymbol{\nu}} \|\hat{u}_{\boldsymbol{\nu}}\|_V)^2 < \infty \quad \text{with} \quad (p_{\boldsymbol{\nu}}(\tau, \lambda) \sigma_{\boldsymbol{\nu}}^{-1})_{\boldsymbol{\nu} \in \mathbb{F}_m} \in \ell_{q/m}(\mathbb{F}_m).$$

*Proof.* This lemma is derived from Theorem 3.5 in a manner similar to the proofs of Theorem 3.25 and of Remark 3.16 in [15] by using [13, Lemma 5.3].  $\square$

For  $m \in \mathbb{N}_0$ , let  $Y_m = (y_{m;k})_{k \in \pi_m}$  be the increasing sequence of the  $m + 1$  roots of the Hermite polynomial  $H_{m+1}$ , ordered as

$$\begin{aligned} y_{m;-j} < \cdots < y_{m;-1} < y_{m;0} = 0 < y_{m;1} < \cdots < y_{m;j} & \text{if } m = 2j, \\ y_{m;-j} < \cdots < y_{m;-1} < y_{m;1} < \cdots < y_{m;j} & \text{if } m = 2j - 1, \end{aligned}$$

where

$$\pi_m := \begin{cases} \{-j, -j+1, \dots, -1, 0, 1, \dots, j-1, j\} & \text{if } m = 2j; \\ \{-j, -j+1, \dots, -1, 1, \dots, j-1, j\} & \text{if } m = 2j - 1. \end{cases}$$

(in particular,  $Y_0 = (y_{0;0})$  with  $y_{0;0} = 0$ ).

For a continuous function  $v : \mathbb{R} \rightarrow V$  and for  $m \in \mathbb{N}_0$ , we define the Lagrange interpolation operator  $I_m$  in the Hermite nodes by

$$(45) \quad I_m(v) := \sum_{k \in \pi_m} v(y_{m;k}) L_{m;k}, \quad L_{m;k}(y) := \prod_{j \in \pi_m, j \neq k} \frac{y - y_{m;j}}{y_{m;k} - y_{m;j}},$$

(in particular,  $I_0(v) = v(y_{0;0})L_{0;0}(y) = v(0)$  and  $L_{0;0}(y) = 1$ ). Notice that  $I_m(v)$  is a function on  $\mathbb{R}$  taking values in  $X$  and interpolating  $v$  at  $y_{m;k}$ , i.e.,  $I_m(v)(y_{m;k}) = v(y_{m;k})$ . We define the univariate operator  $\Delta_m$  for  $m \in \mathbb{N}_0$  by

$$\Delta_m := I_m - I_{m-1},$$

with the convention  $I_{-1} = 0$ , and the univariate operator  $\Delta_m^*$  for even  $m \in \mathbb{N}_0$  by

$$\Delta_m^* := I_m - I_{m-2},$$

with the convention  $I_{-2} = 0$ . For a function  $v : \mathbb{R}^{\mathbb{N}} \rightarrow X$ , we introduce the tensor product operator  $\Delta_{\boldsymbol{\nu}}$ ,  $\boldsymbol{\nu} \in \mathbb{F}$ , by

$$(46) \quad \Delta_{\boldsymbol{\nu}}(v) := \bigotimes_{j \in \mathbb{N}} \Delta_{\nu_j}(v),$$

where the univariate operator  $\Delta_{\nu_j}$  is successively applied to the univariate function  $\bigotimes_{i < j} \Delta_{\nu_i}(v)$  by considering it as a function of variable  $y_j$  with the other variables held fixed. We define for  $\boldsymbol{\nu} \in \mathbb{F}$  in the same manner as  $\Delta_{\boldsymbol{\nu}}$

$$(47) \quad I_{\boldsymbol{\nu}} := \bigotimes_{j \in \mathbb{N}} I_{\nu_j}, \quad L_{\boldsymbol{\nu}; \mathbf{k}} := \bigotimes_{j \in \mathbb{N}} L_{\nu_j; k_j}, \quad \pi_{\boldsymbol{\nu}} := \prod_{j \in \mathbb{N}} \pi_{\nu_j}.$$

For  $\boldsymbol{\nu} \in \mathbb{F}$  and  $\mathbf{k} \in \pi_{\boldsymbol{\nu}}$ , let  $E_{\boldsymbol{\nu}} \subset \mathbb{F}$  be the subset of all  $\mathbf{e}$  such that  $e_j$  is either 1 or 0 if  $\nu_j > 0$ , and  $e_j$  is 0 if  $\nu_j = 0$ , and let, for  $U_0 \subset \mathbb{C}^{\mathbb{N}}$  in Section 3.2  $\mathbf{y}_{\boldsymbol{\nu}; \mathbf{k}} := (y_{\nu_j; k_j})_{j \in \mathbb{N}} \in U_0$ . Recall  $|\boldsymbol{\nu}|_1 := \sum_{j \in \mathbb{N}} \nu_j$  for  $\boldsymbol{\nu} \in \mathbb{F}$ . It is easy to check that the interpolation operator  $\Delta_{\boldsymbol{\nu}}$  can be represented in the ‘‘combination’’ form

$$(48) \quad \Delta_{\boldsymbol{\nu}}(v) = \sum_{\mathbf{e} \in E_{\boldsymbol{\nu}}} (-1)^{|\mathbf{e}|_1} I_{\boldsymbol{\nu}-\mathbf{e}}(v) = \sum_{\mathbf{e} \in E_{\boldsymbol{\nu}}} (-1)^{|\mathbf{e}|_1} \sum_{\mathbf{k} \in \pi_{\boldsymbol{\nu}-\mathbf{e}}} v(\mathbf{y}_{\boldsymbol{\nu}-\mathbf{e}; \mathbf{k}}) L_{\boldsymbol{\nu}-\mathbf{e}; \mathbf{k}}.$$

Let  $0 < q < \infty$  and  $(\sigma_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathbb{F}}$  be a set of positive numbers. For  $\xi > 1$ , define the thresholded multi-index set

$$\Lambda(\xi) := \{\boldsymbol{\nu} \in \mathbb{F} : \sigma_{\boldsymbol{\nu}}^q \leq \xi\}.$$

With  $\Lambda(\xi)$ , we introduce the sparse-grid interpolation operator  $I_{\Lambda(\xi)}$  by

$$(49) \quad I_{\Lambda(\xi)} := \sum_{\boldsymbol{\nu} \in \Lambda(\xi)} \Delta_{\boldsymbol{\nu}}.$$

By (48), we can represent the operator  $I_{\Lambda(\xi)}$  in the form (see (47))

$$I_{\Lambda(\xi)}(v) = \sum_{(\boldsymbol{\nu}, \mathbf{e}, \mathbf{k}) \in G(\xi)} (-1)^{|\mathbf{e}|_1} v(\mathbf{y}_{\boldsymbol{\nu}-\mathbf{e}; \mathbf{k}}) L_{\boldsymbol{\nu}-\mathbf{e}; \mathbf{k}},$$

where the ‘‘sparse grid’’ is given by

$$(50) \quad G(\Lambda(\xi)) := \{(\boldsymbol{\nu}, \mathbf{e}, \mathbf{k}) \in \mathbb{F} \times \mathbb{F} \times \mathbb{F} : \boldsymbol{\nu} \in \Lambda(\xi), \mathbf{e} \in E_{\boldsymbol{\nu}}, \mathbf{k} \in \pi_{\boldsymbol{\nu}-\mathbf{e}}\}.$$

The following theorem gives a bound for the convergence rate of the semi-discrete approximation of the parametric solution  $u$  by the sparse-grid interpolation  $I_{\Lambda(\xi)}u$ .

**Theorem 4.2.** *Under the assumptions of Lemma 4.1, consider the sparse-grid interpolation operator  $I_{\Lambda(\xi)}$  for  $q := 2p/(2-p)$  and the weight sequence  $(\sigma_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathbb{F}}$  defined by (44).*

*Then there exists a constant  $C > 0$  such that for each  $n > 1$ , we can determine a number  $\xi_n$  so that for the set of points  $(\mathbf{y}_{\boldsymbol{\nu}-\mathbf{e}; \mathbf{k}})_{(\boldsymbol{\nu}, \mathbf{e}, \mathbf{k}) \in G(\Lambda(\xi_n))}$ , it holds that  $|G(\Lambda(\xi_n))| \leq n$  and*

$$\|\hat{u} - I_{\Lambda(\xi_n)}\hat{u}\|_{L^2(\mathbb{R}^{\mathbb{N}}, V; \gamma)} \leq Cn^{-(1/p-1)}.$$

*Proof.* By Lemma 4.1, the assumptions of [13, Corollary 3.1] hold for  $X = V$  with  $0 < q < 2$  and  $q := 2p/(2-p)$  for  $p \in (0, 1)$  given as in the assumptions of Lemma 4.1. Therefore, by applying [13, Corollary 3.1] for  $q$  and taking account of  $1/q - 1/2 = 1/p - 1$ , we prove the theorem.  $\square$

We present corresponding results for sparse-grid (Smolyak) quadrature over  $\mathbb{R}^{\mathbb{N}}$  with respect to  $\gamma$ . The symmetry of the Gaussian measure  $\gamma$  with respect to coordinate reflection  $y_j \mapsto -y_j$  for all  $j \in \mathbb{N}$  implies vanishing of Hermite-pc terms with at least one odd-degree Hermite polynomial. To exploit such cancellations in Hermite-pc expansions due to parity, we use the index set

$$\mathbb{F}_{\text{ev}} := \{\boldsymbol{\nu} \in \mathbb{F} : \nu_j \text{ even, } j \in \mathbb{N}\} \subset \mathbb{F}_2.$$

The interpolation operators  $\Delta_{\boldsymbol{\nu}}^*$  for  $\boldsymbol{\nu} \in \mathbb{F}_{\text{ev}}$ ,  $I_{\Lambda}^*$  for a finite set  $\Lambda \subset \mathbb{F}_{\text{ev}}$  are defined by replacing  $\Delta_{\nu_j}$  with  $\Delta_{\nu_j}^*$ ,  $j \in \mathbb{N}$ .

If  $v$  is a function defined on  $\mathbb{R}$  taking values in the space  $V$ , the function  $I_m(v)$  in (45) generates the quadrature formula defined as

$$Q_m(v) := \int_{\mathbb{R}} I_m(v)(y) \, d\gamma(y) = \sum_{k=0}^m \omega_{m;k} v(y_{m;k}),$$

where

$$\omega_{m;k} := \int_{\mathbb{R}} L_{m;k}(y) \, d\gamma(y) = \frac{1}{(m+1)H_m^2(y_{m;k})}.$$

We define the univariate quadrature-increment  $\Delta_m^{\text{Q}}$  for even  $m \in \mathbb{N}_0$  by

$$\Delta_m^{\text{Q}} := Q_m - Q_{m-2},$$

with the convention  $Q_{-2} := 0$ . For a function  $v$  defined on  $\mathbb{R}^{\mathbb{N}}$  taking values in  $V$ , we introduce the operator

$$\Delta_{\boldsymbol{\nu}}^{\text{Q}} := \bigotimes_{j \in \mathbb{N}} \Delta_{\nu_j}^{\text{Q}},$$

in the same manner as  $\Delta_{\boldsymbol{\nu}}$ . For  $\xi > 1$ , let the set  $\Lambda_{\text{ev}}^*(\xi)$  be defined by

$$(51) \quad \Lambda_{\text{ev}}^*(\xi) := \{\boldsymbol{\nu} \in \mathbb{F}_{\text{ev}} : \sigma_{\boldsymbol{\nu}}^{q/2} \leq \xi\}.$$

We introduce the sparse-grid quadrature operator  $Q_{\Lambda_{\text{ev}}^*(\xi)}$  which is generated by the sparse-grid interpolation operator  $I_{\Lambda_{\text{ev}}^*(\xi)}^*$  as follows

$$(52) \quad Q_{\Lambda_{\text{ev}}^*(\xi)} := \sum_{\boldsymbol{\nu} \in \Lambda_{\text{ev}}^*(\xi)} \Delta_{\boldsymbol{\nu}}^{\text{Q}}(v) = \int_{\mathbb{R}^{\mathbb{N}}} I_{\Lambda_{\text{ev}}^*(\xi)}^*(v)(\mathbf{y}) \, d\gamma(\mathbf{y}).$$

**Theorem 4.3.** *Under the assumptions of Lemma 4.1, with the index set  $\Lambda_{\text{ev}}^*(\xi)$  as in (51), consider the sparse-grid Quadrature operator  $Q_{\Lambda_{\text{ev}}^*(\xi)}$  defined by the weight-sequence  $(\sigma_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathbb{F}}$  in (44).*

*Then there exists a constant  $C$  such that for each  $n \in \mathbb{N}$ , there exists a number  $\xi_n$  such that the number of quadrature points  $|G(\Lambda_{\text{ev}}^*(\xi_n))|$  is at most  $n$  and*

$$(53) \quad \left\| \int_{\mathbb{R}^{\mathbb{N}}} \hat{u}(\mathbf{y}) \, d\gamma(\mathbf{y}) - Q_{\Lambda_{\text{ev}}^*(\xi_n)} \hat{u} \right\|_V \leq C n^{-(2/p-3/2)},$$

and if  $\phi \in V'$

$$(54) \quad \left| \int_{\mathbb{R}^{\mathbb{N}}} \langle \phi, \hat{u}(\mathbf{y}) \rangle \, d\gamma(\mathbf{y}) - \langle \phi, Q_{\Lambda_{\text{ev}}^*(\xi_n)} \hat{u} \rangle \right| \leq C \|\phi\|_{V'} n^{-(2/p-3/2)}.$$

*Proof.* Note that  $\mathbb{F}_{\text{ev}} \subset \mathbb{F}_2$ . By Lemma 4.1 the assumptions of [13, Corollary 3.1] hold for  $X = V$  with  $0 < q/2 < 1$  and  $q := 2p/(2-p)$  for  $p \in (0, 1)$  given as in the assumptions of Lemma 4.1. Therefore, by applying [13, Corollary 4.1] for  $q/2$  and taking account  $2/q - 1/2 = 2/p - 3/2$ , we prove the theorem.  $\square$

Observe that Theorems 4.2 and 4.3 provide convergence rates free from the ‘‘curse of dimensionality’’. *Arbitrary high convergence rates* are possible, for sufficiently small summability exponent  $p$ , i.e., with sufficient sparsity of the Hermite expansion coefficients  $(\hat{u}_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathbb{F}}$ .

**4.2. Quasi-Monte Carlo integration.** We shall next show that our regularity results also allow for convergence rate bounds of quasi-Monte Carlo quadrature to compute the expectation of the random solution of moments of it. Since the underlying random variables are Gaussian and hence unbounded, the application of the quasi-Monte Carlo method requires some special care, compare [12, 18]. We follow here the approach of [20] and apply the *Halton sequence* as points of integration.

**Definition 4.4.** Let  $b_1, \dots, b_m$  denote the first  $m$  prime numbers. The  $m$ -dimensional Halton sequence is given by

$$\boldsymbol{\xi}^i = [h_{b_1}(i), \dots, h_{b_m}(i)]^\top, \quad i = 0, 1, 2, \dots,$$

where  $h_{b_j}(i)$  denotes the  $i$ -th element of the van der Corput sequence with respect to  $b_j$ . That is, if  $i = \dots c_3 c_2 c_1$  in radix  $b_j$ , then  $h_{b_j}(i) = 0.c_1 c_2 c_3 \dots$  in radix  $b_j$ .

The associated quasi-Monte Carlo quadrature rule for a given function  $v$  defined on  $[0, 1]^m$  is then defined by

$$(55) \quad \mathbf{Q}_N v := \frac{1}{N} \sum_{i=1}^N v(\boldsymbol{\xi}^i),$$

where  $N$  denotes the number of samples and  $\boldsymbol{\xi}^i \in [0, 1]^m$  denotes the  $i$ -th Halton point. The quadrature rule (55) is known to give an approximation to the integral

$$\mathbf{I}v := \int_{(0,1)^m} v(\mathbf{z}) \, d\mathbf{z}$$

provided that the integrand  $v$  is smooth enough.

To obtain a quasi-Monte Carlo method for the integration domain  $\mathbb{R}^m$ , we map the quadrature points to  $\mathbb{R}^m$  by the inverse distribution function. This is equivalent to the transformation of the integrals under consideration to the unit cube. To that end, we define the cumulative normal distribution

$$\Phi: \mathbb{R} \rightarrow (0, 1), \quad y \mapsto \Phi(y) := \int_{-\infty}^y \rho(y') \, dy'$$

where the density  $\rho$  is as in (26) and its inverse

$$\Phi^{-1}: (0, 1) \rightarrow \mathbb{R}.$$

Then, for a function  $g \in L^1(\mathbb{R}; \gamma)$ , it is well-known that with  $\rho$  as in (26)

$$\int_{\mathbb{R}} g(y) d\gamma(y) = \int_{\mathbb{R}} g(y) \rho(y) \, dy = \int_0^1 g(\Phi^{-1}(z)) \, dz$$

upon the substitution  $z = \Phi(y)$ . Especially, we have  $g \circ \Phi^{-1} \in L^1((0, 1))$ . By defining  $\boldsymbol{\Phi}(\mathbf{y}) := [\Phi(y_1), \dots, \Phi(y_m)]^\top$ , we may extend the above integral transform to the multivariate case, i.e.  $g \in L^1_\rho(\mathbb{R}^m)$  and

$$\int_{\mathbb{R}^m} g(\mathbf{y}) \rho(\mathbf{y}) \, d\mathbf{y} = \int_{(0,1)^m} g(\boldsymbol{\Phi}^{-1}(\mathbf{z})) \, d\mathbf{z}.$$

In our application, the integrand  $g(\mathbf{y}) = \hat{u}(\boldsymbol{\xi}, \mathbf{y})$  is the solution to (10)–(12) and itself depends in addition also on the spatial variable  $\boldsymbol{\xi} \in D_{\text{ref}, \kappa}$  as we are interested in the expectation  $\mathbb{E}[\hat{u}(\boldsymbol{\xi})]$ . Moreover, in order to be able to apply the quasi-Monte

Carlo quadrature rule, we need to truncate the stochastic dimension in accordance with

$$(56) \quad \hat{u}(\boldsymbol{\xi}, \mathbf{y}) = \hat{u}_m(\boldsymbol{\xi}, y_1, \dots, y_m) + \hat{u}_m^c(\boldsymbol{\xi}, \mathbf{y}).$$

Here,  $\hat{u}_m$  denotes the solution of the boundary value problem (10)–(12) for the  $m$ -dimensional model, i.e., the model where the series in (25) is truncated after  $m$  terms. This means that  $\hat{u}_m(\boldsymbol{\xi}, \mathbf{y})$  is understood via zero padding, i.e.

$$\hat{u}_m(\boldsymbol{\xi}, y_1, \dots, y_m) = \hat{u}(\boldsymbol{\xi}, y_1, \dots, y_m, \mathbf{0}).$$

The function  $\hat{u}_m^c(\boldsymbol{\xi}) := \hat{u}(\boldsymbol{\xi}) - \hat{u}_m(\boldsymbol{\xi})$  reflects the so-called dimension truncation error. In view of (56), we have with  $\mathbb{E}$  denoting expectation with respect to  $\boldsymbol{\gamma}$

$$\mathbb{E}[\hat{u}(\boldsymbol{\xi})] = \int_{(0,1)^m} \hat{u}_m(\boldsymbol{\xi}, \Phi^{-1}(\mathbf{z})) \, d\mathbf{z} + \mathbb{E}[\hat{u}_m^c(\boldsymbol{\xi})],$$

which we are going to approximate by

$$\mathbb{E}[\hat{u}(\boldsymbol{\xi})] \approx \mathbf{Q}_N(\hat{u}_m(\boldsymbol{\xi}, \Phi^{-1}(\cdot))) + \mathbb{E}[\hat{u}_m^c(\boldsymbol{\xi})].$$

Here, the dimension truncation parameter  $m$  has to be chosen appropriately such that the truncation error  $\|\mathbb{E}[\hat{u}_m^c(\boldsymbol{\xi})]\|_V$  is sufficiently small. The rate of convergence of this error as  $m \rightarrow \infty$  depends solely on the summability of the coefficients in the series (25), compare [19, Thm. 4.1].

As shown in [21], the convergence rate of the quasi-Monte Carlo quadrature based on Halton points for the determination of the expectation  $\mathbb{E}[\hat{u}_m]$  depends only mildly on the dimensionality  $m$  of the random parameter under certain properties of the sequence  $\boldsymbol{\rho} = (\rho_k)_{k \in \mathbb{N}}$  from (40).

**Theorem 4.5.** *The quasi-Monte Carlo quadrature using Halton points for approximating the expectation  $\mathbb{E}[\hat{u}]$  of the solution  $\hat{u}$  to (25) provides a convergence rate which depends only linearly on the dimensionality  $m$  if the sequence  $\boldsymbol{\rho} = (\rho_k)_{k \in \mathbb{N}}$  from (38) satisfies  $\rho_k \gtrsim k^{4+\varepsilon}$ . More precisely, for each  $\delta > 0$ , the error of the quasi-Monte Carlo quadrature with  $N$  Halton points satisfies*

$$(57) \quad \|\mathbb{E}[\hat{u}] - \mathbf{Q}_N \hat{u}\|_V \lesssim (m+1)N^{-1+\delta} \|f\|_{L^2(\mathbb{R}^2)} + \|\mathbb{E}[\hat{u}_m^c]\|_V.$$

The constant hidden in this error estimate depends on  $\varepsilon$  and  $\delta$ , but neither on the dimension  $m \in \mathbb{N}$  nor on the number of points  $N \in \mathbb{N}$ .

*Proof.* Upon observing that  $\boldsymbol{\nu}! \leq |\boldsymbol{\nu}|!$  for  $\boldsymbol{\nu} \in \mathbb{F}$ , estimate (42) (or (40) with (36)) implies [21, Eq. (14)] with  $(\gamma_k)_{k \in \mathbb{Z}} = (\rho_k^{-1})_{k \in \mathbb{Z}}$ , or, equivalently, [21, Eq. (18)] with  $p = 2$  (the proof given there for  $p > 2$  remains valid verbatim then). As  $\gamma_k \lesssim k^{-4-\varepsilon}$  implies the error estimate in (57) by using [21, Thm. 4.3], we conclude the assertion.  $\square$

## 5. FINITE ELEMENT DISCRETIZATION

**5.1. Discretization in the reference domain  $D_{\text{ref}}$ .** We investigate in this subsection the discretization of a general, parameter-dependent, second order, elliptic boundary value problem by the finite element method. To this end, let  $D_{\text{ref}} \subset \mathbb{R}^2$  denote a generic bounded smooth reference domain (such as  $D_{\text{ref}} = D_{\text{ref}, \kappa}$  in (4), for  $0 < \kappa < 1$  but is not necessary for the definition of the finite element discretization). Throughout this section, the parameters  $\mathbf{y}$  will be real-valued. We recall the set  $U_0 \subset \mathbb{R}^{\mathbb{N}}$  from (37).

In  $D_{\text{ref}}$ , we introduce a quasi-uniform, regular, simplicial partition  $\mathcal{T}_h$  and consider continuous, piecewise linear Lagrangian finite element spaces spanned by the usual “hat”-function bases  $\{\varphi_i\}_{i=1}^N$  defined on  $\mathcal{T}_h$ . The finite element space is denoted by

$$V_h = \text{span}\{\varphi_i : i = 1, \dots, N\} \subset V.$$

In order to avoid errors by the geometry approximation, we consider Zenisek’s curved triangles in case of a non-polygonal boundary, compare [30]. We however mention that a piecewise linear approximation of the boundary is consistent and the subsequent theory can be extended to this situation straightforwardly by using standard arguments from [6] or [4, Thm. III.1.7]. We also assume for now exact integration in the stiffness matrix and the load vector. With these assumptions, given  $\mathbf{y} \in U_0 \subset \mathbb{R}^N$ , we define the parametric bilinear form  $B(\cdot, \cdot; \mathbf{y}) : V \times V \rightarrow \mathbb{R}$

$$(58) \quad B(\hat{u}, v; \mathbf{y}) := \int_{D_{\text{ref}}} M(\mathbf{y}) \nabla \hat{u}(\mathbf{y}) \cdot \nabla v$$

and the parametric linear form  $L(\cdot; \mathbf{y}) : V \rightarrow \mathbb{R}$

$$(59) \quad L(v; \mathbf{y}) := \langle f(\mathbf{y}), v \rangle.$$

**Remark 5.1.** *In the analysis of first order finite element approximation errors in  $D_{\text{ref}, \kappa}$ , convergence rates require higher Sobolev regularity of parametric solutions  $\hat{u}(\mathbf{y})$  on  $D_{\text{ref}, \kappa}$ . This, in turn, imposes stronger regularity requirements on the map  $F(a)$  and also on the map  $\mathbf{y} \mapsto a(\mathbf{y})$ .*

*For  $a$  as in (25), we have  $a(\mathbf{y}) \in W_{\infty}^2(\mathbb{S}^1)$  for all  $\mathbf{y} \in U_1 \subset \mathbb{R}^N$  where  $U_1$  is measurable and of full measure with respect to the tensor-product Gaussian measure  $\gamma$  on  $\mathbb{R}^N$  if, in addition to the conditions in Lemma 3.3, there also holds*

$$(60) \quad \left\| \sum_{k \in \mathbb{N}} \tau_k |\psi_k''| \right\|_{C(\mathbb{S}^1)} < \infty.$$

*For  $a(\mathbf{y}) \in W^{2, \infty}(\mathbb{S}^1)$ , the diffusion coefficient  $M(\mathbf{y})$  in (31) has entries  $M_{ij}(\mathbf{y})(x)$  which are (also  $\gamma$ -a.s.) Lipschitz continuous functions of  $x \in \overline{D_{\text{ref}, \kappa}}$ , for  $0 < \kappa < 1$ . Due to  $f \in L^2(D_{\text{ref}, \kappa})$ , for  $0 < \kappa < 1$  then  $\hat{u}(\mathbf{y}) \in H^2(D_{\text{ref}, \kappa})$ . For this regularity and for  $a$  as in (25), the condition  $\kappa > 0$  is essential. The case  $\kappa = 0$  is discussed in Section 5.3 ahead.*

We assume the following that the eigenvalues of diffusion matrix  $M(\mathbf{y})(\boldsymbol{\xi})$  from (32) satisfy for  $\mathbf{y} \in U_0$  and  $\boldsymbol{\xi} \in D_{\text{ref}}$

$$(61) \quad 0 < \lambda_{\min}(\mathbf{y}) \leq \lambda_{\min}(M(\mathbf{y})(\boldsymbol{\xi})) \leq \lambda_{\max}(M(\mathbf{y})(\boldsymbol{\xi})) \leq \lambda_{\max}(\mathbf{y}) < \infty.$$

Also, the right-hand side  $f(\mathbf{y}) \in L^2(D_{\text{ref}})$ .

**Remark 5.2.** *The preceding assumption holds for (25). From the bounds (19), (20) we find with  $a = \exp(b)$  as in (25) that (cf. (6))  $h(\mathbf{y}) := a'(\mathbf{y})/a(\mathbf{y}) = b'(\mathbf{y})$  and definition (37) of  $U_0$  that*

$$(62) \quad \forall \mathbf{y} \in U_0 : \max\{\lambda_{\min}(\mathbf{y})^{-1}, \lambda_{\max}(\mathbf{y})\} \leq 2 + \|b'(\mathbf{y})\|_{L^{\infty}(\mathbb{S}^1)}^2 < \infty.$$

*For any  $r > 0$  exists  $C_r > 0$  such that*

$$(63) \quad \forall \mathbf{y} \in U_0 : \max\{\lambda_{\min}(\mathbf{y})^{-r}, (\lambda_{\max}(\mathbf{y}))^r\} \leq C_r (1 + \|b'(\mathbf{y})\|_{L^{\infty}(\mathbb{S}^1)}^{2r}) < \infty.$$

For  $\mathbf{y} \in U_0$ , we define  $\hat{u}_h(\mathbf{y})$  as solution of the parametric Galerkin equations:

$$(64) \quad \text{Seek } \hat{u}_h(\mathbf{y}) \in V_h \text{ such that } B(\hat{u}_h(\mathbf{y}), v_h; \mathbf{y}) = L(v_h, \mathbf{y}) \text{ for all } v_h \in V_h.$$

**Theorem 5.3.** For  $\mathbf{y} \in U_1$  as defined in Remark 5.1, the finite element solution  $\hat{u}_h(\mathbf{y})$  to (58)–(64) satisfies the error estimate

$$(65) \quad \|\hat{u}(\mathbf{y}) - \hat{u}_h(\mathbf{y})\|_V \lesssim \sqrt{\frac{\lambda_{\max}(\mathbf{y})}{\lambda_{\min}(\mathbf{y})}} \frac{c(\mathbf{y})}{\lambda_{\min}^2(\mathbf{y})} h \|f(\mathbf{y})\|_{L^2(D_{\text{ref}})}, \quad 0 < h \leq h_0,$$

for some  $h_0 > 0$ . Here the constant hidden in  $\lesssim$  is independent of  $\mathbf{y}$ , and on

$$(66) \quad c(\mathbf{y}) := \|M(\mathbf{y})\|_{W_{\infty}^1(D_{\text{ref}})}.$$

Here,  $\|M(\mathbf{y})\|_{W_{\infty}^1(D_{\text{ref}})}$  is a norm of the matrix of  $W_{\infty}^1(D_{\text{ref}})$ -norms of elements  $M_{ij}(\mathbf{y})$  of  $M(\mathbf{y})$ .

*Proof.* In view of the ellipticity and continuity constants  $\lambda_{\min}(\mathbf{y})$  and  $\lambda_{\max}(\mathbf{y})$ , respectively, we obtain that for every  $\mathbf{y} \in U_0$  there exists a unique solution  $\hat{u}_h(\mathbf{y}) \in V_h$  of (64). Céa's lemma implies for every fixed  $\mathbf{y} \in U_0$  for the Galerkin solution  $\hat{u}_h(\mathbf{y}) \in V_h$  of (58)–(64) the error estimate

$$\|\hat{u}(\mathbf{y}) - \hat{u}_h(\mathbf{y})\|_V \lesssim \sqrt{\frac{\lambda_{\max}(\mathbf{y})}{\lambda_{\min}(\mathbf{y})}} \inf_{v_h \in V_h} \|\hat{u}(\mathbf{y}) - v_h\|_V$$

with a generic constant in  $\lesssim$  that is independent of  $\mathbf{y}$ . If  $\hat{u}(\mathbf{y}) \in H^2(D_{\text{ref}})$ , we thus obtain

$$(67) \quad \|\hat{u}(\mathbf{y}) - \hat{u}_h(\mathbf{y})\|_V \lesssim \sqrt{\frac{\lambda_{\max}(\mathbf{y})}{\lambda_{\min}(\mathbf{y})}} h \|\hat{u}(\mathbf{y})\|_{H^2(D_{\text{ref}})}, \quad 0 < h \leq h_0,$$

for some  $h_0 > 0$ , by standard interpolation error estimates, see [4, 6] for example. Since the problem under consideration is  $H^2(D_{\text{ref}})$  regular if the coefficient matrix satisfies  $M(\mathbf{y}) \in W_{\infty}^1(D_{\text{ref}})$ , we find by tracking carefully the constants in the proofs of [17, Thms. 8.8 & 8.12] or [16, Sect. 6.3, Thms. 1 & 4]

$$(68) \quad \|\hat{u}(\mathbf{y})\|_{H^2(D_{\text{ref}})} \lesssim \frac{c(\mathbf{y})}{\lambda_{\min}(\mathbf{y})} \left( \|f(\mathbf{y})\|_{L^2(D_{\text{ref}})} + \|\hat{u}(\mathbf{y})\|_{H^1(D_{\text{ref}})} \right).$$

Herein, the constant hidden in  $\lesssim$  is again independent of  $\mathbf{y}$ , and  $c(\mathbf{y})$  has finite expectation under  $\gamma$ . By using the standard stability estimate

$$\|\hat{u}(\mathbf{y})\|_V \lesssim \frac{1}{\lambda_{\min}(\mathbf{y})} \|f(\mathbf{y})\|_{L^2(D_{\text{ref}})},$$

with constant in  $\lesssim$  depending only in  $D_{\text{ref}}$  we finally arrive at

$$\|\hat{u}(\mathbf{y})\|_{H^2(D_{\text{ref}})} \lesssim \frac{c(\mathbf{y})}{\lambda_{\min}^2(\mathbf{y})} \|f(\mathbf{y})\|_{L^2(D_{\text{ref}})}.$$

By inserting this estimate into (67) we arrive at (65).  $\square$

**5.2. Fully discrete finite element method.** If the expectation of the  $\mathbf{y}$ -dependent right-hand side of the estimate (65) is finite with respect to the Gaussian measure, then (65) implies that we can solve the problem under consideration on a fixed finite element mesh with a sufficiently small mesh size  $h$ . Especially, due to Galerkin orthogonality, the solution by piecewise linear Lagrangian finite elements does not change if we replace the diffusion matrix  $M(\mathbf{y})$  on each finite element triangle  $T$  by its mean. Nonetheless, in practice, we have to apply numerical quadrature.

In what follows, we investigate the impact of numerical quadrature in form of the midpoint rule. This means that we replace the original variational formulation (58)–(64) by

$$(69) \quad \text{Seek } \hat{u}_h(\mathbf{y}) \in V_h \text{ such that } B_h(\hat{u}_h(\mathbf{y}), v_h; \mathbf{y}) = L_h(v_h, \mathbf{y}) \text{ for all } v_h \in V_h$$

with the bilinear form  $B_h(\cdot, \cdot; \mathbf{y})$  which is derived from the bilinear form  $B(\cdot, \cdot; \mathbf{y})$  by approximating all integrals over triangles  $T \in \mathcal{T}_h$  with the midpoint rule and likewise for the linear form  $L_h(\cdot; \mathbf{y})$ .

**Theorem 5.4.** *Let  $B_h(\cdot, \cdot; \mathbf{y})$  and  $L_h(\cdot; \mathbf{y})$  be the fully discrete versions of  $B(\cdot, \cdot; \mathbf{y})$  and  $L(\cdot; \mathbf{y})$  obtained by application of the midpoint rule.*

*Then for  $\mathbf{y} \in U_1$  as in Remark 5.1, the parametric FE-solution  $\hat{u}_h(\mathbf{y}) \in V_h$  to the fully discrete problem (69) satisfies the error estimate*

$$(70) \quad \|\hat{u}(\mathbf{y}) - \hat{u}_h(\mathbf{y})\|_V \lesssim C(\mathbf{y})h\|f(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}})}$$

with the constant hidden in  $\lesssim$  independent of  $\mathbf{y}$  and

$$C(\mathbf{y}) = \left(1 + \frac{\lambda_{\max}(\mathbf{y})}{\lambda_{\min}(\mathbf{y})}\right) \left(\sqrt{\frac{\lambda_{\max}(\mathbf{y})}{\lambda_{\min}(\mathbf{y})}} \frac{c(\mathbf{y})}{\lambda_{\min}^2(\mathbf{y})} + c(\mathbf{y}) + 1\right).$$

*Proof.* The midpoint rule applied to the restriction of bilinear form  $B(v_h, v_h; \mathbf{y})$  to an arbitrary triangle  $T \in \mathcal{T}_h$  of the triangulation  $\mathcal{T}_h$  is given by

$$B_h|_T(v_h, v_h; \mathbf{y}) = |T|\nabla v_h(\boldsymbol{\xi}_T)^\top M(\mathbf{y})(\boldsymbol{\xi}_T)\nabla v_h(\boldsymbol{\xi}_T),$$

where the size  $|T|$  of the triangle  $T$  is the quadrature weight and the barycenter  $\boldsymbol{\xi}_T \in T$  of  $T$  is the only quadrature point. Observe that  $\nabla v_h|_T$  is constant in  $T$ . We find

$$(71) \quad B_h|_T(v_h, v_h; \mathbf{y}) \geq |T|\lambda_{\min}(\mathbf{y})\|\nabla v_h(\boldsymbol{\xi}_T)\|_2^2 \simeq \lambda_{\min}(\mathbf{y})\|\nabla v_h\|_{L^2(T)}^2$$

with a constant hidden in  $\simeq$  that is independent of the mesh size  $h$  and of the parameter  $\mathbf{y}$ . Summing (71) over all  $T \in \mathcal{T}_h$ , the bilinear form  $B_h(\cdot, \cdot; \mathbf{y})$  is uniformly elliptic, i.e.,

$$\forall v_h \in V_h : B_h(v_h, v_h; \mathbf{y}) \gtrsim \lambda_{\min}(\mathbf{y})\|v_h\|_V^2$$

independent of the mesh size  $h$ . The parametric matrix of the linear system of equations which corresponds to (69) is, for fixed  $\mathbf{y} \in U_0$  positive definite and symmetric uniformly with respect to  $h$ , with smallest eigenvalue bounded from below by a constant that depends only on the quotient of  $\lambda_{\max}(\mathbf{y})$  and  $\lambda_{\min}(\mathbf{y})$ . Hence, for every  $\mathbf{y} \in U_0$  exists a unique solution  $\hat{u}_h \in V_h$  of (69). Strang's first lemma (e.g. [4, 6, 9]) implies

$$(72) \quad \|\hat{u}(\mathbf{y}) - \hat{u}_h(\mathbf{y})\|_V \lesssim \left(1 + \frac{\lambda_{\max}(\mathbf{y})}{\lambda_{\min}(\mathbf{y})}\right) \inf_{v_h \in V_h} \left\{ \|\hat{u}(\mathbf{y}) - v_h\|_V + \sup_{w_h \in V_h} \frac{|B(v_h, w_h; \mathbf{y}) - B_h(v_h, w_h; \mathbf{y})|}{\|w_h\|_V} + \sup_{w_h \in V_h} \frac{|L(w_h; \mathbf{y}) - L_h(w_h; \mathbf{y})|}{\|w_h\|_V} \right\}.$$

We bound the three terms on the right-hand side. The first term is the usual best-approximation error which is  $\mathcal{O}(h)$  due to (68) using approximation properties of first order Lagrangian Finite Elements on regular, quasi-uniform triangulations  $\mathcal{T}_h$  of  $D_{\text{ref}}$ , and the (assumed) regularity  $\hat{u}(\mathbf{y}) \in H^2(D_{\text{ref}})$ .

For  $\|M(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}})} < \infty$  the midpoint rule on  $T \in \mathcal{T}_h$  has the accuracy<sup>1</sup>

$$\left| B|_T(v_h, w_h; \mathbf{y}) - B_h|_T(v_h, w_h; \mathbf{y}) \right| \lesssim h|T| \|M(\mathbf{y})\|_{W_\infty^1(T)} \|\nabla v_h\|_{W_\infty^1(T)} \|\nabla w_h\|_{W_\infty^1(T)}.$$

Summing over  $T \in \mathcal{T}_h$  bounds the second term as

$$(73) \quad |B(v_h, w_h; \mathbf{y}) - B_h(v_h, w_h; \mathbf{y})| \lesssim hc(\mathbf{y}) \|v_h\|_V \|w_h\|_V$$

with the constant  $c(\mathbf{y})$  defined in (66).

For the third term in the bound (72) the midpoint rule for  $T \in \mathcal{T}_h$  yields

$$L_h|_T(w_h; \mathbf{y}) = |T|(f(\mathbf{y}))(\boldsymbol{\xi}_T)w_h(\boldsymbol{\xi}_T).$$

Hence, we arrive at the element-wise error bound

$$\left| L|_T(w_h; \mathbf{y}) - L_h|_T(w_h; \mathbf{y}) \right| \lesssim h|T| \|f(\mathbf{y})\|_{W_\infty^1(T)} \|w_h\|_{W_\infty^1(T)}.$$

Summing over  $T \in \mathcal{T}_h$ , we arrive at

$$(74) \quad |L(w_h; \mathbf{y}) - L_h(w_h; \mathbf{y})| \lesssim h \|f(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}})} \|w_h\|_V.$$

Inserting (65), (73), and (74) into (72) and employing the bound

$$\|f(\mathbf{y})\|_{L^2(D_{\text{ref}})} \lesssim \|f(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}})}$$

yields (70). □

Again, if the expectation of the right-hand side of the error estimate (70) is finite with respect to the Gaussian measure, the proposed fully discrete finite element method is sufficient to approximate the parametric solution  $\hat{u}(\mathbf{y})$  in expectation.

**5.3. Application to PDEs on log-Gaussian domain.** The aforementioned theory of the fully discrete finite element method applies for instance directly to the solution of (3) with log-Gaussian scaling  $a$  as in (25), i.e.  $a(\mathbf{y})(\theta) = \exp(b(\mathbf{y})(\theta))$ .

This means that it has been proven that piecewise linear ansatz functions on a fixed finite element mesh in combination with a midpoint rule is sufficient to compute the expected solution at optimal rate  $\mathcal{O}(h)$  with respect to the energy norm, in expectation with respect to the Gaussian measure  $\gamma$  over the parameter set  $U_1$  defined in Remark 5.1. In particular, *numerical integration order* and *Karhunen-Loève truncation* need not be path-dependent. Due to (61) and the bounds (62), for Example 3.1, with small probability large values for  $\lambda_{\max}(\mathbf{y})/\lambda_{\min}(\mathbf{y})$  can arise, with corresponding large values of the condition number of the matrix corresponding to the parametric bilinear form  $B_h(\cdot, \cdot; \mathbf{y})$  in (69).

In our particular situation, we use the domain parametrization (25) in (2). The generic reference domain  $D_{\text{ref}}$  in Theorems 5.3, 5.4 is any of  $D_{\text{ref}, \kappa}$  for  $0 \leq \kappa < 1$ . By Remark 5.2 and (11), for  $\mathbf{y} \in U_1 \subset \mathbb{R}^N$  it holds  $M_{ij}(\mathbf{y}) \in C(\overline{D_{\text{ref}, \kappa}})$  for  $0 < \kappa < 1$  and  $i, j = 1, 2$ . Furthermore, for  $c(\mathbf{y})$  in (66) holds  $\mathbb{E}[c(\cdot)] < \infty$ . Also (11) shows that in this case  $M_{ij}(\mathbf{y}) \in C(\mathbb{S}^1)$  for  $\mathbf{y} \in U_1$ .

The polar parametrization (5) of the random geometry map which underlies the homothetic transformation can induce singularities in  $M(a)$  at the origin when

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<sup>1</sup> We exploit here only the first order derivatives of the integrand instead of the second order derivatives as this is sufficient to get  $\mathcal{O}(h)$  accuracy.

$\kappa = 0$ . On the one hand, we can estimate for  $\mathbf{y} \in U_1$  in Remark 5.1

$$\begin{aligned} \|f(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}})} &= \|J(\mathbf{y})(f \circ F)(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}}, \kappa)} \\ &\lesssim \|J(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}}, \kappa)} \|(f \circ F)(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}}, \kappa)} \\ &\lesssim \|a(\mathbf{y})\|_{W_\infty^1(\mathbb{S}^1)}^2 \|f\|_{W_\infty^1(\mathbb{R}^2)} \end{aligned}$$

which bounds  $\|f(\mathbf{y})\|_{W_\infty^1(D_{\text{ref}})}$  in (74) deterministically for  $\mathbf{y} \in U_1$ .

On the other hand, for  $\kappa = 0$ , the domain mapping (5) does not amount to a coefficient matrix  $M(a)$  that is in  $W_\infty^1(D_{\text{ref}, 0})$  – only for  $0 < \kappa < 1$  it is in  $C^1(\overline{D_{\text{ref}, \kappa}})$  provided that  $a \in C^2(\mathbb{S}^1)$ . Indeed, when differentiating (11) (with respect to  $\boldsymbol{\xi}$ ), one arrives at a matrix that contains the factor  $1/r$ , which implies an  $1/r$ -singularity at the origin, compare (11). Hence, the constant  $c(\mathbf{y})$  in (66) depends on  $1/\kappa$  and the estimate (65) holds only in  $D_{\text{ref}, \kappa}$  for  $\kappa > 0$ . As a consequence, on  $D_{\text{ref}, 0}$  a uniform mesh  $\mathcal{T}_h$  yields only a reduced order of convergence of the finite element method. In our numerical experiments, we refine the finite element mesh towards the origin to overcome this obstruction.

**Remark 5.5.** *For random shapes generated by a log-Gaussian parametrization (29),  $\mathbf{y}$ -uniform bounds in the FE error analysis with iterative solvers can not be expected. However, the numerical complexity scales optimally in expectation (i.e. on average with respect to  $\gamma$ ), i.e., linearly with respect to the number  $N$  of degrees of freedom, when using multigrid solvers [23].*

## 6. NUMERICAL EXPERIMENT

In our numerical experiment, we consider the right-hand side  $f(\mathbf{x}) = \exp(-\|\mathbf{x}\|_2)$  and the domain perturbation (5) with  $\kappa = 0$  and

$$(75) \quad a(\mathbf{y})(\theta) = \exp\left(y_0 \lambda_0 + \sum_{k=1}^{\infty} y_k \lambda_k \cos(k\theta) + y_{-k} \lambda_{-k} \sin(k\theta)\right)$$

as in Example 3.1. The particular series  $(\lambda_k)_{k \in \mathbb{Z}}$  we prescribe is

$$(76) \quad \lambda_0 = \frac{1}{8} \quad \text{and} \quad \lambda_k = \frac{1}{(|k| + 1)^3} \quad \text{for } k \in \mathbb{Z} \setminus \{0\}.$$

The sum in the Fourier series (75) is truncated after  $K = 199$  terms, which yields the dominant 399 terms of the series, i.e. we used

$$a^K(\mathbf{y})(\theta) := \exp\left(b^K(\mathbf{y})(\theta)\right)$$

in our numerical experiments. This was sufficiently accurate for our tests, and choosing larger values of  $K$  does not change the results. Remark that in Theorem 4.5,  $m = \mathcal{O}(K)$ .

Accordingly, besides the errors due to FE discretization error and numerical integration in computing the bilinear form  $B_h(\cdot, \cdot; \mathbf{y})$  in (69), we incur a further error due to replacing  $a$  by  $a^K$ , resulting in the matrix  $M^K(\mathbf{y}) := M(a^K(\mathbf{y}))$  with  $M(a^K)$  as defined in (11).

The summability condition is satisfied with  $B_\rho \approx 0.876$  if we plug  $\rho_k = 1$  for all  $k \in \mathbb{Z}$  into (41). However, the decay of the series  $(\lambda_k)_{k \in \mathbb{Z}}$  is slower than required to satisfy the assumptions of Theorem 4.5 for the convergence of the quasi-Monte

Carlo method. Indeed, it has already been observed in [21] that the resulting condition for  $(\rho_k)_{k \in \mathbb{Z}}$  is not sharp.

We introduce a shape regular triangular mesh of the unit disk  $D_{\text{ref},0}$  by using Zenisek's curved triangles, see [30]. The mesh is a-priorily graded towards the origin as seen in Figure 3. On this triangulation, we define piecewise linear Lagrangian finite element functions. For our experiments, we use a triangulation with about 20000 triangles which leads to about 10000 finite element basis functions. The solver we use is based on a conjugate gradient method which is preconditioned with the BPX preconditioner, compare [5, 6]. Four realizations of the random domain under consideration and the associated solution of the Poisson problem are found in Figure 3, where the mesh refinement towards the origin are clearly visible.

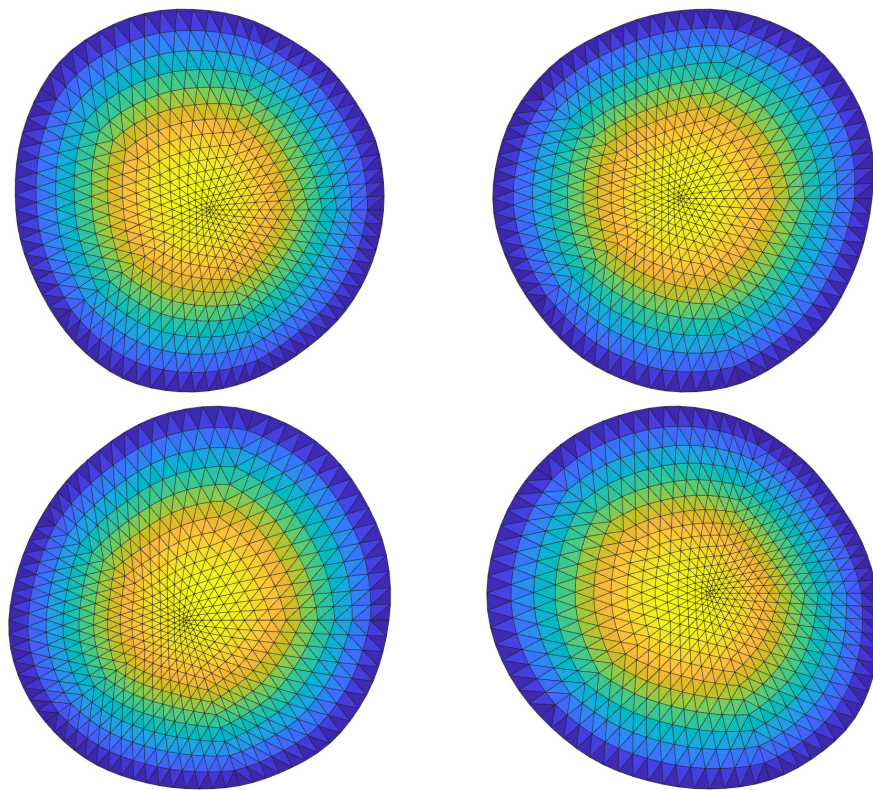


FIGURE 3. Four realizations of the random domain under consideration with the associated solution of the underlying Poisson equation. Shape parametrization (29) with  $\kappa = 0$  and (76). The triangulation is the mapped version of the triangulation of the unit disk. The grading towards the origin is clearly visible.

We run our experiments and measure the rate of convergence realized by the quasi-Monte Carlo method. As we do not know the exact solution, we compute a numerical reference solution using  $2 \cdot 10^6$  sample points which serves as reference solution. Indeed, as one can see in Figure 4, one observes a convergence rate that is nearly linear convergence (57) as predicted from the theory in [21]. The computed

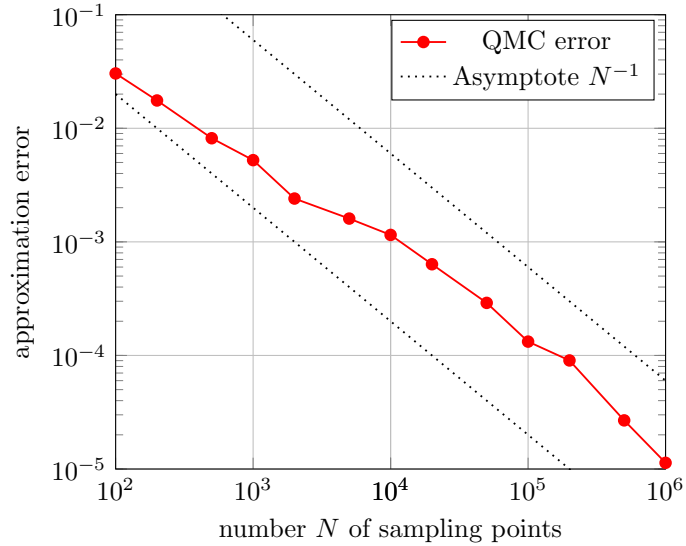


FIGURE 4. Error of the approximation to  $\|\mathbb{E}[\hat{u}]\|_{H^1(D_{\text{ref},0})}$  of the quasi-Monte Carlo method versus the number  $N$  of sampling points.

expectation of the random solution is shown in Figure 5. Note that all 2 million domain samples are contained in the disk of radius 2 centered at the origin.

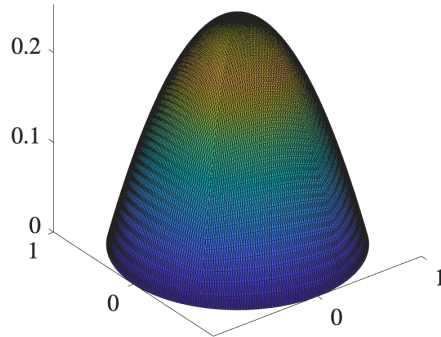


FIGURE 5. The expectation of the random solution on the unit disk.

## 7. CONCLUSION

In the present article, we studied second order elliptic boundary value problems on a class of random domains. The domains we considered are homothetic with respect to a reference domain via a lognormally distributed radial dilation function. Since the domain variations can be large with small probability we employed the domain mapping method to transform the problem posed on the random domain into a problem on the deterministic reference domain. We then have shown the

well-posedness of the pullback problem and verified the analytic dependency of the random solution on the random input parameter by means of holomorphy arguments.

We emphasize that our particular model of the random domain can be straightforwardly extended. For example, the radial function can be modified by

$$a(\mathbf{y})(\theta) = r(\theta) + \exp\left(\sum_{k \in \mathbb{N}} y_k \psi_k(\theta)\right)$$

to account for domains whose boundaries are homothetic, lognormal scalings with respect to the radial coordinate function  $r(\theta)$  of a reference domain  $D_{\text{ref}, \kappa}$  with Lipschitz boundary  $\partial D_{\text{ref}, \kappa}$ , such as  $(-1, 1)^2 \setminus [-1/2, 1/2]^2$ .

Although the presently considered Gaussian geometric shape uncertainty model could be viewed as rather special, it served to elucidate the mathematical and computational issues in numerical approximation. The present findings are expected to be relevant also in other contexts. E.g., one can also consider the unit square  $(0, 1)^2$  as reference domain and the random domain

$$\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < a(\mathbf{y})(x_1)\}$$

with

$$a(\mathbf{y})(x_1) = \exp\left(\sum_{k \in \mathbb{N}} y_k \psi_k(x_1)\right),$$

where  $\{\psi_k\}$  are (non-periodic) functions defined on the interval  $(0, 1)$ . Similarly, the trigonometric uncertainty parametrization in Example 3.1 may be replaced by a localized one, e.g. of Lévy-Cieselski type. An extension to the three dimensional setting is also possible.

We also discussed the numerical solution of the random boundary value problem by using finite elements in space. We have shown that a fixed discretization suffices for the spatial variable. In the random parameter, we used here a quasi-Monte Carlo method, but also a sparse grid quadrature or higher order quasi-Monte Carlo methods are applicable, see [12, 13, 18] for example. By some numerical experiments, we finally validated our theoretical findings.

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DINH DŨNG, INFORMATION TECHNOLOGY INSTITUTE, VIETNAM NATIONAL UNIVERSITY, 144 XUAN THUY, CAU GIAY, HANOI, VIETNAM  
*E-mail address:* `dinhzung@gmail.com`

HELMUT HARBRECHT, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BASEL, SPIEGELGASSE 1, 4051 BASEL, SWITZERLAND  
*E-mail address:* `helmut.harbrecht@unibas.ch`

VAN KIEN NGUYEN, DEPARTMENT OF MATHEMATICAL ANALYSIS, UNIVERSITY OF TRANSPORT AND COMMUNICATIONS, NO. 3 CAU GIAY STREET, HANOI, VIETNAM  
*E-mail address:* `kiennv@utc.edu.vn`

CHRISTOPH SCHWAB, SEMINAR FOR APPLIED MATHEMATICS, ETH ZURICH, RÄMISTRASSE 101, 8092 ZURICH, SWITZERLAND  
*E-mail address:* `schwab@math.ethz.ch`